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On the density of integers with consecutive divisors

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Abstract. The density of positive integers which have divisors of the form $x(x+1)$ with $x \in \mathbb{Z}$ and $x \geq K$ is near $1/K$ as K tends to infinity. Different generalizations of this result are also studied.

1. Introduction

Paul Erdős asked several problems concerning divisors. In particular, P. Erdős and R. R. Hall [4] initiated the study of the number of consecutive divisors. They defined $\tau_k(n)$ by the number of positive divisors of n of the form

$$x(x+1)\dots(x+k-1)$$

with $x \in \mathbb{Z}$. In the case $k=2$ an equivalent definition is

$$\tau_2(n) = |\{i : d_{i+1} - d_i = 1\}|,$$

where $1 = d_1 < d_2 < \dots < d_{\tau(n)} = n$ denote the all positive divisors of n. In [4] it is proved that

$$\tau_k(n) > (\log n)^{e^{1/k}-\varepsilon}$$

holds for infinitely many n. P. Erdős and R. R. Hall also estimated the average value of $\tau_k(n)$ for $k \geq 2$ by proving

$$\frac{1}{x}\sum_{n\leq x}\tau_k(n) = \frac{1}{(k-1)(k-1)!} + O(x^{-(k-1)/k}).$$

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For $k=2$, this gives

$$\frac{1}{x}\sum_{n\le x}\tau_2(n)=1+O(x^{-1/2}).$$

Although the expected value of $\tau_2(n)$ is 1, it is not true that almost all integers n have a divisor of the form $x(x+1)$ with $x\in\mathbb{Z}$. It is easy to see that an integer has such a divisor if and only if it is even. Indeed, $x(x+1)$ is always even, thus if $x(x+1)\mid n$, then n is also even. On the other hand if n is even, then n has a divisor of the form $x(x+1)$, namely $1\cdot 2\mid n$.

In the present paper I will estimate the density of integers n which have at least one divisor of the form $x(x+1)$ with $x\in\mathbb{Z}$ and $x\ge K$. Define $A(K)$ by

$$A(K)\stackrel{\text{def}}{=}\lim_{N\to\infty}\frac{1}{N}\left|\{n:1\le n\le N,\ \exists\, x\ge K,\ \text{such that } x(x+1)\mid n\}\right|.$$

As it was shown above $A(1)=0.5$. For $K\ge 2$ I have not been able to determine the exact value of $A(K)$, but I will prove that it is near $1/K$ as $K\to+\infty$.

Throughout the paper the number of distinct positive prime divisors of n will be denoted by $\omega(n)$, the number of positive divisors by $\tau(n)$.

The first upper bound for $\tau_2(n)$ is due to Tenenbaum [17, Theorem 2], who proved that

$$\tau_2(n)\ll\tau(n)^c \tag{1}$$

holds with $c=0.93974\ldots$. R. de la Bretèche [3, Theorem 2] improved on the exponent c and obtained (1) with $c=0.91829\ldots$. $\tau_2(n)$ was studied by several authors, see in [1], [2], [3], [4], [5] and [17]. R. de la Bretèche [2] extended the problem to other polynomials.

I will also study the question for other polynomials of degree 2.

Definition 1. For $P(X)\in\mathbb{Z}[X]$, let

$$A_P(K)\stackrel{\text{def}}{=}\lim_{N\to\infty}\frac{1}{N}\left|\{n:1\le n\le N,\ \exists\, x\ge K \text{ such that } P(x)\mid n\}\right|.$$

If $P(X)=X(X+1)$ then $A_P(K)=A(K)$. I will prove the following.

Theorem 1. *Let $P(X)=a_2X^2+a_1X+a_0\in\mathbb{Z}[X]$ be a polynomial of degree 2 with non-zero discriminant, so $a_1^2-4a_2a_0\ne 0$. Then for $K\ge 1$ we have*

$$A_P(K)=\frac{1}{|a_2|\,K}+O((\log K)^{(5+2\sqrt{2})/3}K^{-4/3}), \tag{2}$$

where the implied constant factor in the $O(\ldots)$ term depends only on the polynomial $P(X)$.

R. DE LA BRETÈCHE [2] gave upper bound for the maximal number of divisors of n of the form $P(x)$. Here, in Theorem 1 I estimate the average value $A_P(K)$.

If $K \to +\infty$ then (2) provides a sharp estimate for $A_P(K)$, but for fixed K does not give any bounds for it. When $P(X) = X(X+1)$, so $A_P(K) = A(K)$ here I give some estimates calculated by a computer program:

$$0.221 \leq A(2) \leq 0.225,$$

$$0.166 \leq A(3) \leq 0.187,$$

$$0.127 \leq A(4) \leq 0.153,$$

$$0.102 \leq A(5) \leq 0.130,$$

$$0.088 \leq A(6) \leq 0.119,$$

$$0.076 \leq A(7) \leq 0.110.$$

These results are obtained by sieve method. In order to get $A(K)$ we calculated the density of integers which is divisible at least by one number of $K(K+1),\ (K+1)(K+2), \ldots, T(T+1)$, where T depends on K. This gives a lower bound for $A(K)$. In order to obtain an upper bound we need to add to the lower bound the density of integers which is divisible at least by one number of $(T+1)(T+2),\ (T+2)(T+3), \ldots$. For this number we use the trivial estimate $\frac{1}{(T+1)(T+2)} + \frac{1}{(T+2)(T+3)} + \cdots = \frac{1}{T+1}$.

In [6] EVERTSE has proved that S-unit equations have only finite solutions. In the rational case he obtained that if $S = \{p_1, p_2, \ldots, p_s\}$ is a set of s distinct primes, $\lambda, \mu \in \mathbb{Z}$ then the equation

$$\lambda a + \mu b = 1$$

has at most

$$3 \times 7^{2s+3}$$

solutions in a and b such that all prime divisors of a and b are from S. GYŐRY [9] gave an upper bound for the absolute value of such solutions a and b. Moreover, there are reasonably efficient algorithms to determine these solutions. For example, DE WEGER [18] in his thesis determined all solutions of the equation $a + b = 1$, where both $a, b \in \{2^{z_1}3^{z_2}5^{z_3}7^{z_4}11^{z_5}13^{z_6} : z_i \in \mathbb{Z}\}$. Using these results, by computer it is also possible to calculate the density of those integers n which have a divisor of the form $x(x+1)$ such that all prime factors of $x(x+1)$ belong to a fixed set of primes. Indeed, then writing $a = x+1$ and $b = x$ it is clear that

first we need to find the all solutions of the equation

$$a - b = 1,$$

where all prime divisors of a and b are from certain fixed set S containing only few small fixed primes. Knowing all such solutions, by the exclusion-inclusion principle we easily could determine the density asked. This method could give reasonable estimates calculated by computer, but can not be used in the proof of Theorem 1, which is based on generalized Pell-equations.

Pell-equations have been extensively studied for long time, but on generalized Pell-equations slightly less is known. S. Lange [13] gave asymptotic formulas for the number of solutions. Later K. Győry, A. Pethő and recently G. R. Everest generalized this result to norm form equations, see in [7], [10], [11], [12], [15] and [16]. In these asymptotics the main term depends on the number of certain "fundamental solutions", but usually their number is not given in terms of the coefficients of the norm form equation. In [10] Győry and Pethő gave upper and lower bound in terms of the coefficients. Here, in the case of generalized Pell-equations we need a stronger bound than the one in [10]. I will prove

Theorem 2. *Let a, b, c be positive integers. Denote by S the number of the solutions of the generalized Pell-equation*

$$ax^2 - by^2 = c \tag{3}$$

with positive integers x, y and $1 \le y \le N$. Then

a) $S < \frac{\log N}{\log(4ab/c^2)} + 1$ *if* $c^2 < 4ab$.

b) *Let $m \mid c$ such that $(m, ab) = 1$. Then*

$$S \le \left\lceil \frac{ec}{\sqrt{ab}m} \right\rceil \tau(m)\,(\log N + 2) \ll \left(1 + \frac{c}{\sqrt{ab}m}\right) \tau(m) \log N.$$

The usual estimates for the number of solutions of the generalized Pell-equations only handle the case $ac < \sqrt{ab}$, and there are algorithms for finding the solutions but without estimates for their number.

In fact in the proof of Theorem 1 I need the following sum:

Corollary 1. *Suppose that the conditions of Theorem 2 b) hold and A is a positive number. Then*

$$\sum_{\substack{ax^2 - by^2 = c \\ x \ge 0,\ y \ge A}} \frac{1}{y^2} \ll \left(1 + \frac{c}{\sqrt{ab}m}\right) \frac{\tau(m)}{A^2}.$$

Difficult problem is the estimate of the density of natural numbers which have at least t distinct divisors of the form $x(x+1)$ with $x \in \mathbb{N}$. A trivial estimate for this density is $\frac{1}{[1,2,\dots,t+1]} = \frac{1}{e^{(1+o(1))t}}$, since the numbers n with $[1,2,\dots,t+1] \mid n$ are divisible by $1\cdot 2, 2\cdot 3, \dots, t\cdot(t+1)$. By an argument of Erdős and Hall [4] the exponent can be improved. Namely the density asked is

$$\gg \frac{1}{e^{\left(t^{(e^{-1/2}+\varepsilon)}\right)}} \tag{4}$$

as $t \to +\infty$. $[1,2,\dots,y]$ has more than y positive divisors of the form $x(x+1)$ with $x \in \mathbb{N}$. Indeed, Erdős and Hall in [4, Theorem 2] proved that for $A < e^{1/2}$ there exists $\varepsilon > 0$ such that $n = [1,2,\dots,y]$ has at least εy^A divisors of the form $x(x+1)$. The prime number theorem implies that $[1,2,\dots,y] = e^{(1+o(1))y}$, and writing $y = \left(\frac{1}{\varepsilon}t\right)^{1/A}$, we get that there exists a constant $k > 0$ such that $n = [1,2,\dots,y] \ll e^{kt^{1/A}}$ has at least t distinct positive divisors of the form $x(x+1)$ with $x \in \mathbb{N}$, which leads to (4).

2. Proof of Theorem 1

We may suppose that the leading coefficient of $P(X)$ is positive: $a_2 > 0$. We choose f so that

$$a_2(2f-1) \le a_1 < a_2(2f+1)$$

holds, then f depends only on the polynomial $P(X)$. It is clear that there exists a constant K_0 depending on the polynomial $P(X)$ such that for $x > K_0$ we have

$$a_2(x+f-1)(x+f) \le P(x) \le a_2(x+f)(x+f+1). \tag{5}$$

Throughout the proof of Theorem 1 we may suppose that $K > K_0$ where K_0 is a large enough constant depending only on the polynomial $P(X)$, since for $K \le K_0$ the theorem is trivial because of the $O(\dots)$ term in (2).

For the density of the positive integers divisible by at least one of the numbers $P(K),\ P(K+1),\ P(K+2),\dots$ an upper bound is

$$A_P(K) \le \frac{1}{P(K)} + \frac{1}{P(K+1)} + \frac{1}{P(K+2)} + \dots.$$

By (5)

$$A_P(K) \le \frac{1}{a_2(K+f-1)(K+f)} + \frac{1}{a_2(K+f)(K+f+1)} + \dots$$

$$\le \frac{1}{a_2}\left(\left(\frac{1}{K+f-1}-\frac{1}{K+f}\right)+\left(\frac{1}{K+f}-\frac{1}{K+f+1}\right)+\dots\right)$$
$$=\frac{1}{a_2(K+f-1)}=\frac{1}{a_2K}+O(K^{-2}).$$

Next we will give a lower bound for $A_P(K)$. For positive integers T and K let $A_P(K,T)$ the density of positive integers divisible by at least one of the numbers $P(K),\ P(K+1),\ P(K+2),\dots,\ P(T)$:

$$A_P(K,T)\stackrel{\text{def}}{=}\lim_{N\to\infty}\frac{1}{N}\left|\{n:1\le n\le N,\ \exists\ K\le x\le T,\ \text{such that } P(x)\mid n\}\right|.$$

Clearly for $T\ge K$ we have

$$A_P(K)\ge A_P(K,T). \tag{6}$$

We will give a lower bound for $A_P(K,T)$ by the exclusion-inclusion principle. For $K\le i_1<i_2<\dots<i_k\le T$ let

$$N_{i_1,i_2,\dots,i_k}\stackrel{\text{def}}{=}\lim_{N\to\infty}\frac{1}{N}\left|\{n:1\le n\le N,\ P(i_1),\ P(i_2),\dots,\ P(i_k)\mid n\}\right|$$
$$=\frac{1}{[P(i_1),\ P(i_2),\dots,\ P(i_k)]}.$$

Then

$$A_P(K,T)\ge\sum_{K\le i_1\le T}N_{i_1}-\sum_{K\le i_1<i_2\le T}N_{i_1,i_2}. \tag{7}$$

Here by (5)

$$\sum_{K\le i_1\le T}N_{i_1}=\sum_{K\le i_1\le T}\frac{1}{P(i_1)}\ge\sum_{K\le i_1\le T}\frac{1}{a_2(i_1+f)(i_1+f+1)}$$
$$=\frac{1}{a_2}\sum_{K\le i_1\le T}\left(\frac{1}{i_1+f}-\frac{1}{i_1+f+1}\right)=\frac{1}{a_2(K+f)}-\frac{1}{a_2(T+f+1)}$$
$$=\frac{1}{a_2K}-\frac{1}{a_2T}+O(K^{-2}).$$

By this, (6) and (7) we have

$$A_P(K)\ge A_P(K,T)\ge\frac{1}{a_2K}-\frac{1}{a_2T}-\sum_{K\le i_1<i_2\le T}N_{i_1,i_2}+O(K^{-2}). \tag{8}$$

Thus we need to give an upper bound for $\sum_{K\le i_1<i_2\le T}N_{i_1,i_2}$. Let S be this sum:

$$S\stackrel{\text{def}}{=}\sum_{K\le i_1<i_2\le T}N_{i_1,i_2}=\sum_{K\le i_1<i_2\le T}\frac{1}{[P(i_1),P(i_2)]}=\sum_{K\le i_1<i_2\le T}\frac{(P(i_1),P(i_2))}{P(i_1)P(i_2)}. \tag{9}$$

We split the sum in S into two parts according to that the greatest common divisor of $P(i_1)$ and $P(i_2)$ is less or greater than a fixed integer H. We will give the exact value of H at the end of the proof (H will depend on K). Let

$$S_1 \stackrel{\text{def}}{=} \sum_{\substack{K\le i_1<i_2\le T\\ (P(i_1),P(i_2))\le H}} \frac{(P(i_1),P(i_2))}{P(i_1)P(i_2)},$$

$$S_2 \stackrel{\text{def}}{=} \sum_{\substack{K\le i_1<i_2\le T\\ (P(i_1),P(i_2))> H}} \frac{(P(i_1),P(i_2))}{P(i_1)P(i_2)}.$$

Then

$$S = S_1 + S_2.$$

Throughout the proof we will use the standard notations $\ll$ and $\gg$ in the sense that the implied constant factors only depend on the polynomial $P(X)$.

We will prove the following two lemmas:

Lemma 1. *If $K(\log K)^{1/2} \le H$ then*

$$S_1 \ll \frac{H^2(\log H)^3}{K^4}.$$

Lemma 2. *For $H \ge K \ge 1$*

$$S_2 \ll \frac{(\log T)^{\sqrt{2}+1}}{H}.$$

From Lemma 1 and Lemma 2 easily follows Theorem 1 since if $K(\log K)^{1/2} \le H$ then

$$S \ll \frac{H^2(\log H)^3}{K^4} + \frac{(\log T)^{\sqrt{2}+1}}{H}. \tag{10}$$

Fix $T = K^2$ and $H = \frac{K^{4/3}}{(\log K)^{(2-\sqrt{2})/3}}$. Then from (8), (9) and (10) we get the theorem. It remains to prove Lemma 1 and Lemma 2.

Proof of Lemma 1. Trivially

$$(P(i_1),P(i_2)) \le \sum_{\substack{d\mid P(i_1),P(i_2)\\ d>0}} d,$$

since d in the sum also takes $(P(i_1), P(i_2))$. Thus

$$S_1 \le \sum_{K\le i_1<i_2\le T} \frac{\sum\limits_{\substack{d\mid P(i_1),P(i_2)\\ d\le H}} d}{P(i_1)P(i_2)} \le \frac{1}{2}\sum_{d=1}^{H} d\Bigg(\sum_{\substack{K\le i_1\le T\\ d\mid P(i_1)}}\frac{1}{P(i_1)}\Bigg)\Bigg(\sum_{\substack{K\le i_2\le T\\ d\mid P(i_2)}}\frac{1}{P(i_2)}\Bigg)$$
$$\le \frac{1}{2}\sum_{d=1}^{H} d\Bigg(\sum_{\substack{K\le i\le T\\ d\mid P(i)}}\frac{1}{P(i)}\Bigg)^2. \tag{11}$$

The congruence

$$P(x)\equiv 0 \pmod d$$

has at least $2^{\omega(d)}$ solutions in x modulo d, denote them by $s_1, s_2, \dots, s_r$ where

$$r \le 2^{\omega(d)}. \tag{12}$$

Thus

$$\sum_{\substack{K\le i\le T\\ d\mid P(i)}}\frac{1}{P(i)} = \sum_{j=1}^{r}\sum_{\substack{K\le i\le T\\ i\equiv s_j \pmod d}}\frac{1}{P(i)}. \tag{13}$$

By (5) for $x\ge K_0$

$$P(x)\ge a_2(x+f-1)(x+f)\ge a_2(x+f-d)(x+f)$$

for all positive integer d. Then for fixed s_j we have

$$\sum_{\substack{K\le i\le T\\ i\equiv s_j \pmod d}}\frac{1}{P(i)} \le \frac{1}{P(K)} + \sum_{\substack{K+d\le i\le T\\ i\equiv s_j \pmod d}}\frac{1}{P(i)}$$
$$\ll \frac{1}{(K+f-1)(K+f)} + \sum_{\substack{K+d\le i\le T\\ i\equiv s_j \pmod d}}\frac{1}{(i+f-d)(i+f)}$$
$$\ll \frac{1}{K^2} + \frac{1}{d}\sum_{\substack{K+d\le i\le \infty\\ i\equiv s_j \pmod d}}\left(\frac{1}{i+f-d}-\frac{1}{i+f}\right)$$
$$\ll \frac{1}{K^2}+\frac{1}{d(K+f)} \ll \frac{1}{K^2}+\frac{1}{dK}.$$

By this, (12) and (13) we have:

$$\sum_{\substack{K\le i\le T\\ d\mid P(i)}}\frac{1}{P(i)}\ll\frac{2^{\omega(d)}}{K^2}+\frac{2^{\omega(d)}}{dK}.$$

By the condition of the lemma $H\ge K$ so by (11) we have

$$\begin{aligned}S_1&\ll\sum_{d=1}^{H}d\left(\frac{2^{\omega(d)}}{K^2}+\frac{2^{\omega(d)}}{dK}\right)^2\ll\sum_{d=1}^{K}d\left(\frac{2^{\omega(d)}}{dK}\right)^2+\sum_{d=K+1}^{H}d\left(\frac{2^{\omega(d)}}{K^2}\right)^2\\&=\frac{1}{K^2}\sum_{d=1}^{K}\frac{4^{\omega(d)}}{d}+\frac{1}{K^4}\sum_{d=K+1}^{H}d4^{\omega(d)}.\end{aligned}\tag{14}$$

Here the first sum can be written as a product over primes. By this, $1+x\le e^x$, using an explicit form of Mertens' theorem [14] and $K(\log K)^{1/2}\le H$ we have

$$\begin{aligned}\frac{1}{K^2}\sum_{d=1}^{K}\frac{4^{\omega(d)}}{d}&\ll\frac{1}{K^2}\prod_{\substack{2\le p\le K\\ \text{prime } p}}\left(1+\frac{4}{p}+\frac{4}{p^2}+\frac{4}{p^3}+\dots\right)\le\frac{1}{K^2}\prod_{\substack{2\le p\le K\\ \text{prime } p}}e^{\left(\frac{4}{p}+\frac{4}{p^2}+\dots\right)}\\&\ll\frac{1}{K^2}e^{\left(\sum\limits_{\substack{2\le p\le K\\ \text{prime } p}}\frac{4}{p}\right)}\ll\frac{(\log K)^4}{K^2}\ll\frac{H^2}{K^4}(\log H)^3.\end{aligned}\tag{15}$$

Next we estimate the second sum in S_1:

$$\begin{aligned}\frac{1}{K^4}\sum_{d=K+1}^{H}d4^{\omega(d)}&\le\frac{H}{K^4}\sum_{d=1}^{H}4^{\omega(d)}\le\frac{H}{K^4}\sum_{d=1}^{H}(\tau(d))^2\le\frac{H}{K^4}\sum_{d=1}^{H}\sum_{x\mid d}\sum_{y\mid d}1\\&\le\frac{H}{K^4}\sum_{x=1}^{H}\sum_{y=1}^{H}\sum_{\substack{d=1\\ [x,y]\mid d}}^{H}1\le\frac{H}{K^4}\sum_{a=1}^{H}\sum_{p=1}^{[H/a]}\sum_{\substack{q=1\\ (p,q)=1}}^{[H/a]}\sum_{\substack{d=1\\ [ap,aq]\mid d}}^{H}1\\&\le\frac{H}{K^4}\sum_{a=1}^{H}\sum_{p=1}^{H}\sum_{q=1}^{H}\frac{H}{apq}\le\frac{H^2}{K^4}\sum_{a=1}^{H}\frac{1}{a}\sum_{p=1}^{H}\frac{1}{p}\sum_{q=1}^{H}\frac{1}{q}\ll\frac{H^2}{K^4}(\log H)^3.\end{aligned}\tag{16}$$

By (14), (15) and (16) we get

$$S_1\ll\frac{H^2(\log H)^3}{K^4}$$

which was to be proved.

Proof of Lemma 2. Let $K \le i_1 < i_2 \le T$ such that $(P(i_1), P(i_2)) = d > H$. Then $\frac{(P(i_1),P(i_2))}{P(i_1)P(i_2)}$ is a term in S_2. So there exist integers $b < a$ such that $(a, b) = 1$ and

$$P(i_1) = bd, \tag{17}$$

$$P(i_2) = ad. \tag{18}$$

By the definition of d we may assume that

$$(a, b) = 1. \tag{19}$$

Then

$$b < a = \frac{P(i_2)}{d} \le \frac{P(i_2)}{H} \ll \frac{T^2}{H}.$$

So there exist a constant c_2 only depending on the polynomial $P(x)$ such that

$$b < a \le \frac{c_2 T^2}{H}.$$

From (17) and (18) follows

$$\frac{(P(i_1), P(i_2))}{P(i_1)P(i_2)} = \frac{1}{abd}.$$

Let

$$\mathcal{H}(a, b) = \{d > H : \exists\ K \le i_1 < i_2 \le T \text{ with } P(i_1) = bd,\ P(i_2) = ad\}.$$

Then

$$S_2 \le \sum_{1 \le b < a \le \frac{c_2 T^2}{H}} \frac{1}{ab} \sum_{d \in \mathcal{H}(a,b)} \frac{1}{d}. \tag{20}$$

First we need an upper bound for

$$\sum_{d \in \mathcal{H}(a,b)} \frac{1}{d}.$$

Again we suppose that $d \in \mathcal{H}(a, b)$, so there exist $K \le i_1 < i_2 \le T$ with (17) and (18). $P(X)$ is a polynomial of degree 2, so write it of the form

$$P(X) = a_2 X^2 + a_1 X + a_0.$$

Then

$$4a_2P(X) = (2a_2X + a_1)^2 + 4a_2a_0 - a_1^2.$$

Write

$$x_1 = 2a_2i_1 + a_1, \qquad y_1 = 2a_2i_2 + a_1,$$
$$c = (a_1^2 - 4a_2a_0)(a - b).$$

By this, (17) and (18) we have

$$4a_2bd = 4a_2P(i_1) = x_1^2 - c/(a - b),$$
$$4a_2ad = 4a_2P(i_2) = y_1^2 - c/(a - b).$$

So

$$a{x_1}^2 - b{y_1}^2 = c. \tag{21}$$

Since the discriminant of $P(X)$ is non-zero, $c \neq 0$. Here c can be both positive and negative. If c is negative in place of (21) we consider

$$b{y_1}^2 - a{x_1}^2 = -c. \tag{22}$$

Using (17) and (18)

$$d = \frac{P(i_2)}{a} = \frac{y_1^2 - \frac{c}{a-b}}{4a_2a} = \frac{P(i_1)}{b} = \frac{x_1^2 - \frac{c}{a-b}}{4a_2b}.$$

From $d > H$ follows that there exists a constant c_3 only depending on the polynomial $P(X)$ such that

$$y_1 \geq c_3\sqrt{Ha}$$
$$x_1 \geq c_3\sqrt{Hb}.$$

So

$$\sum_{d\in\mathcal{H}(a,b)} \frac{1}{d} \leq \sum_{\substack{y_1\geq c_3\sqrt{Ha}:\exists x_1 \\ a{x_1}^2 - b{y_1}^2 = c}} \frac{4a_2a}{y_1^2 - \frac{c}{a-b}} \ll \sum_{\substack{y_1\geq c_3\sqrt{Ha}:\exists x_1 \\ a{x_1}^2 - b{y_1}^2 = c}} \frac{a}{y_1^2}, \tag{23}$$

and similarly

$$\sum_{d\in\mathcal{H}(a,b)} \frac{1}{d} \leq \sum_{\substack{x_1\geq c_3\sqrt{Hb}:\exists y_1 \\ a{x_1}^2 - b{y_1}^2 = c}} \frac{4a_2b}{x_1^2 - \frac{c}{a-b}} \ll \sum_{\substack{x_1\geq c_3\sqrt{Hb}:\exists y_1 \\ a{x_1}^2 - b{y_1}^2 = c}} \frac{b}{x_1^2}. \tag{24}$$

Using Corollary 1 for (23) if c is positive and for (24) if c is negative, for all $m \mid c$, $(m, ab) = 1$ we obtain

$$\sum_{d \in \mathcal{H}(a,b)} \frac{1}{d} \ll \left(1 + \frac{|c|}{\sqrt{ab}m}\right) \frac{\tau(m)}{H}. \tag{25}$$

We would like to choose m so that we obtain an optimal estimate for $\sum_{d \in \mathcal{H}(a,b)} \frac{1}{d}$ in (25).

Lemma 3. *Let $g \in \mathbb{N}$. Then for $1 \le x < \tau(g)$, there exist positive integers $m_1,\ m_2 \mid g$ such that*

$$\tau(m_1) \le x,\ \tau(m_2) \le \frac{4\tau(g)}{x},$$

and every $d \mid g$ can be written of the form $d = d_1 d_2$ with $d_1 \mid m_1$ and $d_2 \mid m_2$.

Proof of Lemma 3. This is Lemma 4 in [8]

Let $g = a - b$ in Lemma 3. For $x = 2\tau(g)^{1/2}$ we obtain $\tau(m_1), \tau(m_2) \le 2\tau(g)^{1/2}$. If $d = |g|$ in Lemma 3 we see that $|g| = d_1 d_2 \le m_1 m_2$ so for $m = \max\{m_1, m_2\}$ we get $m \ge |g|^{1/2}$. By (19) we also have $(ab, m) \le (ab, g) = (ab, a - b) = 1$. For this m (25) gives

$$\sum_{d \in \mathcal{H}(a,b)} \frac{1}{d} \ll \left(1 + \frac{|c|}{\sqrt{ab}g^{1/2}}\right) \frac{\tau(g)^{1/2}}{H} \ll \left(1 + \frac{|c|}{\sqrt{ab(a-b)}}\right) \frac{\tau(a-b)^{1/2}}{H} \tag{26}$$

Here by (2)

$$\frac{|c|}{\sqrt{ab(a-b)}} \ll \frac{a-b}{\sqrt{ab(a-b)}} \ll \sqrt{\frac{a-b}{ab}} \ll \sqrt{\frac{a}{ab}} \ll 1.$$

By this and (26) we have

$$\sum_{d \in \mathcal{H}(a,b)} \frac{1}{d} \ll \frac{\tau(a-b)^{1/2}}{H}.$$

By this and (20) we have

$$S_2 \ll \frac{1}{H} \sum_{b < a \le \frac{c_1 T^2}{H}} \frac{\tau(a-b)^{1/2}}{ab} \ll \frac{1}{H} \sum_{b,s \le \frac{c_1 T^2}{H}} \frac{\tau(s)^{1/2}}{b(b+s)}$$

$$\ll \frac{1}{H} \sum_{s \le \frac{c_1 T^2}{H}} \tau(s)^{1/2} \sum_{b<\infty} \frac{1}{b(b+s)}$$
$$\ll \frac{1}{H} \sum_{s \le \frac{c_1 T^2}{H}} \frac{\tau(s)^{1/2}}{s} \sum_{b<\infty} \left(\frac{1}{b} - \frac{1}{b+s} \right)$$
$$= \frac{1}{H} \sum_{s \le \frac{c_1 T^2}{H}} \frac{\tau(s)^{1/2}}{s} \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{s} \right)$$
$$\ll \frac{1}{H} \sum_{s \le \frac{c_1 T^2}{H}} \frac{\tau(s)^{1/2}}{s} \log s \ll \frac{\log T}{H} \sum_{s \le \frac{c_1 T^2}{H}} \frac{\tau(s)^{1/2}}{s}.$$

Here the last sum can be estimated by a product over primes.

$$S_2 \ll \frac{\log T}{H} \prod_{p \le \frac{c_1 T^2}{H}} \left(\sum_{\nu \ge 0} \frac{(\nu+1)^{1/2}}{p^\nu} \right) \ll \frac{\log T}{H} \prod_{p \le \frac{c_1 T^2}{H}} \left(1 + \frac{2^{1/2}}{p} \right) \left(\sum_{\substack{\nu \ge 0 \\ \nu \ne 1}} \frac{\nu+1}{p^\nu} \right)$$
$$\ll \frac{\log T}{H} \prod_{p \le \frac{c_1 T^2}{H}} \left(1 + \frac{2^{1/2}}{p} \right) \left(1 + \frac{3}{(p-1)^2} \right) \ll \frac{(\log T)^{1+\sqrt{2}}}{H}$$

which was to be proved.

Proof of Theorem 2. Denote the positive integer solutions of (3) by $(x_1, y_1),\ (x_2, y_2), \ldots, (x_S, y_S)$ where $1 \le y_1 < y_2 < y_3 < \cdots < y_S \le N$. Then for $1 \le i < j \le S$ we have

$$a{x_i}^2 - b{y_i}^2 = c, \tag{27}$$
$$a{x_j}^2 - b{y_j}^2 = c. \tag{28}$$

Multiplying (27) by y_j^2, (28) by y_i^2 and taking the difference we get

$$a\left(x_i y_j - x_j y_i\right)\left(x_i y_j + x_j y_i\right) = c\left(y_j^2 - y_i^2\right). \tag{29}$$

The right-hand-side of (29) is positive, so the left-hand-side is also positive. It follows from this that

$$1 \le x_i y_j - x_j y_i, \qquad 2x_j y_i \le x_i y_j + x_j y_i. \tag{30}$$

Using this and (29) we get

$$2a x_j y_i < c y_j^2. \tag{31}$$

From (28)

$$ax_j{}^2 = c + by_j{}^2 \geq by_j{}^2, \qquad x_j \geq \sqrt{\frac{b}{a}}y_j.$$

By this and (31) we get

$$2\sqrt{ab}\cdot y_i y_j < cy_j^2, \qquad \sqrt{\frac{4ab}{c^2}}y_i < y_j. \tag{32}$$

For $j = i+1$ we obtain

$$\sqrt{\frac{4ab}{c^2}}y_i < y_{i+1}.$$

Thus

$$\left(\sqrt{\frac{4ab}{c^2}}\right)^{S-1} y_1 < y_S \leq N,$$

from which part a) follows.

We will prove part b) by induction on c. So suppose that $c = 1$ or we have verified the statement for $1, 2, \ldots, c-1$ in place of c in (3). Then throughout the proof we may suppose that ab is a perfect square modulo m. So there is an integer u such that

$$ab \equiv u^2 \pmod{m}. \tag{33}$$

Indeed, otherwise, there is a prime factor p of m (so $p \mid m \mid c$) such that the Legendre symbol $\left(\frac{ab}{p}\right) = -1$. $(ab, m) = 1$ thus $p \nmid a, b$, so from $\left(\frac{ab}{p}\right) = -1$ then $p \mid x, y$ and $p^2 \mid c$ follows. Thus from the Pell equation

$$ax^2 - by^2 = c$$

follows that

$$a\left(\frac{x}{p}\right)^2 - b\left(\frac{y}{p}\right)^2 = \frac{c}{p^2}.$$

$\frac{m}{(m,p^2)} \mid \frac{c}{p^2}$, thus by the induction we have

$$S \leq \left\lceil \frac{ec/p^2}{\sqrt{ab}m/(m,p^2)} \right\rceil \tau(m/(m,p^2))\,(\log N + 2) \leq \left\lceil \frac{ec}{\sqrt{ab}m} \right\rceil \tau(m)\,(\log N + 2)\,.$$

and we are done. m is a divisor of c, thus

$$ax^2 - by^2 \equiv 0 \pmod{m}.$$

Multiplying it by a and using (33) we get

$$(ax - uy)(ax + uy) \equiv 0 \pmod{m}.$$

So there is an $n \mid m$ such that

$$n \mid ax - uy \quad \text{and} \quad \frac{m}{n} \mid ax + uy. \tag{34}$$

For $n \mid m$ let $G(n)$ denote the set the solutions of (3) such that $1 \le x$, $1 \le y \le N$ and (34) holds. Then we get

$$S = \sum_{n|m} |G(n)| = \tau(m) \max |G(n)|. \tag{35}$$

Consider a fixed n. We will give an upper bound for $G(n)$. In this case we denote by (x_1, y_1), $(x_2, y_2), \dots, (x_T, y_T)$ the all solutions of (3) such that $(x_i, y_i) \in G(n)$ for $1 \le i \le T$, where

$$T = |G(n)|.$$

Again we may assume that $1 \le y_1 < y_2 < \cdots < y_T \le N$. For (x_i, y_i), $(x_j, y_j) \in G(n)$, $i < j$ we have

$$n \mid ax_i - uy_i \quad \text{and} \quad \frac{m}{n} \mid ax_i + uy_i.$$
$$n \mid ax_j - uy_j \quad \text{and} \quad \frac{m}{n} \mid ax_j + uy_j.$$

Thus

$$m \mid (ax_i - uy_i)(ax_j + uy_j) - (ax_j - uy_j)(ax_i + uy_i) = 2au\,(x_iy_j - x_jy_i).$$

Here $(au, m) \le (au^2, m) = (a^2b, m) = 1$. So:

$$m \mid 2\,(x_iy_j - x_jy_i). \tag{36}$$

We fix i. The sequence $\{2\,(x_iy_j - x_jy_i)\}_{j=i+1,i+2,\dots,T}$ is strictly monotone increasing since

$$(x_iy_j - x_jy_i) = y_iy_j\left(\frac{x_i}{y_i} - \frac{x_j}{y_j}\right) = y_iy_j\left(\frac{x_i}{y_i} - \sqrt{\frac{b}{a} + \frac{c}{ay_j^2}}\right)$$

and here both $\{y_j\}_{j=i+1,\dots,T}$, $\left\{\frac{x_i}{y_i}-\sqrt{\frac{b}{a}+\frac{c}{ay_j^2}}\right\}_{j=i+1,\dots,T}$ are strictly monotone increasing sequences. By (36) we also know that the elements of the sequence $\{2\,(x_iy_j-y_jx_i)\}_{j=i+1,\dots,T}$ are divisible by m.

Let t be a positive integer. Thus in place of (30) even

$$\frac{tm}{2}\le x_iy_{i+t}-x_{i+t}y_i$$

holds. Similarly to (32) we get

$$tm\sqrt{\frac{ab}{c^2}}y_i<y_{i+t}.$$

Fix

$$t=\left\lceil\frac{ce}{\sqrt{ab}m}\right\rceil. \tag{37}$$

Then

$$ey_i<y_{i+t}. \tag{38}$$

Thus

$$e^{[(T-1)/t]}y_1<y_{1+t[(T-1)/t]}\le y_T\le N.$$

So

$$T=|G(n)|<t\log N+t+1\le t(\log N+2).$$

Using this, (35) and (37) we get part b).

Proof of Corollary 1. We will use the groups $G(n)$ defined in the proof of Theorem 2. Again we fix t by (37). By (38) for the solutions $(x_1,y_1),\dots,(x_T,y_T)\in G(n)$ of (3) with $0\le x_i$ and $A\le y_1<y_2<\dots<y_T$ we have

$$\frac{1}{{y_i}^2}+\frac{1}{{y_{i+t}}^2}+\frac{1}{{y_{i+2t}}^2}+\dots\ll\frac{1}{A^2}.$$

So

$$\frac{1}{{y_1}^2}+\frac{1}{{y_2}^2}+\frac{1}{{y_3}^2}+\dots\ll\frac{t}{A^2}=\frac{\left\lceil\frac{ce}{\sqrt{ab}m}\right\rceil}{A^2}.$$

Since we defined $\tau(m)$ groups $G(n)$, the corollary follows.

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References

[1] A. Balog, P. Erdős and G. Tenenbaum, On Arithmetic Functions Involving Consecutive Divisors, Analytic number theory, (Urbana, 1989), Prog. Math. 85, 77–90, (B. Berndt, H. Diamond, H. Halberstam, A. Hildebrand, eds.), *Birkäuser*, 1990.

[2] R. de la Bretèche, Nombre de valeurs polynomials qui divisent un entier, *Math. Proc. Camb. Phil. Soc.* **131** (2001), 193–209.

[3] R. de la Bretèche, Sur une classe de fonctions arithmétiques liées aux diviseurs d'un entier, *Indag. Mathem.* New Ser. **11** (2000), 437–452.

[4] P. Erdős and R. R. Hall, On some unconventional problems on the divisors of integers, *J. Austral. Math. Soc.* (Series A) **25** (1978), 479–485.

[5] P. Erdős and G. Tenenbaum, Sur les fonctions arithmétiques liées aux diviseurs consécutifs, *J. Number Theory* **31** (1989), 285–311.

[6] J. H. Evertse, On equations in S-units and the Thue-Mahler equation, *Invent. Math.* **75** (1984), 561–584.

[7] G. R. Everest and K. Győry, Counting solutions of decomposable form equations, *Acta Arith.* **79**, no. 2 (1997), 173–191.

[8] K. Gyarmati, On the number of divisors which are values of a polynomial, *The Ramanujan Journal* (*to appear*).

[9] K. Győry, On the number of solutions of linear equations in units of an algebraic number field, *Comment. Math. Helv.* **54** (1979), 583–600.

[10] K. Győry and A. Pethő, Sur la distribution des solutions des équations du type "norm–forme", *Acta Math. Acad. Sci. Hung.* **26** (1975), 135–142.

[11] K. Győry and A. Pethő, Über die Verteilung der Lösungen von Normformen Gleichungen II., *Acta Arith.* **32** (1977), 349–363.

[12] K. Győry and A. Pethő, Über die Verteilung der Lösungen von Normformen Gleichungen III., *Acta Arith.* **37** (1980), 143–165.

[13] S. Lang, Asymptotic approximation to quadratic irrationalities I.,II., *Amer. J. Math.* **87** (1965), 481–496, and in Introduction to diophantine approximation, *Addison Wesley Publ. Comp.*, 1966.

[14] F. Mertens, Ein Beitrag zur analytyschen Zahlentheorie, *J. Reine Angew. Math.* **78** (1874), 46–62.

[15] A. Pethő, Über die Darstellung rationalen Zahlen durch zerlegbare Formen, *Publ. Math. Debrecen* **21** (1974), 31–38.

[16] A. Pethő, "Uber die Verteilung der Lösungen von S-Normformen Gleichungen, *Publ. Math. Debrecen* **29** (1982), 1–17.

[17] G. Tenenbaum, Une inégalité de Hilbert pour les diviseurs, *Indag. Mathem.* New Ser. **2**, (1) (1991), 105–114.

[18] B. M. M. de Weger, Algorithms for Diophantine Equations, Ph.D.-thesis, *Leiden*, 1988.

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