

Limits of random iterates

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Dedicated to Professor Zoltán Daróczy on his 70th birthday

Abstract. Given a probability space $(\Omega, \mathcal{A}, P)$, a nonempty subset X of a separable Banach space Y and an rv-function $f : X \times \Omega \rightarrow X$, we assume that the sequence of iterates of f converges to a function $\xi : X \times \Omega^\infty \rightarrow Y$. We give conditions on f and types of convergence which imply continuity of ξ with respect to the first variable. A possible application of obtained results to iterative equations is presented.

1. Introduction

Throughout this paper we assume that $(\Omega, \mathcal{A}, P)$ is a probability space, X is a non-empty subset of a separable Banach space $(Y, \|\cdot\|)$. By $\mathcal{B}(X)$ we denote the σ -algebra of all Borel subsets of X . Following [6] we say that $f : X \times \Omega \rightarrow X$ is a *random valued vector function* (shortly *an rv-function*) if it is measurable with respect to the product σ -algebra $\mathcal{B}(X) \otimes \mathcal{A}$. Given an rv-function f define the sequence $(f^m)_{m \in \mathbb{N}}$ of its *iterates* as follows (see [6]; cf. [7]):

$$f^1(x, \omega_1, \omega_2, \dots) = f(x, \omega_1), \quad f^{m+1}(x, \omega_1, \omega_2, \dots) = f(f^m(x, \omega_1, \omega_2, \dots), \omega_{m+1}),$$

for all $x \in X$ and $(\omega_1, \omega_2, \dots) \in \Omega^\infty$. Since, in fact, $f^m(\cdot, \omega)$ depends only on the first m coordinates of $\omega = (\omega_1, \omega_2, \dots)$, we may (and we do) consider this iterate

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as a function defined on $X \times \Omega^\infty$ or, alternatively, on $X \times \Omega^m$. It may be shown that these iterates form a random dynamical system (see [2]) and a homogeneous MARKOV chain (see [18]). The basic property of iterates of rv-functions says that they are rv-functions on the product probability space $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$. Further properties of iterates of rv-functions were studied in [3], [14] in the scalar case and in [10], [12] in the vector case.

Iteration is one of the basic technique for solving functional equations in a single variable which usually leads to solutions expressed by limits of iterates (see [4], [15], [16]). It turns out that the above definition of iterates of rv-functions can be successfully adopted in this technique (see [5]). However, we still do not know much on regularity of limits of convergent iterates of rv-functions.

Assume that for every $x \in X$ the sequence $(f^m(x, \cdot))_{m \in \mathbb{N}}$ of an rv-function f converges, in some sense, to a function $\xi(x, \cdot)$. In general $\xi : X \times \Omega^\infty \rightarrow Y$ is measurable with respect to the second variable. The problem is: What we need assume on f and which type of convergence should holds to get an additional information on ξ . In the present paper we focus on the problem of continuity of ξ with respect to the first variable.

The paper is organized as follows. At the beginning we introduce definitions of some types of continuity of rv-functions. In the third section we show that if the sequence of iterates of a given rv-function consists of continuous functions, then its limit function ξ so is, if suitable kind of convergence holds. Next, in the section four, we study conditions under which continuity of a given rv-function implies continuity of its sequence of iterates. In the last section we give a possible application of obtained results to iterative equations.

2. Notation

In the remainder of this paper we assume that $p \in [1, +\infty)$ is fixed.

Let $(f_m)_{m \in \mathbb{N}_0}$ be a sequence of measurable functions, acting from Ω to X . By $P\text{-}\lim_{m \rightarrow \infty} f_m = f_0$ we denote the convergence in probability P and, if moreover, $f_m \in L^p(\Omega, \mathcal{A}, P)$ for all $m \in \mathbb{N}_0$, then by $L^p\text{-}\lim_{m \rightarrow \infty} f_m = f_0$ we denote the convergence in L^p . Here and later we consider integrability in the Bochner sense.

Motivated by [5] we introduce a few kinds of regularity of rv-functions.

We say that an rv-function $f : X \times \Omega \rightarrow X$ is:

- *P-continuous at $x_0 \in X$* if $P\text{-}\lim_{j \rightarrow \infty} f(x_j, \cdot) = f(x_0, \cdot)$ for every sequence $(x_j)_{j \in \mathbb{N}}$ of points from X convergent to x_0 ;
- *P-continuous* if f is *P-continuous* at every point from X ;

- *Uniformly P -continuous* if for any $\varepsilon, \beta > 0$ there exists a $\delta > 0$ such that $P(\|f(x, \cdot) - f(y, \cdot)\| \geq \beta) \leq \varepsilon$ for all $x, y \in X$ with $\|x - y\| \leq \delta$.

Assume that $f : X \times \Omega \rightarrow X$ is an rv-function such that $f(x, \cdot) \in L^p$ for all $x \in X$. We say that f is:

- *L^p -continuous at $x_0 \in X$* if $L^p\text{-}\lim_{j \rightarrow \infty} f(x_j, \cdot) = f(x_0, \cdot)$ for every sequence $(x_j)_{j \in \mathbb{N}}$ of points from X convergent to x_0 ;
- *L^p -continuous* if f is L^p -continuous at every point from X ;
- *Uniformly L^p -continuous* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_{\Omega} \|f(x, \cdot) - f(y, \cdot)\|^p dP \leq \varepsilon$ for all $x, y \in X$ with $\|x - y\| \leq \delta$.

It is clear that every uniformly P -continuous rv-function is P -continuous, every L^p -continuous at x_0 rv-function is P -continuous at x_0 , and every uniformly L^p -continuous rv-function is uniformly P -continuous and L^p -continuous.

Remark 2.1. Assume that $f : X \times \Omega \rightarrow X$ is an rv-function and there exists an integrable function $\psi : \Omega \rightarrow [0, +\infty)$ such that $\|f(x, \cdot)\|^p \leq \psi$ for all $x \in X$.

- (i) If f is P -continuous at $x_0 \in X$, then f is L^p -continuous at x_0 .
- (ii) If f is uniformly P -continuous, then f is uniformly L^p -continuous.

PROOF. We will prove assertion (i) only. The proof of assertion (ii) is similar.

Fix $\varepsilon > 0$ and a sequence $(x_j)_{j \in \mathbb{N}}$ convergent to x_0 . By P -continuity of f at x_0 we choose a $j_0 \in \mathbb{N}$ such that $\int_{\|f(x_j, \cdot) - f(x_0, \cdot)\|^p \geq \varepsilon} \psi dP \leq \varepsilon$ for all $j \geq j_0$. Hence

$$\int_{\Omega} \|f(x_j, \cdot) - f(x_0, \cdot)\|^p dP \leq 2^{p+1} \int_{\|f(x_j, \cdot) - f(x_0, \cdot)\|^p \geq \varepsilon} \psi dP + \varepsilon \leq (2^{p+1} + 1)\varepsilon$$

for all $j \geq j_0$. □

We say that a sequence $(f_m)_{m \in \mathbb{N}}$ of rv-functions is P -continuous at x_0 , P -continuous, etc., if f_m is P -continuous at x_0 , P -continuous, etc., for all $m \in \mathbb{N}$.

We finish this section with two examples. In both of them we assume that $(\Omega, \mathcal{A}, P) = ((0, 1), \mathcal{B}(0, 1), l_1|_{\mathcal{B}(0, 1)})$.

Example 2.2. Consider a function $g : (0, +\infty) \times (0, 1) \rightarrow (0, +\infty)$ given by $g(x, \omega) = \frac{\omega}{x}$. A simple verification shows that g is L^p -continuous, but $g^m(x, \cdot) \notin L^p$ for all $x \in (0, 1)$ and $m \geq 2$.

The second example shows that L^p -continuity is stronger than P -continuity.

Example 2.3. Consider a function $f : [0, +\infty) \times (0, 1) \rightarrow [0, +\infty)$ given by

$$f(x, \omega) = \begin{cases} x^{-\frac{1}{p}} \chi_{[0, x)}(\omega), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

It is easy to check that f is P -continuous at 0. Since $\int_0^1 |f(x, \cdot)|^p dP = 1$ for all $x \neq 0$, we conclude that $f(x, \cdot) \in L^p$ for all $x \in [0, +\infty)$ and that f is not L^p -continuous at 0.

For $p = 1$ we can get, by induction,

$$f^m(x, \omega_1, \dots, \omega_m) = \begin{cases} x^{(-1)^n}, & \text{if } x \neq 0 \text{ and } \omega_k < x^{(-1)^{k+1}} \text{ for all } k \in \{1, \dots, n\}, \\ 0, & \text{otherwise,} \end{cases}$$

for all $m \in \mathbb{N}$. It follows easily that for every $m \in \mathbb{N}$ the iterate f^{2m+1} is not L^1 -continuous at 0, whereas the sequence $(f^{2m})_{m \in \mathbb{N}}$ is uniformly L^1 -continuous and the sequence $(f^m)_{m \in \mathbb{N}}$ is P^∞ -continuous.

3. Continuity of limits

Throughout this section we assume that $f : X \times \Omega \rightarrow X$ and $\xi : X \times \Omega^\infty \rightarrow Y$ are rv-functions. We show that continuity of $(f^m)_{m \in \mathbb{N}}$ can be transferred to continuity of its limit function if suitable local uniform convergence holds with respect to the first variable; i.e., every point $x \in X$ has a neighbourhood on which convergence is uniform.

Proposition 3.1. Assume that $(f^m)_{m \in \mathbb{N}}$ is P^∞ -continuous at $x_0 \in X$.

(i) If

for every $\varepsilon, \beta > 0$ and for each sequence $x_j \rightarrow x_0$ there exist $m, j_0 \in \mathbb{N}$ such that, for all $j \geq j_0$, we have $P^\infty(\|f^m(x_j, \cdot) - \xi(x_j, \cdot)\| \geq \beta) \leq \varepsilon$,
then ξ is P^∞ -continuous at x_0 . (3.1)

(ii) If ξ is P^∞ -continuous at x_0 and if

for every $\varepsilon, \beta > 0$ there exists m such that $P^\infty(\|f^m(x_0, \cdot) - \xi(x_0, \cdot)\| \geq \beta) \leq \varepsilon$,
then (3.1) holds. (3.2)

PROOF. Fix $\varepsilon, \beta > 0$ and a sequence $(x_j)_{j \in \mathbb{N}}$ convergent to x_0 .

(i) Applying (3.1), to a sequence $(y_j)_{j \in \mathbb{N}}$ defined by $y_{2n-1} = x_0$ and $y_{2n} = x_n$ for all $n \in \mathbb{N}$, we conclude that there exist $m, j_0 \in \mathbb{N}$ such that

$$P^\infty(\|f^m(x_0, \cdot) - \xi(x_0, \cdot)\| \geq \beta) \leq \varepsilon, \quad (3.3)$$

$$P^\infty(\|f^m(x_j, \cdot) - \xi(x_j, \cdot)\| \geq \beta) \leq \varepsilon \quad \text{for all } j \geq j_0. \quad (3.4)$$

By P^∞ -continuity of f^m at x_0 we choose a $j_1 \geq j_0$ such that

$$P^\infty(\|f^m(x_j, \cdot) - f^m(x_0, \cdot)\| \geq \beta) \leq \varepsilon \quad \text{for all } j \geq j_1. \quad (3.5)$$

Now (3.4), (3.5) and (3.3) imply

$$\begin{aligned} P^\infty(\|\xi(x_j, \cdot) - \xi(x_0, \cdot)\| \geq 3\beta) &\leq P^\infty(\|\xi(x_j, \cdot) - f^m(x_j, \cdot)\| \geq \beta) \\ &+ P^\infty(\|f^m(x_j, \cdot) - f^m(x_0, \cdot)\| \geq \beta) + P^\infty(\|f^m(x_0, \cdot) - \xi(x_0, \cdot)\| \geq \beta) \leq 3\varepsilon \end{aligned}$$

for all $j \geq j_1$.

(ii) Applying (3.2) we choose an $m \in \mathbb{N}$ such that (3.3) holds. By P^∞ -continuity of f^m at x_0 we choose a $j_1 \in \mathbb{N}$ such that (3.5) holds, and by P^∞ -continuity of ξ at x_0 we choose a $j_2 \geq j_1$ such that

$$P^\infty(\|\xi(x_j, \cdot) - \xi(x_0, \cdot)\| \geq \beta) \leq \varepsilon \quad \text{for all } j \geq j_2. \quad (3.6)$$

From (3.5), (3.3) and (3.6) we conclude that $P^\infty(\|f^m(x_j, \cdot) - \xi(x_j, \cdot)\| \geq 3\beta) \leq 3\varepsilon$ for all $j \geq j_2$. $\square$

If $(f^m)_{m \in \mathbb{N}}$ is P^∞ -continuous at x_0 and if $P^\infty\text{-}\lim_{m \rightarrow \infty} f^m(x_0, \cdot) = \xi(x_0, \cdot)$, then Proposition 3.1 shows that P^∞ -continuity of ξ at x_0 is equivalent to condition (3.1). Hence we have the following corollary.

Corollary 3.2. *Assume that $(f^m)_{m \in \mathbb{N}}$ is P^∞ -continuous. If $P^\infty\text{-}\lim_{m \rightarrow \infty} f^m(x, \cdot) = \xi(x, \cdot)$ locally uniformly, then ξ is P^∞ -continuous.*

Note that in the case where X has an ordered structure the sequence $(f^m(x, \cdot))_{m \in \mathbb{N}}$ forms a submartingale provided the mean $m(x) = \mathbb{E}f(x, \cdot)$ satisfies $m(x) \geq x$ for all $x \in X$ (see [10]; cf. [5]). Consequently, convergence of iterates follows from a submartingale convergence theorem (see [17]). However, a uniform convergence theorem holds only for positive parts of $(f^m(x, \cdot) - \xi(x, \cdot))_{m \in \mathbb{N}}$ (see [9]).

The next proposition may be proved in the same way as Proposition 3.1.

Proposition 3.3. Assume that $(f^m)_{m \in \mathbb{N}}$ is L^p -continuous at $x_0 \in X$ and $\xi(x, \cdot) \in L^p$ for all $x \in X$.

(i) If

for every $\varepsilon > 0$ and for each sequence $x_j \rightarrow x_0$ there exist $m, j_0 \in \mathbb{N}$ such that, for all $j \geq j_0$, we have $\int_{\Omega^\infty} \|f^m(x_j, \cdot) - \xi(x_j, \cdot)\|^p dP^\infty \leq \varepsilon$, then ξ is L^p -continuous at x_0 . (3.7)

(ii) If ξ is L^p -continuous at x_0 and if for every $\varepsilon > 0$ there exists an $m \in \mathbb{N}$ such that $\int_{\Omega^\infty} \|f^m(x_0, \cdot) - \xi(x_0, \cdot)\|^p dP^\infty \leq \varepsilon$, then (3.7) holds.

Corollary 3.4. Assume that $(f^m)_{m \in \mathbb{N}}$ is L^p -continuous. If $L^p\text{-}\lim_{m \rightarrow \infty} f^m(x, \cdot) = \xi(x, \cdot)$ locally uniformly, then ξ is L^p -continuous.

Concerning uniform continuity we have the following propositions.

Proposition 3.5. Assume that $(f^m)_{m \in \mathbb{N}}$ is uniformly P^∞ -continuous. If $P^\infty\text{-}\lim_{m \rightarrow \infty} f^m(x, \cdot) = \xi(x, \cdot)$ uniformly, then ξ is uniformly P^∞ -continuous.

Proposition 3.6. Assume that $(f^m)_{m \in \mathbb{N}}$ is uniformly L^p -continuous. If $L^p\text{-}\lim_{m \rightarrow \infty} f^m(x, \cdot) = \xi(x, \cdot)$ uniformly, then ξ is uniformly L^p -continuous.

4. Continuity of iterates

In this section we are interested in conditions under which sequences of iterates of continuous rv-functions are continuous. For this purpose we will formulate a more general problem.

Until the end we assume that $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{P})$ is a probability space, $\tilde{X}$ is a non-empty subset of a separable Banach space $(\tilde{Y}, \|\cdot\|)$, $f : X \times \Omega \rightarrow X$ and $g : \tilde{X} \times \tilde{\Omega} \rightarrow \tilde{X}$ are rv-functions, $\Phi : \tilde{X} \rightarrow X$ is a continuous function. Define a Φ -composition function $f \circ_\Phi g : \tilde{X} \times \tilde{\Omega} \times \Omega \rightarrow X$ putting

$$f \circ_\Phi g(\tilde{x}, \tilde{\omega}, \omega) = f(\Phi(g(\tilde{x}, \tilde{\omega})), \omega).$$

It is easy to check that $f \circ_\Phi g$ is an rv-function with respect to the product space $(\tilde{\Omega} \times \Omega, \tilde{\mathcal{A}} \otimes \mathcal{A}, \tilde{P} \otimes P)$. Clearly, $f \circ_{Id_X} f$ coincides with the second iterate f^2 of f . The problem reads: Under which assumptions the Φ -composition function is continuous?

We begin with a result which can be proved by adapting the proof of Lemma 2.2 from [5]; that lemma is just the first part of our Corollary 4.2 in the case where $X = [0, 1]$.

Theorem 4.1. *If g is $\tilde{P}$ -continuous at $\tilde{x}_0 \in \tilde{X}$ and if f is P -continuous at points of the set $\Phi \circ g(\{\tilde{x}_0\} \times \tilde{\Omega})$, then $f \circ_{\Phi} g$ is $\tilde{P} \otimes P$ -continuous at $\tilde{x}_0$.*

If there exists a $\tilde{P} \otimes P$ -integrable function $\psi : \tilde{\Omega} \times \Omega \rightarrow [0, +\infty]$ such that $\|f \circ_{\Phi} g(\tilde{x}, \cdot)\|^p \leq \psi$ for all $\tilde{x} \in \tilde{X}$, then from Remark 2.1(i) we conclude that $\tilde{P} \otimes P$ -continuity in the assertion of Theorem 4.1 can be replaced by L^p -continuity. Note that such a ψ exists if f is bounded, and in particular, if X is bounded.

Corollary 4.2. *If f is P -continuous, then $(f^m)_{m \in \mathbb{N}}$ is P^∞ -continuous. Moreover, if f is bounded, then $(f^m)_{m \in \mathbb{N}}$ is L^p -continuous.*

Concerning uniform continuity in probability we have the following result.

Theorem 4.3. *Assume that there exists an $a > 0$ such that*

$$\|\Phi(\tilde{x}) - \Phi(\tilde{y})\| \leq a\|\tilde{x} - \tilde{y}\| \quad \text{for all } \tilde{x}, \tilde{y} \in g(\tilde{X} \times \tilde{\Omega}). \quad (4.1)$$

If g is uniformly $\tilde{P}$ -continuous and if f is uniformly P -continuous, then $f \circ_{\Phi} g$ is uniformly $\tilde{P} \otimes P$ -continuous.

PROOF. Fix $\varepsilon, \beta > 0$. By uniform P -continuity of f we choose a $\gamma > 0$ such that $P(\|f(x, \cdot) - f(y, \cdot)\| \geq \beta) \leq \varepsilon$ for all $x, y \in X$ with $\|x - y\| \leq a\gamma$. This jointly with (4.1) gives

$$\begin{aligned} P(\|f \circ_{\Phi} g(\tilde{x}, \tilde{\omega}, \cdot) - f \circ_{\Phi} g(\tilde{y}, \tilde{\omega}, \cdot)\| \geq \beta) &\leq \varepsilon && \text{for all } \tilde{x}, \tilde{y} \in \tilde{X}, \tilde{\omega} \in \tilde{\Omega} \\ &&& \text{with } \|g(\tilde{x}, \tilde{\omega}) - g(\tilde{y}, \tilde{\omega})\| \leq \gamma. \end{aligned}$$

Now, by uniform $\tilde{P}$ -continuity of g we choose a $\delta \leq \gamma$ such that

$$\tilde{P}(\|g(\tilde{x}, \cdot) - g(\tilde{y}, \cdot)\| \geq \gamma) \leq \varepsilon \text{ for all } \tilde{x}, \tilde{y} \in \tilde{X} \text{ with } \|\tilde{x} - \tilde{y}\| \leq \delta.$$

Fix $\tilde{x}, \tilde{y} \in \tilde{X}$ such that $\|\tilde{x} - \tilde{y}\| \leq \delta$. Then, by the Fubini theorem, we have

$$\begin{aligned} &(\tilde{P} \otimes P)(\|f \circ_{\Phi} g(\tilde{x}, \cdot) - f \circ_{\Phi} g(\tilde{y}, \cdot)\| \geq \beta) \\ &= \int_{\|g(\tilde{x}, \tilde{\omega}) - g(\tilde{y}, \tilde{\omega})\| \geq \gamma} P(\|f \circ_{\Phi} g(\tilde{x}, \tilde{\omega}, \cdot) - f \circ_{\Phi} g(\tilde{y}, \tilde{\omega}, \cdot)\| \geq \beta) d\tilde{P}(\tilde{\omega}) \\ &\quad + \int_{\|g(\tilde{x}, \tilde{\omega}) - g(\tilde{y}, \tilde{\omega})\| < \gamma} P(\|f \circ_{\Phi} g(\tilde{x}, \tilde{\omega}, \cdot) - f \circ_{\Phi} g(\tilde{y}, \tilde{\omega}, \cdot)\| \geq \beta) d\tilde{P}(\tilde{\omega}) \\ &\leq \tilde{P}(\|g(\tilde{x}, \cdot) - g(\tilde{y}, \cdot)\| \geq \gamma) + \int_{\|g(\tilde{x}, \tilde{\omega}) - g(\tilde{y}, \tilde{\omega})\| < \gamma} \varepsilon d\tilde{P}(\tilde{\omega}) \leq 2\varepsilon, \end{aligned}$$

which completes the proof. $\square$

As an immediate consequence of Theorem 4.3 and Remark 2.1(ii) we get the following corollaries.

Corollary 4.4. *Assume that (4.1) holds with some $a > 0$ and there exists a $\tilde{P} \otimes P$ -integrable function $\psi : \tilde{\Omega} \times \Omega \rightarrow [0, +\infty]$ such that $\|f \circ_{\Phi} g(\tilde{x}, \cdot)\|^p \leq \psi$ for all $\tilde{x} \in \tilde{X}$. If g is uniformly $\tilde{P}$ -continuous and if f is uniformly P -continuous, then $f \circ_{\Phi} g$ is uniformly L^p -continuous.*

Corollary 4.5. *If f is uniformly P -continuous, then $(f^m)_{m \in \mathbb{N}}$ is uniformly P^∞ -continuous. Moreover, if f is bounded, then $(f^m)_{m \in \mathbb{N}}$ is uniformly L^p -continuous.*

Now we pass to L^p -continuity.

Theorem 4.6. *Assume that there exist $a, b, c, d, \alpha, \beta \geq 0$ such that*

$$\int_{\Omega} \|f(x, \cdot)\|^p dP \leq a\|x\|^\alpha + b \quad \text{for all } x \in \Phi \circ g(\tilde{X} \times \tilde{\Omega}), \quad (4.2)$$

$$\|\Phi(\tilde{x})\|^\alpha \leq c\|\tilde{x}\|^\beta + d \quad \text{for all } \tilde{x} \in g(\tilde{X} \times \tilde{\Omega}), \quad (4.3)$$

and for every countable and bounded set $K \subset \tilde{X}$ the function

$$“\tilde{\Omega} \ni \tilde{\omega} \longmapsto \sup \{ \|g(\tilde{x}, \tilde{\omega})\|^\beta : \tilde{x} \in K \} \in [0, +\infty]” \quad \text{is } \tilde{P}\text{-integrable.} \quad (4.4)$$

If g is $\tilde{P}$ -continuous at $\tilde{x}_0 \in \tilde{X}$ and if f is L^p -continuous at points of the set $\Phi \circ g(\{\tilde{x}_0\} \times \tilde{\Omega})$, then $f \circ_{\Phi} g$ is L^p -continuous at $\tilde{x}_0$.

PROOF. By (4.2), (4.3) and (4.4) we have $f \circ_{\Phi} g(\tilde{x}, \cdot) \in L^p$ for all $\tilde{x} \in \tilde{X}$.

Fix a sequence $(\tilde{x}_j)_{j \in \mathbb{N}}$ convergent to $\tilde{x}_0$ and for every $j \in \mathbb{N}$ put

$$I_j = \int_{\tilde{\Omega} \times \Omega} \|f \circ_{\Phi} g(\tilde{x}_j, \cdot) - f \circ_{\Phi} g(\tilde{x}_0, \cdot)\|^p d(\tilde{P} \otimes P).$$

The proof will be completed if we show that every strictly increasing sequence $(j_k)_{k \in \mathbb{N}}$ of positive integers has a subsequence $(j'_k)_{k \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow \infty} I_{j'_k} = 0. \quad (4.5)$$

Fix a strictly increasing sequence $(j_k)_{k \in \mathbb{N}}$ of positive integers. By $\tilde{P}$ -continuity of g we choose its subsequence $(j'_k)_{k \in \mathbb{N}}$ such that $g(\tilde{x}_{j'_k}, \cdot)$ converges to $g(\tilde{x}_0, \cdot)$ almost everywhere and put $A = \{\tilde{\omega} \in \tilde{\Omega} : \lim_{k \rightarrow \infty} g(\tilde{x}_{j'_k}, \tilde{\omega}) = g(\tilde{x}_0, \tilde{\omega})\}$. Clearly, $\tilde{P}(A) = 1$. Next, for every $k \in \mathbb{N}$ define a function $\psi_k : A \rightarrow [0, +\infty]$ putting

$$\psi_k(\tilde{\omega}) = \int_{\Omega} \|f \circ_{\Phi} g(\tilde{x}_{j'_k}, \tilde{\omega}, \cdot) - f \circ_{\Phi} g(\tilde{x}_0, \tilde{\omega}, \cdot)\|^p dP.$$

It is clear that

$$I_{j'_k} = \int_A \psi_k d\tilde{P}. \quad (4.6)$$

Continuity of Φ implies $\lim_{k \rightarrow \infty} \Phi(g(\tilde{x}_{j'_k}, \tilde{\omega})) = \Phi(g(\tilde{x}_0, \tilde{\omega}))$ for all $\tilde{\omega} \in A$, and then L^p -continuity of f yields

$$\lim_{k \rightarrow \infty} \psi_k(\tilde{\omega}) = 0 \quad \text{for all } \tilde{\omega} \in A. \quad (4.7)$$

Put $K = \{\tilde{x}_{j'_k} : k \in \mathbb{N}\} \cup \{\tilde{x}_0\}$. Fix $k \in \mathbb{N}$ and $\tilde{\omega} \in A$. From (4.2) and (4.3) we get

$$\begin{aligned} \psi_k(\tilde{\omega}) &\leq 2^p \int_{\Omega} \|f \circ_{\Phi} g(\tilde{x}_{j'_k}, \tilde{\omega}, \cdot)\|^p dP + 2^p \int_{\Omega} \|f \circ_{\Phi} g(\tilde{x}_0, \tilde{\omega}, \cdot)\|^p dP \\ &\leq 2^p ac(\|g(\tilde{x}_{j'_k}, \tilde{\omega})\|^{\beta} + \|g(\tilde{x}_0, \tilde{\omega})\|^{\beta}) + 2^{p+1}(ad + b) \\ &\leq 2^{p+1}ac \sup \{\|g(\tilde{x}, \tilde{\omega})\|^{\beta} : \tilde{x} \in K\} + 2^{p+1}(ad + b). \end{aligned}$$

This jointly with (4.4), (4.6) and (4.7) imply (4.5). $\square$

Applying Theorem 4.6 to $g = f$ with $\alpha = \beta = p$ we get the following corollary.

Corollary 4.7. *Assume that there exist $a, b \geq 0$ such that $\int_{\Omega} \|f(x, \cdot)\|^p dP \leq a\|x\|^p + b$ for all $x \in X$ and for every bounded set $K \subset X$ there exists an integrable function $\psi : \Omega \rightarrow [0, +\infty)$ such that $\|f(x, \cdot)\|^p \leq \psi$ for all $x \in K$. If f is L^p -continuous, then $(f^m)_{m \in \mathbb{N}}$ is L^p -continuous.*

Observe that if f and g are non expansive in L^p and if Φ is non expansive, then $f \circ_{\Phi} g$ does. In particular, we have the following proposition.

Proposition 4.8. *If $f(x, \cdot) \in L^p$ for all $x \in X$ and if there exists an $a \geq 0$ such that $\int_{\Omega} \|f(x, \cdot) - f(y, \cdot)\|^p dP \leq a\|x - y\|^p$ for all $x, y \in X$, then $(f^m)_{m \in \mathbb{N}}$ is uniformly L^p -continuous.*

5. Iterative equations

Fix an rv-function $\xi : X \times \Omega^{\infty} \rightarrow X$ and define a function $\pi : X \times \mathcal{B}(X) \rightarrow [0, 1]$ putting

$$\pi(x, B) = P^{\infty}(\xi(x, \cdot) \in B). \quad (5.1)$$

It is easy to see that for every $x \in X$ the function $\pi(x, \cdot)$ is a probability measure. Assume now that ξ is P^{∞} -continuous at $x_0 \in X$. Fix a continuous and bounded function $g : X \rightarrow \mathbb{R}$ and a sequence $(x_j)_{j \in \mathbb{N}}$ convergent

to x_0 . Then $P^\infty\text{-}\lim_{j \rightarrow \infty} g \circ \xi(x_j, \cdot) = g \circ \xi(x_0, \cdot)$ and $g \circ \xi$ is bounded. Hence $\lim_{j \rightarrow \infty} \int_X g(x) \pi(x_j, dx) = \int_X g(x) \pi(x_0, dx)$, and in consequence, the sequence $(\pi(x_j, \cdot))_{j \in \mathbb{N}}$ converges weakly to $\pi(x_0, \cdot)$. This jointly with Corollary 3.2 and [11, Theorem 2] gives following proposition concerning solutions of iterative equations.

Proposition 5.1. *Assume that f is P -continuous and for every $x \in X$ the sequence $(f^m(x, \cdot))_{m \in \mathbb{N}}$ converges locally uniformly in probability to a random variable $\xi(x, \cdot)$, and $\pi(x, \cdot)$ is the measure given by (5.1). If $\pi(x, \cdot) \neq \pi(y, \cdot)$ for some $x, y \in X$, then there exists a continuous and bounded function $g : X \rightarrow \mathbb{R}$ such that the function $\varphi : X \rightarrow \mathbb{R}$ defined by*

$$\varphi(x) = \int_X g(y) \pi(x, dy)$$

is a bounded, continuous and non-constant solution of the equation

$$\varphi(x) = \int_\Omega \varphi(f(x, \cdot)) dP.$$

The next example shows a possible application of Proposition 5.1.

Example 5.2. Fix measurable functions $L, M : \Omega \rightarrow \mathbb{R}$ and continuous functions $F, G : \mathbb{R} \rightarrow \mathbb{R}$. Define a function $f : \mathbb{R}^2 \times \Omega \rightarrow \mathbb{R}^2$ putting

$$f(x, y, \omega) = (G(x)L(\omega), F(x)M(\omega) + y).$$

It is clear that f is P -continuous. Iterating f we get

$$f^m(x, y, \omega) = \left(G_m(x, \omega), \sum_{k=1}^m F(G_{k-1}(x, \omega)) M_k(\omega) + y \right)$$

for all $(x, y, \omega) \in \mathbb{R}^2 \times \Omega^\infty$, where $G_k : \mathbb{R} \times \Omega^\infty \rightarrow \mathbb{R}$ and $M_k : \Omega^\infty \rightarrow \mathbb{R}$ are defined by $G_0(x, \omega) = x$, $G_k(x, \omega) = G(G_{k-1}(x, \omega))L(\omega_k)$, $M_k(\omega) = M(\omega_k)$ for all $k \in \mathbb{N}$.

Assume now that $|G(x)| \leq |x|$ for all $x \in \mathbb{R}$ and there exists an $\alpha > 0$ such that $|F(x)| \leq \alpha|x|$ for all $x \in \mathbb{R}$. Then

$$\begin{aligned} & \|f^{m+n}(x, y, \omega) - f^m(x, y, \omega)\| \\ &= |G_{m+n}(x, \omega) - G_m(x, \omega)| + \left| \sum_{k=m+1}^{m+n} F(G_{k-1}(x, \omega)) M_k(\omega) \right| \end{aligned}$$

$$\leq |x| \left(\prod_{k=1}^{m+n} |L_k(\omega)| + \prod_{k=1}^n |L_k(\omega)| \right) + \alpha |x| \sum_{k=m+1}^{m+n} |M_k(\omega)| \prod_{i=1}^{k-1} |L_i(\omega)| \quad (5.2)$$

for all $m, n \in \mathbb{N}$ and $(x, y, \omega) \in \mathbb{R}^2 \times \Omega^\infty$, where $L_i : \Omega^\infty \rightarrow \mathbb{R}$ is defined by $L_i(\omega) = L(\omega_i)$ for all $i \in \mathbb{N}$. According to the Kolmogorov law of large numbers the first summand in (5.2) converges to zero provided $P(L = 0) = 0$ and $-\infty < \mathbb{E} \log |L| < 0$. If additionally $\mathbb{E} \log \max\{|M|, 1\} < +\infty$, then we get the desired convergence of $(f^m(x, y, \cdot))_{m \in \mathbb{N}}$ (see [13]; cf. [8]). It is not difficult to check that the convergence is locally uniform and that the limit function is not constant in (x, y) . Proposition 5.1 now shows that the equation

$$\varphi(x, y) = \int_{\Omega} \varphi(G(x)L, F(x)M + y) dP$$

has a bounded, continuous and non-constant solution $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Remark 5.3. Assume additionally that in Example 5.2 functions $L, M : \Omega \rightarrow \mathbb{R}$ are in L^p . Then f is L^p -continuous, and, by Corollary 4.7, we conclude that $(f^m(x, y, \cdot))_{m \in \mathbb{N}}$ is L^p -continuous. Moreover, if $\mathbb{E}|L|^p < 1$, then $(f^m(x, y, \cdot))_{m \in \mathbb{N}}$ converges in L^p (see [19]; cf. [1]); this convergence is locally uniform. Corollary 3.4 now implies L^p -continuity of the limit function.

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