

Bernstein–Doetsch type results for s -convex functions

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Dedicated to 70th birthday of Professor Z. Daróczy

Abstract. As a possible generalization of the concept of s -convexity due to BRECKNER [2], we introduce the so-called (H, s) -convexity. Besides collecting some facts on this type of functions, the main goal of this paper is to prove some regularity properties of (H, s) -convex functions.

1. Introduction

Let D be a convex, open, nonempty subset of a real (complex) linear space X . BERNSTEIN and DOETSCH [1] (see [11] further references) proved that if a function $f : D \rightarrow \mathbb{R}$ is locally bounded from above at a point of D , then the Jensen-convexity of the function yields its local boundedness and continuity as well, which implies the convexity of the function f . This result has been generalized by several authors. The first such type results are due to NIKODEM and NG [13] for the approximately Jensen-convex functions (the so-called ε -Jensen-convexity), which was extended by PÁLES ([14], [15]) to approximately t -convex functions. Further generalizations can be found in papers of MROWIEC [12], HÁZY ([6], [7]), HÁZY and PÁLES ([8], [9]). In the paper of GILÁNYI, NIKODEM and PÁLES [5] there are some Bernstein–Doetsch type results for quasiconvex functions.

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The concept of s -convexity was introduced by BRECKNER [2]. A real valued function $f : D \rightarrow \mathbb{R}$ is called *Breckner s -convex* (or shortly *s -convex*, in notation $f \in \mathcal{K}^s$), if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y) \quad (1)$$

for every $x, y \in D$ and $\lambda \in [0, 1]$, where $s \in]0, 1]$ is a fixed number (see also [3], [10], [17]). The case $s = 1$ means the usual convexity of f .

Let $H \subseteq [0, 1]$ be a nonempty set. A real valued function $f : D \rightarrow \mathbb{R}$ is called *Breckner (H, s) -convex*, (or shortly *(H, s) -convex*, in notation $f \in \mathcal{K}_H^s$), if it fulfills (1) for all $\lambda \in H$.

In the special cases when $H = \{\frac{1}{2}\}$, $H = \{\lambda\}$ or $H = \mathbb{Q} \cap [0, 1]$, the corresponding Breckner (H, s) -convex functions are said to be *Breckner Jensen s -convex*, *Breckner (λ, s) -convex* and *Breckner rationally s -convex*, respectively (or shortly *Jensen s -convex*, *(λ, s) -convex* and *rationally s -convex*).

In [2] and [3] it was proved a Bernstein–Doetsch type result on rationally s -convex functions, moreover, the s -Hölder property of s -convex functions. PYCIA [17] gives a new proof of the latter statement, when f is defined on a nonempty, convex subset of a finite dimensional vector space. In [10] the authors collect some properties of s -convex functions defined on the nonnegative reals.

The main goal of this paper is to prove some regularity properties of (H, s) -convex functions, besides we also collect some facts on such functions.

2. Some elementary properties of s -convex functions

In this section we collect some interesting, easily-proved properties of Breckner s -convex functions.

Proposition 1. *If $\lambda, s \in]0, 1[$ and $f : D \rightarrow \mathbb{R}$ is an (λ, s) -convex function, then f is nonnegative.*

PROOF. Let x be an arbitrary element of D . Using (λ, s) -convexity of f

$$f(x) = f(\lambda x + (1 - \lambda)x) \leq \lambda^s f(x) + (1 - \lambda)^s f(x) = (\lambda^s + (1 - \lambda)^s) f(x),$$

which implies

$$0 \leq (\lambda^s + (1 - \lambda)^s - 1) f(x).$$

Since $\lambda^s + (1 - \lambda)^s - 1 > 0$ for all $\lambda, s \in]0, 1[$, we have that $f(x) \geq 0$, as desired. $\square$

Remark 1. According to the previous proposition, (H, s) -convex functions are also nonnegative when $0 < s < 1$ and $H \setminus \{0, 1\} \neq \emptyset$. This is not true for $s = 1$.

Proposition 2. *Let $H \subseteq [0, 1]$. If $f, g \in \mathcal{K}^s$ (or $\mathcal{K}_H^s$), then $f + g$, cf (with $c > 0$), and $\max\{f, g\}$ are also in $\mathcal{K}^s$ (resp. $\mathcal{K}_H^s$).*

PROOF. Easy calculation. $\square$

The next two propositions imply that the set of s -convex functions is strictly increasing as s tends to zero.

Proposition 3. *Let $0 < s_2 \leq s_1 < 1$. If $f \in \mathcal{K}^{s_1}$ (or $\mathcal{K}_H^{s_1}$), then f is also in $\mathcal{K}^{s_2}$ (resp. $\mathcal{K}_H^{s_2}$).*

PROOF. Assume that $f \in \mathcal{K}^{s_1}$, and let first $\lambda \in]0, 1[$. Then, by Proposition 1, $f(x)$ and $f(y)$ are nonnegative for all $x, y \in D$. Furthermore, $\lambda^{s_1} \leq \lambda^{s_2}$ and $(1 - \lambda)^{s_1} \leq (1 - \lambda)^{s_2}$, thus

$$f(\lambda x + (1 - \lambda)y) \leq \lambda^{s_1} f(x) + (1 - \lambda)^{s_1} f(y) \leq \lambda^{s_2} f(x) + (1 - \lambda)^{s_2} f(y).$$

The above inequalities hold for $\lambda \in \{0, 1\}$, too, therefore $f \in \mathcal{K}^{s_2}$. $\square$

Proposition 4. *Let $0 < s_1 < s_2 \leq 1$. Then there exists a function f such that $f \in \mathcal{K}_{\frac{1}{2}}^{s_1}$ but $f \notin \mathcal{K}_{\frac{1}{2}}^{s_2}$.*

PROOF. Let the function $f :]0, \infty[\rightarrow \mathbb{R}$ be defined by $f(x) := x^{s_1}$. First we show that f is a Jensen s_1 -convex function. To this we may assume that $x \leq y$ without loss of generality. Then the Jensen s_1 -convexity of f is equivalent to the inequality

$$(u + 1)^{s_1} \leq u^{s_1} + 1, \quad u \in]0, 1],$$

where $u := \frac{x}{y}$. The above inequality is equivalent to the nonnegativity of the function

$$g(u) = \log(u^{s_1} + 1) - s_1 \log(u + 1), \quad u \in [0, 1].$$

Because of $g(0) = 0$ and of g being monotone increasing on $[0, 1]$ (first derivative test), we get the Jensen s_1 -convexity of f .

Now we prove $f \notin \mathcal{K}_{\frac{1}{2}}^{s_2}$. Assume to the contrary that $f \in \mathcal{K}_{\frac{1}{2}}^{s_2}$. Then

$$\left(\frac{x + y}{2} \right)^{s_1} \leq \frac{x^{s_1} + y^{s_1}}{2^{s_2}}, \quad x, y \in]0, \infty[.$$

We can assume again that $x \leq y$. Divide by y^{s_1} both sides of the above inequality and substitute $u := \frac{x}{y}$. After some rearranging we get

$$1 \leq 2^{s_1-s_2} \frac{u^{s_1} + 1}{(u+1)^{s_1}}, \quad u \in]0, 1].$$

Here the right-hand side tends to $2^{s_1-s_2} < 1$ as u tends to zero, which is a contradiction. $\square$

We give a simple characterization of s -convex functions, which is analogous to the characterization of convex functions.

Theorem 1. *Let $I \subset \mathbb{R}$ be a nonempty, open interval. A function $f : I \rightarrow \mathbb{R}$ is s -convex if and only if*

$$(z-x)^s f(y) \leq (z-y)^s f(x) + (y-x)^s f(z), \quad (2)$$

for every $x < y < z$, $x, y, z \in I$.

PROOF. Assume that f is s -convex and let x, y and z be arbitrary elements of I such that $x < y < z$. Then

$$f(y) = f\left(\frac{z-y}{z-x}x + \frac{y-x}{z-x}z\right) \leq \left(\frac{z-y}{z-x}\right)^s f(x) + \left(\frac{y-x}{z-x}\right)^s f(z),$$

which is equivalent to (2). One can prove the converse assertion in a similar manner. $\square$

3. Regularity properties of (λ, s) -convex functions

In this section we assume that $(X, \|\cdot\|)$ is a real (complex) normed space instead of a real (complex) linear space. We recall that a function $f : D \rightarrow \mathbb{R}$ is called locally bounded from above on D if, for each point of $p \in D$, there exist $\varrho > 0$ and a neighborhood $U(p, \varrho) := \{x \in X : \|x - p\| < \varrho\}$ such that f is bounded from above on $U(p, \varrho)$.

Theorem 2. *Let $D \subset X$ be convex, open, nonempty and $f : D \rightarrow \mathbb{R}$. Let $\lambda \in]0, 1[$ be fixed. If $f \in \mathcal{K}_\lambda^s$ is locally bounded from above at a point $p \in D$, then f is locally bounded at every point of D .*

PROOF. First we prove that f is locally bounded from above on D . Define the sequence of sets D_n by

$$D_0 := \{p\}, \quad D_{n+1} := \lambda D_n + (1 - \lambda)D.$$

Using induction on n , we prove that f is locally upper bounded at each point of D_n . By assumption, f is locally upper bounded at $p \in D_0$. Assume that f is locally upper bounded at each point of D_n . For $x \in D_{n+1}$, there exist $x_0 \in D_n$ and $y_0 \in D$ such that $x = \lambda x_0 + (1 - \lambda)y_0$. By the inductive assumption, there exist $r > 0$ and a constant $M_0 \geq 0$ such that $f(x') \leq M_0$ for $\|x_0 - x'\| < r$. Then, by the (λ, s) -convexity of f , for $x' \in U_0 := U(x_0, r)$ we have

$$f(\lambda x' + (1 - \lambda)y_0) \leq \lambda^s f(x') + (1 - \lambda)^s f(y_0) \leq \lambda^s M_0 + (1 - \lambda)^s f(y_0) =: M.$$

Therefore, for

$$y \in U := \lambda U_0 + (1 - \lambda)y_0 = U(\lambda x_0 + (1 - \lambda)y_0, \lambda r) = U(x, \lambda r),$$

we get that $f(y) \leq M$. Thus f is locally bounded from above on D_{n+1} .

On the other hand, we show that

$$D = \bigcup_{n=1}^{\infty} D_n.$$

From the definition of D_n , it follows by induction that $D_n = \lambda^n p + (1 - \lambda^n)D$. For fixed $x \in D$, define the sequence x_n by

$$x_n := \frac{x - \lambda^n p}{1 - \lambda^n}.$$

Then $x_n \rightarrow x$ if $n \rightarrow \infty$. As D is open, $x_n \in D$ for some n . Therefore

$$x = \lambda^n p + (1 - \lambda^n)x_n \in \lambda^n p + (1 - \lambda^n)D = D_n.$$

Thus f is locally bounded from above on D .

Now, we prove that f is locally bounded from below. Let $q \in D$ be arbitrary. Since f is locally bounded from above at the point q , there exist $\varrho > 0$ and $M > 0$ such that

$$\sup_{U(q, \varrho)} f \leq M.$$

Let $x \in U(q, \lambda \varrho)$ and $y := \frac{q - (1-\lambda)x}{\lambda}$. Then y is in $U(q, \varrho)$. By (λ, s) -convexity,

$$f(q) \leq (1-\lambda)^s f(x) + \lambda^s f(y),$$

which implies

$$f(x) \geq \frac{f(q) - \lambda^s f(y)}{(1-\lambda)^s} \geq \frac{f(q) - \lambda^s M}{(1-\lambda)^s} =: M'.$$

Therefore f is locally bounded from below at any point of D . $\square$

As an immediate consequence of the previous theorem we obtain:

Corollary 1. *Let $f : D \rightarrow \mathbb{R}$ be a Jensen s -convex function. If f is locally bounded from above at a point of D , then f is locally bounded at every point of D .*

The next theorem essentially weakens the local boundedness assumption if the underlying space is of finite dimension. It can be derived from Theorem 2 adopting the argument followed in [8] (that is based on STEINHAUS' and PICCARD's theorems (cf. [18], [16])).

Theorem 3. *Let D be an open convex subset of $\mathbb{R}^n$ and let $f : D \rightarrow \mathbb{R}$ be a (λ, s) -convex function with a fixed $0 < \lambda < 1$. Assume that there exist a Lebesgue-measurable set of positive measure (or a Baire-measurable set of second category) $S \subseteq D$ and a Lebesgue-measurable (resp. Baire-measurable) function $g : S \rightarrow \mathbb{R}$ such that $f \leq g$ on S . Then f is locally bounded on D .*

PROOF. Let

$$S_{k,m} := \{x \in S \mid g(x) \leq k\} \cap U(0, m) \quad m, k \in \mathbb{N}.$$

Then

$$S = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} S_{k,m},$$

therefore, for some k, m , the set $S_{k,m}$ is of positive measure. Therefore, f is bounded by k on $S_{k,m}$, which is a bounded set of positive measure (or a bounded set of second category).

Taking $x, y \in S_{k,m}$, we get that

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \leq (\lambda^s + (1-\lambda)^s)k \leq 2^{1-s}k.$$

That is, f is bounded on $\lambda S_{k,m} + (1-\lambda)S_{k,m}$, which, by the theorem of STEINHAUS [18] (or the theorem of PICCARD [16]) (cf. [11]), contains an interior point. Therefore, f is locally bounded from above at a point of D . As an immediate consequence of the previous theorem we obtain that f is locally bounded on D . $\square$

Remark 2. It is a well-known fact that if a Jensen-convex function f is locally bounded above at a point of its domain (see [1], [11]), then it is continuous on its domain. This is not true for Jensen s -convex functions. Indeed, let $0 < s < 1$ be fixed and

$$f(x) := \begin{cases} 1, & \text{if } x \in](2^s - 1)^{\frac{1}{s}}, 1[\setminus \mathbb{Q}; \\ x^s, & \text{if } x \in](2^s - 1)^{\frac{1}{s}}, 1[\cap \mathbb{Q}, \end{cases}$$

Then f is Jensen s -convex, bounded and nowhere continuous.

Next theorem gives a sufficient condition for local boundedness to imply continuity.

Theorem 4. *Let the sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ be such that $\lambda_n \in]0, 1]$ and λ_n tends to 0 (when $n \rightarrow \infty$). If $f : D \rightarrow \mathbb{R}$ is in $\mathcal{K}_{\{\lambda_n\}_{n \in \mathbb{N}}}^s$ and f is locally bounded from above at a point $x_0 \in D$, then f is continuous at x_0 .*

PROOF. Since f is locally bounded from above at a point $x_0 \in D$, there exists a neighborhood U of x_0 and a constat $K \geq 0$ such that $f(x) \leq K$ for every $x \in U$. Let ε be an arbitrary nonnegative constant. Then there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then

$$\lambda_n^s K + [(1 - \lambda_n)^s - 1] f(x_0) < \varepsilon,$$

whence

$$\frac{\lambda_n^s}{(1 - \lambda_n)^s} K + \left[1 - \frac{1}{(1 - \lambda_n)^s} \right] f(x_0) < \varepsilon.$$

Let V be a neighborhood of 0 such that $x_0 + V \subseteq U$, and let $U' = x_0 + \lambda_n V$. We prove that

$$|f(x) - f(x_0)| < \varepsilon \quad (x \in U').$$

For $x \in U'$ there exist $y, z \in x_0 + V$ such that

$$x = \lambda_n y + (1 - \lambda_n)x_0, \quad x_0 = \lambda_n z + (1 - \lambda_n)x.$$

Indeed,

$$y - x_0 = \frac{1}{\lambda_n}(x - x_0) \in \frac{1}{\lambda_n}\lambda_n V = V,$$

and

$$z - x_0 = \frac{1 - \lambda_n}{\lambda_n}(x_0 - x) \in \frac{1 - \lambda_n}{\lambda_n}\lambda_n V = (1 - \lambda_n)V \subseteq V.$$

According to (λ_n, s) -convexity of f ,

$$f(x) \leq \lambda_n^s f(y) + (1 - \lambda_n)^s f(x_0) \leq \lambda_n^s K + (1 - \lambda_n)^s f(x_0),$$

$$f(x_0) \leq \lambda_n^s f(z) + (1 - \lambda_n)^s f(x) \leq \lambda_n^s K + (1 - \lambda_n)^s f(x).$$

We get

$$f(x) - f(x_0) \leq \lambda_n^s K + [(1 - \lambda_n)^s - 1] f(x_0) < \varepsilon \quad (3)$$

and

$$f(x) \geq \frac{f(x_0) - \lambda_n^s K}{(1 - \lambda_n)^s},$$

which implies

$$f(x) - f(x_0) \geq \left[\frac{1}{(1 - \lambda_n)^s} - 1 \right] f(x_0) - \frac{\lambda_n^s}{(1 - \lambda_n)^s} K > -\varepsilon. \quad (4)$$

The inequalities (3) and (4) show that $|f(x) - f(x_0)| < \varepsilon$, that is f is continuous at x_0 , which was to be proved. $\square$

Corollary 2. *Let $H \subseteq [0, 1]$ and assume that 0 or 1 is a limit point of H . If $f : D \rightarrow \mathbb{R}$ is (H, s) -convex and locally bounded at a point of D , then f is continuous at that point.*

PROOF. Since f is (H, s) -convex, it is also $(1 - H, s)$ -convex, so there exists a sequence in H or in $1 - H$, which tends to zero. Now, we can apply the previous theorem. $\square$

Theorem 5. *Let $H \subseteq [0, 1]$ and assume that 0 or 1 is a limit point of H . If $f : D \rightarrow \mathbb{R}$ is (H, s) -convex and locally bounded at a point of D , then f is continuous on D .*

PROOF. According to Theorem 2, f is locally bounded at every point of D . So, we can use the previous corollary to get the continuity of f at every point of D . $\square$

References

- [1] F. BERNSTEIN and G. DOETSCH, Zur Theorie der konvexen Funktionen, *Math. Annalen* **76** (1915), 514–526.
- [2] W. W. BRECKNER, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen, *Publ. Inst. Math. (Beograd)* **23** (1978), 13–20.
- [3] W. W. BRECKNER and G. ORBÁN, Continuity properties of rationally s -convex mappings with values in ordered topological liner space, “*Babes-Bolyai*” University, Kolozsvár, 1978.
- [4] Z. DARÓCZY and ZS. PÁLES, Convexity with given infinite weight sequences, *Stochastica* **11** (1987), 5–12.
- [5] A. GILÁNYI, K. NIKODEM and ZS. PÁLES, Bernstein-Doetsch type results for quasiconvex functions, *Math. Ineq. and Appl.* **7**, no. 2 (2004), 169–175.
- [6] A. HÁZY, On approximately t -convexity, *Math. Ineq. and Appl.* **8**, no. 3 (2005), 389–402.

- [7] A. HÁZY, On the stability of t -convex functions, *Aequationes Math.* **74** (2007), 210–218.
- [8] A. HÁZY and Zs. PÁLES, On approximately midconvex functions, *Bull. London Math. Soc.* **36** (2004), 339–350.
- [9] A. HÁZY and Zs. PÁLES, On approximately t -convex functions, *Publ. Math. Debrecen* **66**, no. 3–4 (2005), 489–501, Dedicated to the 75th birthday of Professor Heinz König.
- [10] H. HUDZIK and L. MALIGRANDA, Some remarks on s_i -convex functions, *Aequationes Math.* **48** (1994), 100–111.
- [11] M. KUCZMA, An Introduction to the Theory of Functional Equations and Inequalities, *Państwowe Wydawnictwo Naukowe – Uniwersytet Ślęski, Warszawa – Kraków – Katowice*, 1985.
- [12] J. MROWIEC, Remark on approximately Jensen-convex functions, *C. R. Math. Acad. Sci. Soc. R. Canada* **23** (2001), 16–21.
- [13] C. T. NG and K. NIKODEM, On approximately convex functions, *Proc. Amer. Math. Soc.* **118**, no. 1 (1993), 103–108.
- [14] Zs. PÁLES, Bernstein–Doetsch-type results for general functional inequalities, *Rocznik Nauk.-Dydakt. Prace Mat.* **17** (2000), 197–206, Dedicated to Professor Zenon Moszner on his 70th birthday.
- [15] Zs. PÁLES, On approximately convex functions, *Proc. Amer. Math. Soc.* **131** (2003), 243–252.
- [16] S. PICCARD, Sur des Ensembles Parfaits, *Mém. Univ. Neuchâtel, vol. 16, Secrétariat de l'Université, Neuchâtel*, 1942.
- [17] M. PYCIA, A direct proof of the s -Hölder continuity of Breckner s -convex functions, *Aequationes Math.* **61** (2001), 128–130.
- [18] H. STEINHAUS, Sur les distances des points des ensembles de mesure positive, *Fund. Math.* **1** (1920), 93–104.

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