

## On a characterization theorem on Abelian groups

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**Abstract.** Let  $\xi_1, \xi_2, \dots, \xi_n$ ,  $n \geq 2$  be independent identically distributed random variables. It is well known that if  $\bar{\xi} = \frac{1}{n} \sum_{j=1}^n \xi_j$  and  $\mathbf{v} = (\xi_1 - \bar{\xi}, \xi_2 - \bar{\xi}, \dots, \xi_n - \bar{\xi})$  are independent, then all  $\xi_j$  are Gaussian. We give a complete description of second countable locally compact Abelian groups for which a group analogue of this characterization theorem holds true.

### 1. Introduction

It is well known that if  $\xi_1, \xi_2, \dots, \xi_n$ ,  $n \geq 2$  are independent identically distributed Gaussian random variables, then  $\bar{\xi} = \frac{1}{n} \sum_{j=1}^n \xi_j$  and  $\mathbf{v} = (\xi_1 - \bar{\xi}, \xi_2 - \bar{\xi}, \dots, \xi_n - \bar{\xi})$  are independent. Assume now that  $\xi_1, \xi_2, \dots, \xi_n$ ,  $n \geq 2$  are independent identically distributed random variables such that  $\bar{\xi}$  and  $\mathbf{v}$  are independent. Then  $\bar{\xi}$  and  $s^2 = \frac{1}{n} \sum_{j=1}^n (\xi_j - \bar{\xi})^2$  are also independent. This implies by Geary's theorem ([6], [11], [10], [13]) that the random variables  $\xi_j$  are Gaussian. Thus, a Gaussian measure on the real line is characterized by the independence of  $\bar{\xi}$  and  $\mathbf{v}$ .

The article deals with a generalization of this characterization theorem to the case when independent random variables take values in a locally compact Abelian group. Since an arbitrary Abelian group generally is not a group with unique division by  $n$ , instead of  $\bar{\xi}$  and  $\mathbf{v}$  we consider  $S = \sum_{j=1}^n \xi_j$  and  $\mathbf{V} = (n\xi_1 - S, \dots, n\xi_n - S)$ .

We will use in the article the standard results on structure of locally compact Abelian groups and the duality theory ([7]). Agree on notation. For an arbitrary

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locally compact Abelian group  $X$  denote by  $Y = X^*$  its character group, and by  $(x, y)$  the value of a character  $y \in Y$  at an element  $x \in X$ . If  $H$  is a closed subgroup of the group  $Y$ , then denote by  $A(X, H) = \{x \in X : (x, y) = 1 \text{ for all } y \in H\}$  its annihilator. Denote by  $c_X$  the connected component of zero of the group  $X$ , and by  $b_X$  the set of all compact elements of  $X$ . Let  $n$  be a natural number. Put  $X_{(n)} = \{x \in X : nx = 0\}$  and  $X^{(n)} = \{x \in X : x = n\tilde{x}, \tilde{x} \in X\}$ . Denote by  $\mathbb{Z}$  the group of integers, by  $\mathbb{R}$  the group of real numbers and by  $\mathbb{T}$  the circle group (the one-dimensional torus). Let  $Y$  be an arbitrary Abelian group,  $f(y)$  be a function on  $Y$ ,  $h$  be an element of  $Y$ . Denote by  $\Delta_h$  the finite difference operator

$$\Delta_h f(y) = f(y + h) - f(y).$$

A function  $f(y)$  on  $Y$  is called a polynomial if for a nonnegative integer  $m$   $f(y)$  satisfies the equation

$$\Delta_h^{m+1} f(y) = 0, \quad y, h \in Y.$$

The minimal  $m$  for which this equality holds is called the degree of the polynomial  $f(y)$ .

We will assume in the article that  $X$  is a second countable locally compact Abelian group. Denote by  $M^1(X)$  the convolution semigroup of probability distributions on  $X$ . For  $\mu \in M^1(X)$  denote by

$$\hat{\mu}(y) = \int_X (x, y) d\mu(x)$$

its characteristic function. Note that if  $\xi$  is a random variable with values in  $X$  and distribution  $\mu$ , then the characteristic function of the distribution  $\mu$  is the mathematical expectation

$$\hat{\mu}(y) = \mathbf{E}[(\xi, y)].$$

Denote by  $E_x$  the degenerate distribution concentrated at the point  $x \in X$ . The set of all degenerate distributions on the group  $X$  denote by  $D(X)$ . For  $\mu \in M^1(X)$  define the distribution  $\bar{\mu} \in M^1(X)$  by the formula  $\bar{\mu}(B) = \mu(-B)$  for any Borel set  $B$ . Denote by  $\sigma(\mu)$  the support of a distribution  $\mu$ .

A probability measure  $\gamma$  on the group  $X$  is called Gaussian (in the sense of PARTHASARATHY) ([12, Ch. 4.6]), if its characteristic function can be represented in the form

$$\hat{\gamma}(y) = (x, y) \exp\{-\varphi(y)\}, \quad (1)$$

where  $x \in X$ , and  $\varphi(y)$  is a continuous nonnegative function on  $Y$  satisfying the equation

$$\varphi(u + v) + \varphi(u - v) = 2[\varphi(u) + \varphi(v)], \quad u, v \in Y. \quad (2)$$

Taking into account that in the article we will deal only with Gaussian measures in the sense of Parthasarathy we will name them Gaussian. Denote by  $\Gamma(X)$  the set of Gaussian measures on the group  $X$ . Denote by  $m_K$  the normalized Haar measure of a compact subgroup  $K$  of the group  $X$ , and by  $I(X)$  the set of shifts of such measures.

We will say that a distribution  $\mu \in \Gamma_n(X)$  if there exist independent identically distributed random variables  $\xi_j$ ,  $j = 1, 2, \dots, n$ ,  $n \geq 2$  with values in the group  $X$  and a distribution  $\mu$  such that  $S$  and  $\mathbf{V}$  are independent.

It is not difficult to verify that for a group  $X$  the inclusion  $\Gamma(X) \subset \Gamma_n(X)$  holds. In §2 we completely describe groups  $X$  which have the following property: if  $\mu \in \Gamma_n(X)$  and the characteristic function  $\hat{\mu}(y)$  does not vanish, then  $\mu \in \Gamma(X)$ . We apply the results of §2 in §3 to give the complete description of locally compact Abelian groups  $X$  for which any distribution  $\mu \in \Gamma_n(X)$  is invariant with respect to a compact subgroup  $K$  of the group  $X$  and under the natural homomorphism  $X \mapsto X/K$  induces on the factor-group  $X/K$  a Gaussian measure. We can consider the obtained class of groups as the widest subclass of locally compact Abelian groups on which an analogue of the theorem about characterization of a Gaussian measure on the real line by the independence of  $\bar{\xi} = \frac{1}{n} \sum_{j=1}^n \xi_j$  and  $\mathbf{v} = (\xi_1 - \bar{\xi}, \xi_2 - \bar{\xi}, \dots, \xi_n - \bar{\xi})$  holds.

The above mentioned problems are reduced to solving of a functional equation in the class of continuous positive definite functions on the group  $Y = X^*$ .

## 2. The characteristic function $\hat{\mu}(y)$ does not vanish

**Lemma 1.** *A distribution  $\mu \in \Gamma_n(X)$  if and only if its characteristic function  $\hat{\mu}(y)$  satisfies the equation*

$$\begin{aligned} \prod_{j=1}^n \hat{\mu}(u + nv_j - (v_1 + \dots + v_n)) \\ = \hat{\mu}^n(u) \prod_{j=1}^n \hat{\mu}(nv_j - (v_1 + \dots + v_n)), \quad u, v_1, \dots, v_n \in Y. \end{aligned} \quad (3)$$

PROOF. Note that  $S$  and  $\mathbf{V}$  are independent if and only if the equality

$$\mathbf{E}[(S, u)(\mathbf{V}, (v_1, \dots, v_n))] = \mathbf{E}[(S, u)]\mathbf{E}[(\mathbf{V}, (v_1, \dots, v_n))] \quad (4)$$

holds for all  $u, v_1, \dots, v_n \in Y$ . Taking into account that the random variables

$\xi_1, \dots, \xi_n$  are independent, we transform the left-hand side of (4):

$$\begin{aligned} \mathbf{E}[(S, u)(\mathbf{V}, (v_1, \dots, v_n))] &= \mathbf{E}[(\xi_1 + \dots + \xi_n, u)((n\xi_1 - (\xi_1 + \dots + \xi_n), \\ &\quad \dots, n\xi_n - (\xi_1 + \dots + \xi_n), (v_1, \dots, v_n))] \\ &= \mathbf{E} \left[ \prod_{j=1}^n (\xi_j, u + nv_j - (v_1 + \dots + v_n)) \right] = \prod_{i=1}^n \widehat{\mu}(u + nv_i - (v_1 + \dots + v_n)). \end{aligned}$$

Analogously we transform the right-hand side of (4):

$$\begin{aligned} \mathbf{E}[(S, u)]\mathbf{E}[(\mathbf{V}, (v_1, \dots, v_n))] &= \mathbf{E}[(\xi_1 + \dots + \xi_n, u)]\mathbf{E}[(n\xi_1 - (\xi_1 + \dots + \xi_n), \\ &\quad \dots, n\xi_n - (\xi_1 + \dots + \xi_n), (v_1, \dots, v_n))] \\ &= \prod_{j=1}^n \mathbf{E}[(\xi_j, u)]\mathbf{E} \left[ \prod_{j=1}^n (\xi_j, nv_j - (v_1 + \dots + v_n)) \right] \\ &= \prod_{j=1}^n \mathbf{E}[(\xi_j, u)] \prod_{j=1}^n \mathbf{E}[(\xi_j, nv_j - (v_1 + \dots + v_n))] \\ &= \widehat{\mu}^n(u) \prod_{i=1}^n \widehat{\mu}(nv_i - (v_1 + \dots + v_n)). \quad \square \end{aligned}$$

Suppose that  $\gamma \in \Gamma(X)$  and the characteristic function  $\widehat{\gamma}(y)$  has representation (1). By the function  $\varphi(y)$  we can construct a symmetric 2-additive function by the formula

$$\psi(u, v) = \frac{1}{2}[\varphi(u + v) - \varphi(u) - \varphi(v)].$$

Then  $\varphi(y) = \psi(y, y)$ . Using this representation for the function  $\varphi(y)$  one can check directly that the characteristic function  $\widehat{\gamma}(y)$  satisfies equation (3). Hence, by Lemma 1 the inclusion

$$\Gamma(X) \subset \Gamma_n(X) \quad (5)$$

holds. The main result of this section is the complete description of groups  $X$  which have the property: if  $\mu \in \Gamma_n(X)$  and the characteristic function  $\widehat{\mu}(y)$  does not vanish, then  $\mu \in \Gamma(X)$ . The following proposition is valid.

**Proposition 1.** *Assume that  $\mu \in \Gamma_2(X)$  and the characteristic function  $\widehat{\mu}(y)$  does not vanish. This implies that  $\mu \in \Gamma(X)$  if and only if the group  $X$  contains no subgroup topologically isomorphic to the two-dimensional torus  $\mathbb{T}^2$ . Assume that  $\mu \in \Gamma_n(X)$ , where  $n \geq 3$ . This implies that  $\mu \in \Gamma(X)$  if and only if the group  $X$  contains no subgroup topologically isomorphic to the circle group  $\mathbb{T}$ .*

We need some lemmas to prove Proposition 1.

**Lemma 2.** *Let  $X = \mathbb{T}$  and  $n \geq 3$ . Then there exists a distribution  $\mu \in \Gamma_n(X)$  such that the characteristic function  $\hat{\mu}(y)$  does not vanish and  $\mu \notin \Gamma(X)$ .*

PROOF. Let  $n = 3$ . Consider on the group  $\mathbb{Z}$  the function

$$l(k) = \begin{cases} 1 & \text{if } k \in \mathbb{Z}^{(3)} \\ \exp\left\{\frac{2\pi i}{9}\right\} & \text{if } k \in 1 + \mathbb{Z}^{(3)} \\ \exp\left\{-\frac{2\pi i}{9}\right\} & \text{if } k \in 2 + \mathbb{Z}^{(3)}. \end{cases} \quad (6)$$

Obviously, (6) implies that

$$l^3(k) = \exp\left\{\frac{2\pi ki}{3}\right\}, \quad k \in \mathbb{Z}. \quad (7)$$

Taking into account (7) and the fact that  $l(k+3p) = l(k)$ ,  $k, p \in \mathbb{Z}$ , we can verify directly that the function  $l(k)$  satisfies equation (3) for  $n = 3$ .

Take  $\sigma > 0$  in such a way that the inequality

$$\sum_{k \in \mathbb{Z}, k \neq 0} \exp\{-\sigma k^2\} < 1 \quad (8)$$

holds. Put

$$\rho(t) = 1 + \sum_{k \in \mathbb{Z}, k \neq 0} l(k) \exp\{-\sigma k^2 - ikt\}, \quad t \in \mathbb{R}.$$

Since  $l(-k) = \overline{l(k)}$ ,  $|l(k)| = 1$ ,  $k \in \mathbb{Z}$ , in view of (8) the inequality

$$\rho(t) > 0, \quad t \in \mathbb{R}$$

is valid. We also have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(t) dt = 1.$$

Let  $\mu$  be the distribution on the group  $\mathbb{T}$  with density  $r(e^{it}) = \rho(t)$  with respect to  $m_{\mathbb{T}}$ . By the construction the characteristic function of the distribution  $\mu$  is of the form

$$\hat{\mu}(k) = l(k) \exp\{-\sigma k^2\}, \quad k \in \mathbb{Z}.$$

Since the function  $l(k)$  satisfies equation (3) for  $n = 3$ , the characteristic function  $\hat{\mu}(k)$  also satisfies equation (3) for  $n = 3$ . By Lemma 1  $\mu \in \Gamma_3(X)$ . On the other hand, since the function  $l(k)$  is not a character of the group  $\mathbb{Z}$ , we have  $\mu \notin \Gamma(X)$ .

Let  $n > 3$ . If  $n = 2p - 1$  we put

$$l(k) = \begin{cases} 1 & \text{if } k \in q + \mathbb{Z}^{(n)}, \quad q = 0, 1, \dots, p-2, p+1, \dots, n-1, \\ \exp\left\{\frac{2\pi i}{n}\right\} & \text{if } k \in p-1 + \mathbb{Z}^{(n)} \\ \exp\left\{-\frac{2\pi i}{n}\right\} & \text{if } k \in p + \mathbb{Z}^{(n)}. \end{cases}$$

If  $n = 2p$  we put

$$l(k) = \begin{cases} 1 & \text{if } k \notin p + \mathbb{Z}^{(n)} \\ -1 & \text{if } k \in p + \mathbb{Z}^{(n)}. \end{cases}$$

Next we argue as in the case when  $n = 3$ . □

*Remark 1.* Let  $n \geq 3$ . It is easily seen that the function  $l(k)$  constructed in the proof of Lemma 2 is the characteristic function of a signed measure on the group  $\mathbb{T}$  concentrated in the subgroup  $\mathbb{Z}(n)$  (the multiplicative group of  $n$ th roots of unity).

**Lemma 3.** Assume that  $\gamma \in \Gamma_n(X)$ , and the characteristic function  $\hat{\gamma}(y) > 0$  for  $y \in Y$ . Then  $\gamma \in \Gamma(X)$ , and the function  $\hat{\gamma}(y)$  can be represented in the form (1), where  $x = 0$ .

PROOF. Put  $\psi(y) = -\log \hat{\gamma}(y)$ . By Lemma 1 the characteristic function  $\hat{\gamma}(y)$  satisfies equation (3). Taking the logarithm of both sides of (3), we get

$$\begin{aligned} \sum_{j=1}^n \psi(u + nv_j - (v_1 + \dots + v_n)) \\ = n\psi(y) + \sum_{j=1}^n \psi(nv_j - (v_1 + \dots + v_n)), \quad u, v_j \in Y. \end{aligned} \quad (9)$$

We use the finite difference method to solve equation (9). Let  $h_1$  be an arbitrary element of the group  $Y$ . Substitute  $u + h_1$  for  $u$  and  $v_j + h_1$  for  $v_j$ ,  $j = 1, 2, \dots, n$  in equation (9). Subtracting equation (9) from the resulting equation we obtain

$$\sum_{j=1}^n \Delta_{h_1} \psi(u + nv_j - (v_1 + \dots + v_n)) = n\Delta_{h_1} \psi(u), \quad u, h_1, v_j \in Y. \quad (10)$$

Let  $h_2$  be an arbitrary element of the group  $Y$ . Substitute  $u + h_2$  for  $u$  and  $v_1 + h_2$  for  $v_1$  in equation (9). Subtracting equation (10) from the resulting equation we get

$$\Delta_{nh_2} \Delta_{h_1} \psi(u + nv_1 - (v_1 + \dots + v_n)) = n\Delta_{h_2} \Delta_{h_1} \psi(u), \quad u, h_1, h_2, v_j \in Y. \quad (11)$$

Let  $h_3$  be an arbitrary element of the group  $Y$ . Substitute  $u + h_3$  for  $u$  and  $v_2 + h_3$  for  $v_2$  in equation (9). Subtracting equation (11) from the resulting equation we find

$$n\Delta_{h_3}\Delta_{h_2}\Delta_{h_1}\psi(u) = 0, \quad u, h_1, h_2, h_3 \in Y. \quad (12)$$

Put in (12)  $h_1 = h_2 = h_3 = h$ . We get

$$\Delta_h^3\psi(u) = 0, \quad u, h \in Y, \quad (13)$$

i.e.  $\psi(y)$  is a continuous polynomial of degree  $\leq 2$ . It is easy to see that each polynomial of degree  $\leq 2$ , in particular  $\psi(y)$ , can be represented in the form

$$\psi(y) = \varphi(y) + l(y) + c, \quad y \in Y, \quad (14)$$

where the function  $\varphi(y)$  satisfies equation (2), the function  $l(y)$  satisfies equation

$$l(u + v) = l(u) + l(v), \quad u, v \in Y,$$

and  $c = \text{const}$ . Since  $\widehat{\gamma}(0) = 1$ , we can assume that  $c = 0$ . Since the function  $\widehat{\gamma}(y)$  is real-valued, we have  $\widehat{\gamma}(-y) = \overline{\widehat{\gamma}(y)} = \widehat{\gamma}(y)$ . Hence,  $\psi(-y) = \psi(y)$ . This implies that in (14)  $l(y) = 0$ ,  $y \in Y$ . So,  $\psi(y) = \varphi(y)$ ,  $y \in Y$ . This proves the lemma.  $\square$

A distribution  $\mu \in M^1(X)$  is called a Gaussian measure in the sense of Bernstein ([9, 5.3]) if  $\mu$  has the following property: if  $\xi_1$  and  $\xi_2$  are independent identically distributed random variables with values in  $X$  and distribution  $\mu$ , then their sum and difference are independent. We denote by  $\Gamma_B(X)$  the set of Gaussian measures in the sense of Bernstein on the group  $X$ .

**Lemma 4** ([9, 5.3]). *A distribution  $\mu \in M^1(X)$  belongs to the class  $\Gamma_B(X)$  if and only if the characteristic function  $\widehat{\mu}(y)$  satisfies the equation*

$$\widehat{\mu}(u + v)\widehat{\mu}(u - v) = \widehat{\mu}^2(u)|\widehat{\mu}(v)|^2, \quad u, v \in Y. \quad (15)$$

**Lemma 5.**  $\Gamma_2(X) = \Gamma_B(X)$ .

PROOF. Let  $\mu \in \Gamma_2(X)$ . By Lemma 1 the characteristic function  $\widehat{\mu}(y)$  satisfies equation (3) which takes the form

$$\widehat{\mu}(u + (v_1 - v_2))\widehat{\mu}(u - (v_1 - v_2)) = \widehat{\mu}^2(u)|\widehat{\mu}(v_1 - v_2)|^2, \quad u, v_1, v_2 \in Y. \quad (16)$$

Substituting  $v_1 = v$ ,  $v_2 = 0$  into (16), we obtain that the characteristic function  $\widehat{\mu}(y)$  satisfies equation (15). Hence, by Lemma 4  $\mu \in \Gamma_B(X)$ . Lemmas 1 and 4 also imply that if  $\mu \in \Gamma_B(X)$ , then  $\mu \in \Gamma_2(X)$ .  $\square$

PROOF OF PROPOSITION 1. Let  $n = 2$ . Applying Lemma 5 we reduce the proof of Proposition 1 to the proof of the corresponding statement for distributions from the class  $\Gamma_B(X)$ , but for such distributions this statement was proved in [2] (see also [5], Lemmas 9.6 and 9.7).

Let  $n \geq 3$  and  $\mu \in \Gamma_n(X)$ . Put  $\nu = \mu * \bar{\mu}$ . It follows from Lemma 1 that  $\nu \in \Gamma_n(X)$ . Since  $\widehat{\nu}(y) = |\widehat{\mu}(y)|^2 > 0$ , by Lemma 3  $\nu \in \Gamma(X)$ . If a group  $X$  contains no subgroup topologically isomorphic to  $\mathbb{T}$ , then by Cramer's theorem for locally compact Abelian groups ([1], see also [5, Theorem 4.6]), we get that  $\mu \in \Gamma(X)$ . Thus, the sufficiency when  $n \geq 3$  is also proved. The necessity follows from Lemma 2.  $\square$

### 3. The main theorem

Put  $I_n(X) = I(X) \cap \Gamma_n(X)$ . It follows from (5) and Lemma 1 that the inclusion

$$\Gamma(X) * I_n(X) \subset \Gamma_n(X)$$

holds. The main problem solved in this section is the following: to describe all groups  $X$  for which

$$\Gamma(X) * I_n(X) = \Gamma_n(X). \quad (17)$$

Observe that if  $\mu \in \Gamma(X) * I(X)$ , then  $\mu$  is invariant with respect to a compact subgroup  $K$  of the group  $X$  and under the natural homomorphism  $X \mapsto X/K$  induces on the factor-group  $X/K$  a Gaussian measure. We formulate now the main theorem.

**Theorem 1.** *Equality (17) holds for a group  $X$ , when  $n = 2$ , if and only if the connected component of zero  $c_X$  of  $X$  contains no more than one element of order 2. Equality (17) holds for a group  $X$ , when  $n \geq 3$ , if and only if  $c_X$  has the property*

$$\{x \in c_X : nx = 0\} = \{0\}. \quad (18)$$

To prove Theorem 1 we need some lemmas. First we will formulate as a lemma the following simple statement, but omit its proof.

**Lemma 6.** *Let  $n$  be a natural number,  $K$  be a compact subgroup of a group  $X$ . Then the following statements are equivalent:*

- (i)  $K^{(n)} = K$ ;
- (ii) if  $ny \in A(Y, K)$ , then  $y \in A(Y, K)$ .

We note that the characteristic function of the Haar measure  $m_K$  is of the form

$$\widehat{m}_K(y) = \begin{cases} 1 & \text{if } y \in A(Y, K) \\ 0 & \text{if } y \notin A(Y, K). \end{cases} \quad (19)$$

**Lemma 7.** *Let  $K$  be a compact subgroup of a group  $X$ . Then the following statements are equivalent:*

- (i)  $m_K \in I_n(X)$ ;
- (ii)  $K^{(n)} = K$ .

PROOF. (i)  $\Rightarrow$  (ii). By Lemma 1 the characteristic function  $\widehat{m}_K(y)$  satisfies equation (3). Substituting  $v_1 = u$ ,  $v_2 = \cdots = v_n = 0$  into (3), we get

$$\widehat{m}_K(nu) = \widehat{m}_K(u)\widehat{m}_K((n-1)u), \quad u \in Y. \quad (20)$$

It follows from (19) and (20) that if  $ny \in A(Y, K)$ , then  $y \in A(Y, K)$ . Hence, by Lemma 6  $K^{(n)} = K$ .

(ii)  $\Rightarrow$  (i). By Lemma 1 it suffices to check that the characteristic function  $\widehat{m}_K(y)$  satisfies equation (3), which takes the form

$$\begin{aligned} \prod_{j=1}^n \widehat{m}_K(u + nv_j - (v_1 + \cdots + v_n)) \\ = \widehat{m}_K(u) \prod_{j=1}^n \widehat{m}_K(nv_j - (v_1 + \cdots + v_n)), \quad u, v_1, \dots, v_n \in Y. \end{aligned} \quad (21)$$

Both sides of equation (21) take the values either 0 or 1. If the right-hand side of (21) is equal to 1, then  $u \in A(Y, K)$ ,  $nv_j - (v_1 + \cdots + v_n) \in A(Y, K)$ ,  $j = 1, 2, \dots, n$ . Hence,  $u + nv_j - (v_1 + \cdots + v_n) \in A(Y, K)$ ,  $j = 1, 2, \dots, n$ . Therefore the left-hand side of (21) is also equal to 1.

Let the left-hand side of (21) be equal to 1. This implies that

$$u + nv_j - (v_1 + \cdots + v_n) \in A(Y, K), \quad j = 1, 2, \dots, n. \quad (22)$$

It follows from (22) that

$$\sum_{j=1}^n (u + nv_j - (v_1 + \cdots + v_n)) = nu \in A(Y, K). \quad (23)$$

By Lemma 6 we get from (23) that  $u \in A(Y, K)$ . Then (22) implies that  $nv_j - (v_1 + \cdots + v_n) \in A(Y, K)$ ,  $j = 1, 2, \dots, n$ . Hence, the right-hand side of (21) is also equal to 1.  $\square$

**Lemma 8** (compare with [4, Lemma 10.4]). *Let  $X$  be a discrete torsion free group. Then  $\Gamma_n(X) = D(X)$ .*

PROOF. If  $X$  is a discrete torsion free group, then  $Y$  is a connected compact group ([7, (24.25)]). Assume first that  $Y \not\cong \mathbb{T}$ . Then there exists a continuous monomorphism  $p : \mathbb{R} \mapsto Y$  such that its image  $p(\mathbb{R})$  is everywhere dense in  $Y$  ([7, (25.18)]). Let  $\mu \in \Gamma_n(X)$ . Consider the restriction of the characteristic function  $\widehat{\mu}(y)$  to  $p(\mathbb{R})$ . Put  $f(t) = \widehat{\mu}(p(t))$ ,  $t \in \mathbb{R}$ . By Lemma 1 the characteristic function  $\widehat{\mu}(y)$  satisfies equation (3). Therefore  $f(t)$  is a characteristic function on  $\mathbb{R}$  satisfying (3) too. Taking into account that  $\Gamma_n(\mathbb{R}) = \Gamma(\mathbb{R})$ , we have

$$f(t) = \exp\{-\sigma t^2 + i\beta t\}, \quad \sigma \geq 0, \beta \in \mathbb{R}.$$

Since  $p$  is a monomorphism and  $\overline{p(\mathbb{R})} = Y$ , for any neighborhood of zero  $V$  of the group  $Y$  we can choose a sequence of real numbers  $t_n \rightarrow \infty$  such that  $p(t_n) \in V$  for all  $n$ . If  $\sigma > 0$ , then

$$|f(t_n)| = |\widehat{\mu}(p(t_n))| = \exp\{-\sigma t_n^2\} \rightarrow 0$$

when  $t_n \rightarrow \infty$ . Taking into account that  $V$  is an arbitrary neighborhood of zero of the group  $Y$  we come to contradiction with the continuity at zero of the function  $\widehat{\mu}(y)$ . Hence,  $\sigma = 0$ , and this implies that  $|\widehat{\mu}(p(t))| = 1$  for  $t \in \mathbb{R}$ . Since  $\overline{p(\mathbb{R})} = Y$ , we have  $|\widehat{\mu}(y)| = 1$  for  $y \in Y$ . It follows from this that  $\mu \in D(X)$ .

If  $Y \cong \mathbb{T}$ , then  $X \cong \mathbb{Z}$ . Since  $\mathbb{Z}$  is a subgroup of  $\mathbb{R}$ , we have  $\Gamma_n(\mathbb{Z}) \subset \Gamma_n(\mathbb{R}) = \Gamma(\mathbb{R})$ . Taking into account that any Gaussian measure on  $\mathbb{R}$  supported in  $\mathbb{Z}$  is degenerated we completely prove the lemma.  $\square$

**Corollary 1.** *Let  $Y$  be a connected compact group,  $f(y)$  be a characteristic function on  $Y$  satisfying equation (3). Then  $|f(y)| = 1$ ,  $y \in Y$ .*

**Lemma 9.** *Let  $\mu \in \Gamma_n(X)$ . Then  $\mu$  is supported in a coset of a subgroup  $G \cong \mathbb{R}^m \times K$ , where  $m \geq 0$ , and  $K$  is a compact subgroup of  $X$  such that  $K^{(n)} = K$ .*

PROOF. Note first that if  $\lambda \in \Gamma_n(X)$ , then for any  $x \in X$  we have  $\lambda * E_x \in \Gamma_n(X)$ . Therefore if it is necessary we can substitute a distribution  $\lambda \in \Gamma_n(X)$  by its shift, and the shift also belongs to the class  $\Gamma_n(X)$ .

Let  $\lambda \in M^1(X)$  be an arbitrary distribution. Consider the set

$$E = \{y \in Y : |\widehat{\lambda}(y)| = 1\}.$$

Then  $E$  is a subgroup of  $Y$ , and there exists an element  $x \in X$  such that  $\widehat{\lambda}(y) = (x, y)$  for  $y \in E$ . It follows from what has been said that we can substitute the distribution  $\mu$  by its shift and assume from the beginning that

$$E = \{y \in Y : |\widehat{\mu}(y)| = 1\} = \{y \in Y : \widehat{\mu}(y) = 1\}. \quad (24)$$

It follows from (24) that  $\sigma(\mu) \subset G$ , where  $G = A(X, E)$ . Put  $H = G^*$ . The distribution  $\mu$  considering as a distribution on  $G$  has the property

$$\{y \in H : |\widehat{\mu}(y)| = 1\} = \{0\}. \quad (25)$$

We will check that  $G$  is the desired subgroup. It follows from the structure theorem for locally compact Abelian group that  $H \cong R^m \times L$ , where  $m \geq 0$ , and  $L$  contains a compact open subgroup [7, (24.30)]. Consider the restriction of equation (3) to the connected component of zero  $c_L$  of the group  $L$ . Since the group  $c_L$  is compact, by Corollary 1  $|\widehat{\mu}(y)| = 1$  for  $y \in c_L$ . Taking into account (25) this implies that  $c_L = \{0\}$ , i.e. the group  $L$  is totally disconnected. We will prove that the group  $L$  is discrete.

By Lemma 1 the characteristic function  $\widehat{\mu}(y)$  satisfies equation (3). Substitute  $v_1 = u$ ,  $v_2 = \dots = v_n = 0$  into (3). From the resulting equation we obtain

$$|\widehat{\mu}(nu)| = |\widehat{\mu}(u)|^{2n-1} |\widehat{\mu}((n-1)u)|, \quad u \in H.$$

This implies the inequality

$$|\widehat{\mu}(nu)| \leq |\widehat{\mu}(u)|^{2n-1}, \quad u \in H. \quad (26)$$

It follows from (26) that for any natural  $p$  the inequality

$$|\widehat{\mu}(n^p u)| \leq |\widehat{\mu}(u)|^{(2n-1)^p}, \quad u \in H \quad (27)$$

holds. Since  $\widehat{\mu}(0) = 1$  and the function  $\widehat{\mu}(y)$  is continuous, there exists a neighborhood of zero  $V$  of the group  $L$  such that  $|\widehat{\mu}(y)| > 0$  for  $y \in V$ . Inasmuch as the group  $L$  is totally disconnected, for any neighborhood of zero of  $L$ , in particular for  $V$ , there exists a compact subgroup  $W$  such that  $W \subset V$  ([7, (7.7)]). Thus, we have

$$|\widehat{\mu}(y)| > 0, \quad y \in W. \quad (28)$$

Assume that at a point  $y_0 \in W$  the inequality

$$|\widehat{\mu}(y_0)| < 1 \quad (29)$$

holds. Since the group  $W$  is compact, the sequence  $n^p y_0$ ,  $p = 1, 2, \dots$  contains a converging subsequence  $n^{p_j} y_0 \rightarrow \tilde{y}$ ,  $\tilde{y} \in W$ . It follows from (27) and (29) that  $\hat{\mu}(\tilde{y}) = 0$ , that contradicts (28). Hence,  $|\hat{\mu}(y)| = 1$  for  $y \in W$ . Taking into account (25), we get that  $W = \{0\}$ , i.e. the group  $L$  is discrete. Put  $K = L^*$ . Since  $L$  is discrete, this implies that  $K$  is compact.

Take  $y_0 \in L_{(n)}$ , i.e.  $ny_0 = 0$ . It follows from (26) that  $|\hat{\mu}(y_0)| = 1$ . Taking into account (25), we obtain that  $y_0 = 0$ . Hence,  $L_{(n)} = \{0\}$ . This implies that  $K^{(n)} = K$ .  $\square$

**Lemma 10.** *Let  $X = R^m \times K$ , where  $m \geq 0$  and  $K$  is a compact group such that  $K^{(n)} = K$ . Then  $X_{(n)} \subset c_X$ .*

PROOF. We have  $A(Y, Y^{(n)}) = X_{(n)}$ ,  $A(Y, b_Y) = c_X$  ([7, (24.17)]). The lemma will be proved if we check that  $Y^{(n)} \supset b_Y$ . Put  $L = K^*$ . Obviously,  $b_Y$  is a discrete torsion group. Since  $K^{(n)} = K$ , we have  $L_{(n)} = \{0\}$ . Take  $y_0 \in b_Y$ . Let  $M$  be a finite cyclic subgroup generated by  $y_0$ . It follows from  $L_{(n)} = \{0\}$  that the restriction of the mapping  $y \mapsto ny$ ,  $y \in Y$ , to  $M$  is a monomorphism. Taking into account that the group  $M$  is finite, this mapping is an isomorphism. Hence,  $y_0 \in Y^{(n)}$ .  $\square$

**Lemma 11.** *Let  $X$  be a locally compact Abelian group. Then the following statements are equivalent:*

- (i) *for any compact subgroup  $K$  of the group  $X$ , satisfying the condition  $K^{(n)} = K$ , the equality  $(K^*)^n = K^*$  holds;*
- (ii) *the connected component of zero  $c_X$  of the group  $X$  has the property (18);*
- (iii) *for any compact subgroup  $K$  of the group  $X$ , satisfying the condition  $K^{(n)} = K$ , the factor group  $X/K$  contains no subgroup topologically isomorphic to  $\mathbb{T}$ .*

PROOF. The equivalence of (i) and (ii) was proved in [3] (see also [4, Lemma 11.15]). To prove the equivalence of (i) and (iii) note first that  $K^{(n)} = K$  if and only if  $K^{(p)} = K$  for any prime divisor  $p$  of the number  $n$ . Hence it suffices to prove the equivalence of (i) and (iii) assuming that  $n$  is a prime number.

Let us prove (iii)  $\Rightarrow$  (i). Let  $n$  be a prime number and  $K$  be a compact subgroup of  $X$  such that  $K^{(n)} = K$ . Put  $L = K^*$ . Then  $L$  is a discrete group satisfying the condition  $L_{(n)} = \{0\}$ . It follows from this that if  $y_0$  is an element of finite order in  $L$ , then  $y_0 \in L^{(n)}$ . We will check that any element  $y_0$  of infinite order in  $L$  also belongs to  $L^{(n)}$ . Thus, the implication (iii)  $\Rightarrow$  (i) will be proved.

Let  $y_0$  be an element of infinite order in  $L$  such that  $y_0 \notin L^{(n)}$ . Consider the factor-group  $L/L^{(n)}$ . Since  $n$  is a prime number, all nonzero elements of the factor-group  $L/L^{(n)}$  have order  $n$ . It is obvious that  $[y_0] \neq 0$ . Therefore  $[y_0]$  has

order  $n$  and hence,  $ky_0 \notin L^{(n)}$ ,  $k = 1, 2, \dots, n-1$ . Denote by  $M$  the subgroup of  $L$  generated by  $y_0$ , i.e.  $M = \{y \in L : y = ly_0, l \in \mathbb{Z}\}$ . Let  $h \in L$  and  $nh \in M$ . Then  $nh = ly_0$ ,  $l \in \mathbb{Z}$ . We have  $l = qn + k$ , where  $q \in \mathbb{Z}$ ,  $k \in \{0, 1, \dots, n-1\}$ . This implies that  $ky_0 = n(h - qy_0) \in L^{(n)}$ . Therefore  $k = 0$ , and hence  $nh = qny_0$ . Since  $L_{(n)} = \{0\}$ , we get  $h = qy_0 \in M$ . Thus, the subgroup  $M$  has the property: if  $nh \in M$ , then  $h \in M$ . By Lemma 6 it follows from this that for the annihilator  $G = A(K, M)$  the equality  $G^{(n)} = G$  holds. Note now that  $(K/G)^* \cong M \cong \mathbb{Z}$ . This implies that  $K/G \cong \mathbb{T}$ . Since  $K/G$  is a subgroup of  $X/G$ , this contradicts to (iii).

The implication (i)  $\Rightarrow$  (iii) for  $n = 2$  was proved in [5, Lemma 7.7]. The proof in the general case is the same as in the case when  $n = 2$ .  $\square$

PROOF OF THEOREM 1. Let  $n = 2$ . By Lemma 5  $\Gamma_2(X) = \Gamma_B(X)$  and the statement of the theorem in this case was proved in [2] (see also [4, Theorem 9.10]).

Assume that  $n \geq 3$  and let us prove the sufficiency. Let  $\mu \in \Gamma_n(X)$ . By Lemma 1 the characteristic function  $\hat{\mu}(y)$  satisfies equation (3). Substituting  $v_1 = v$ ,  $v_2 = -v$ ,  $v_3 = \dots = v_n = 0$  in (3), we get

$$\hat{\mu}(u + nv)\hat{\mu}(u - nv)\hat{\mu}^{n-2}(u) = \hat{\mu}^n(u)|\hat{\mu}(nv)|^2, \quad u, v \in Y. \quad (30)$$

By Lemma 9 we can assume that the group  $X$  is of the form  $X = R^m \times K$ , where  $m \geq 0$ , and  $K$  is a compact group such that  $K^{(n)} = K$ . Applying Lemma 10 we obtain  $X_{(n)} \subset c_X$ . Then (18) implies that  $X_{(n)} = \{0\}$ , and hence  $A(Y, X_{(n)}) = \overline{Y^{(n)}} = Y^{(n)} = Y$ . Taking this into account, (30) implies that the characteristic function  $\hat{\mu}(y)$  satisfies equation

$$\hat{\mu}(u + v)\hat{\mu}(u - v)\hat{\mu}^{n-2}(u) = \hat{\mu}^n(u)|\hat{\mu}(v)|^2, \quad u, v \in Y. \quad (31)$$

It follows from (31) that the set

$$B = \{y \in Y : \hat{\mu}(y) \neq 0\}$$

is an open subgroup of  $Y$ . Put  $K = A(X, B)$ . Since  $B$  is an open subgroup, the group  $K$  is compact. Inasmuch as the characteristic function  $\hat{\mu}(y)$  satisfies equation (3), the function  $\hat{\mu}(y)$  satisfies inequality (26), and it follows from (26) that the group  $B$  has the property: if  $ny \in B$ , then  $y \in B$ . Applying Lemma 6 we get  $K^{(n)} = K$ . By Lemma 11 the factor-group  $X/K$  contains no subgroup topologically isomorphic to  $\mathbb{T}$ . Note now that  $(X/K)^* \cong B$  and consider the restriction of the characteristic function  $\hat{\mu}(y)$  to  $B$ . By Lemma 1 this restriction

is the characteristic function of a distribution from  $\Gamma_n(X/K)$ . Since the factor-group  $X/K$  contains no subgroup topologically isomorphic to  $\mathbb{T}$ , we can apply Proposition 1 to the factor-group  $X/K$ . As a result we obtain the following representation for the characteristic function  $\hat{\mu}(y)$

$$\hat{\mu}(y) = \begin{cases} (x, y) \exp\{-\varphi(y)\} & \text{if } y \in B \\ 0 & \text{if } y \notin B. \end{cases} \quad (32)$$

where  $x \in X$ , and  $\varphi(y)$  is a continuous function on  $B$ , satisfying equation (2). It is well known that the function  $\varphi(y)$  can be extended from the subgroup  $B$  to  $Y$  retaining its properties ([9, Lemma 5.2.5]). Denote by  $\tilde{\varphi}(y)$  the extended function. Let  $\gamma$  be a Gaussian measure on  $X$  with the characteristic function

$$\hat{\gamma}(y) = (x, y) \exp\{-\tilde{\varphi}(y)\}, \quad y \in Y. \quad (33)$$

Obviously, that  $\hat{\mu}(y) = \hat{\gamma}(y)\hat{m}_K(y)$ . Hence,  $\mu = \gamma * m_K$ .

Let us prove the necessity. Assume that (18) is not fulfilled. By Lemma 11 there exists a compact subgroup  $K$  of the group  $X$  such that  $K^{(n)} = K$  and the factor-group  $X/K$  contains a subgroup topologically isomorphic to  $\mathbb{T}$ . Therefore the distribution  $\mu$  on the group  $\mathbb{T}$ , constructed in Lemma 2, can be considered as a distribution on the factor-group  $X/K$ . We retain the notation  $\mu$  for this distribution. Since  $(X/K)^* \cong A(Y, K)$ , we may assume that the characteristic function  $\hat{\mu}(y)$  is defined on  $A(Y, K)$ . Consider on the group  $Y$  the function

$$h(y) = \begin{cases} \hat{\mu}(y) & \text{if } y \in A(Y, K) \\ 0 & \text{if } y \notin A(Y, K). \end{cases}$$

Since the set  $A(Y, K)$  is a subgroup and  $\hat{\mu}(y)$  is a positive definite function,  $h(y)$  is also a positive definite function ([8, (32.43)]). Since  $K$  is a compact group, its annihilator  $A(Y, K)$  is an open subgroup, and hence the function  $h(y)$  is continuous. By Bochner's theorem there exists a distribution  $\lambda \in M^1(X)$  such that  $\hat{\lambda}(y) = h(y)$ . We will check that  $\lambda \in \Gamma_n(X)$ . By Lemma 1 it suffices to verify that the function  $h(y)$  satisfies equation (3).

Take  $u \in A(Y, K)$ . If  $nv_{j_0} - (v_1 + \dots + v_n) \notin A(Y, K)$  holds at least for one  $j = j_0$ , then  $u + nv_{j_0} - (v_1 + \dots + v_n) \notin A(Y, K)$ , and both sides of equation (3) are equal to zero. Assume that  $nv_j - (v_1 + \dots + v_n) \in A(Y, K)$  for all  $j = 1, 2, \dots, n$ . This implies that  $n(v_j - v_1) \in A(Y, K)$ , and hence by Lemma 6  $v_j - v_1 = h_j \in A(Y, K)$ ,  $j = 1, 2, \dots, n$ . Substituting  $v_j = v_1 + h_j$ ,  $j = 1, 2, \dots, n$  in equation (3) and taking into account that the function  $\hat{\mu}(y)$  satisfies equation (3) on  $A(Y, K)$ , we get the equality.

Take  $u \notin A(Y, K)$ . Then the right-hand side of equation (3) is equal to zero. If in this case the left-hand side of equation (3) does not vanish, then the inclusions  $u + nv_j - (v_1 + \dots + v_n) \in A(Y, K)$ ,  $j = 1, 2, \dots, n$  are fulfilled. This implies that  $\sum_{j=1}^n (u + nv_j - (v_1 + \dots + v_n)) = nu \in A(Y, K)$ , and hence by Lemma 6  $u \in A(Y, K)$ , contrary to the choice of  $u$ . So we proved that the function  $h(y)$  satisfies equation (3). Since  $\mu \notin \Gamma(X/K)$ , obviously that  $\lambda \notin \Gamma(X) * I(X)$ . The theorem is completely proved.  $\square$

*Remark 2.* Let  $\mu \in \Gamma_n(\mathbb{R})$ ,  $n \geq 2$ . Put  $\gamma = \mu * \bar{\mu} \in \Gamma_n(\mathbb{R})$ . Then  $\hat{\gamma}(y) = |\hat{\mu}(y)|^2 \geq 0$ . By Lemma 1 the characteristic function  $\hat{\gamma}(y)$  satisfies equation (3), and hence satisfies equation (30) too. Taking into account that  $\mathbb{R}^{(n)} = \mathbb{R}$ , (30) implies (31), and hence the set  $B = \{y \in \mathbb{R} : \hat{\gamma}(y) \neq 0\}$  is an open subgroup of  $\mathbb{R}$ . So,  $B = \mathbb{R}$ . Since  $\hat{\gamma}(y) > 0$ ,  $y \in \mathbb{R}$ , by Lemma 3  $\gamma \in \Gamma(\mathbb{R})$ . This implies by Cramer's theorem that  $\mu \in \Gamma(\mathbb{R})$ . Thus, we proved the equality  $\Gamma(\mathbb{R}) = \Gamma_n(\mathbb{R})$ ,  $n \geq 2$ , which we used in the proof of Lemma 8, and this proof is independent from Gery's theorem.

*Remark 3.* Comparing Proposition 1 and Theorem 1 we see that in both statements we have a particular case  $n = 2$ . If  $n \geq 3$ , then the description of the corresponding class of groups in Proposition 1 does not depend on  $n$ , and does not depend on  $n$  in Theorem 1.

*Remark 4.* Observe that if  $\mu$  is an infinitely divisible distribution and  $\mu \in \Gamma_n(X)$ , then  $\mu \in \Gamma(X) * I_n(X)$ . Indeed, since  $\mu$  is an infinitely divisible distribution, the set  $B = \{y \in Y : \hat{\mu}(y) \neq 0\}$  is an open subgroup of  $Y$  ([12, p. 106]). Put  $\nu = \mu * \bar{\mu}$  and consider the restriction of the characteristic function  $\hat{\nu}(y)$  to  $B$ . Since  $\hat{\nu}(y) = |\hat{\mu}(y)|^2 > 0$  for  $y \in B$ , by Lemma 3  $\hat{\nu}(y) = \exp\{-\varphi(y)\}$ ,  $y \in B$ . Note now that on an arbitrary locally compact Abelian group in the class of infinitely divisible distributions a Gaussian measure has only Gaussian factors ([5, Remark 4.8]). Thus, for the characteristic function  $\hat{\mu}(y)$  we get the representation similar to (32), and the desired statement follows from this.

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