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Discussion on "A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces" by A. Branciari

By BESSEM SAMET (Tunis)

Abstract. The aim of this discussion is to expose incorrect property of the generalized metric space introduced by A. Branciari in previous Publ. Math. Debrecen article.

1. Preliminaries

Recently Branciari [1] introduced the concept of a generalized metric space where the triangular inequality of a metric space has been replaced by a more general inequality involving four points instead of three. More precisely, Let X be a non-empty set and $d : X \times X \to [0, +\infty)$ be a mapping such that for all $x, y \in X$ and for all distinct point $\xi, \eta \in X$, each of them different from x and y, one has

(i) $d(x, y) = 0 \Leftrightarrow x = y$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \leq d(x, \xi) + d(\xi, \eta) + d(\eta, y)$.

Then, (X, d) is called a generalized metric space, (shortly a g.m.s.).

In [1], Branciari said that the fnuction d is continuous in each coordinates. More precisely, if x_n, a, b are distinct points in X $(n \in \mathbb{N})$ and if $\lim_{n\to+\infty} x_n = a$ then we have

$$|d(x_n, b) - d(a, b)| \to 0 \quad \text{as} \quad n \to +\infty. \tag{1}$$

The aim of this paper is to show that (1) is not true in general.A counter-example is given in the next section.

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2. Counter-example

Let $(x_n)_{n\in\mathbb{N}^*}$ be a sequence in $\mathbb{Q}$ and $a,b\in\mathbb{R}\backslash\mathbb{Q}$, $a\neq b$. We put the set $X=\{x_1,x_2,\cdots,x_n,\cdots\}\cup\{a,b\}$ and we consider $d:X\times X\to\mathbb{R}$ defined by

$$\begin{cases} d(x,x)=0, & \forall\, x\in X,\\ d(x,y)=d(y,x), & \forall\, x,y\in X,\\ d(x_n,x_m)=1, & \forall\, n,m\in\mathbb{N}^*,\ n\neq m,\\ d(x_n,b)=\dfrac{1}{n}, & \forall\, n\in\mathbb{N}^*,\\ d(x_n,a)=\dfrac{1}{n}, & \forall\, n\in\mathbb{N}^*,\\ d(a,b)=1. \end{cases}$$

It is not difficult to show that (X,d) is a g.m.s. Here, we have $\lim_{n\to+\infty}x_n=a$ because $d(x_n,a)=\frac{1}{n}\to 0$ as $n\to+\infty$. But

$$|d(x_n,b)-d(a,b)|=1-\frac{1}{n}\to 1\quad\text{as}\quad n\to+\infty.$$

Then, we show that (1) is false in this case. This error can be explained as follows. To obtain (1), the author used that if (x_n) is a convergent sequence in X then $d(x_n,x_m)\to 0$ as $n,m\to+\infty$. In the other manner, the author used that every convergent sequence is a Cauchy sequence. This result is true in the case of a metric space, but in the case of a g.m.s it is false in general as it is shown in the given counter-example.

References

[1] A. Branciari, A fixed point theorem of Banach–Caccioppoli type on a class of generalized metric spaces, *Publ. Math. Debrecen* **57** (2000), 31–37.

BESSEM SAMET
DEPARTEMENT DE MATHEMATIQUES
ECOLE SUPERIEURE DES SCIENCES ET TECHNIQUES DE TUNIS
5, AVENUE TAHA HUSSEIN-TUNIS
B.P.:56, BAB MENARA-1008
TUNISIE

E-mail: bessem.samet@gmail.com

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