

Some rigidity results for Dirac-harmonic maps

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Abstract. Let (ϕ, ψ) be a Dirac-harmonic maps from a Riemannian manifold into another Riemannian manifold. We call (ϕ, ψ) *trivial* if ϕ is harmonic. By using Bochner-type formula and extending Chen–Jost–Li–Wang’ result, we give some sufficient conditions for a Dirac-harmonic map (ϕ, ψ) to be trivial. We also give a structure theorem of Dirac-harmonic maps from a Riemann surface generalizing result previously only known in the case when source manifold is a two sphere.

1. Introduction

Dirac-harmonic maps are a generalization and combination of harmonic maps and harmonic spinors while preserving the essential properties of the former. They arise from the supersymmetric nonlinear sigma model of quantum field theory [6].

Obviously, there are two types of basic examples, a harmonic map together with a vanishing spinor and a constant map together with a harmonic spinor. In [4], the authors constructed Dirac-harmonic maps (ϕ, ψ) from S^2 to S^2 , where ϕ is a harmonic map (or equivalently, a (possible branched) conformal map), ψ could be written in the form

$$\psi = \sum_{\alpha} \epsilon_{\alpha} \cdot \Psi \otimes \phi_*(\epsilon_{\alpha}) \quad (1.1)$$

where ϵ_{α} ($\alpha = 1, 2$) is a local orthonormal basis of S^2 , and Ψ is a twistor spinor.

In the spirit of CHEN–JOST–LI–WANG, recently, JOST–MO–ZHU constructed explicit examples of Dirac-harmonic maps (ϕ, ψ) from an Euclidean space to a hyperbolic space which are non-trivial in the sense that ϕ is not harmonic [4],

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[10]. Precisely, in their examples, $\phi : \mathbb{R}^n \rightarrow H^{n+1}$ is an isometric immersion where $n \geq 3$ and ψ could be written in the form

$$\psi^T = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$$

where $(\dots)^T$ denotes the orthogonal projection into the subbundle $\Sigma \mathbb{R}^n \otimes T\mathbb{R}^n$ and ϵ_α ($\alpha = 1, 2$) is an orthonormal basis of $\mathbb{R}^n$. A natural question then is whether there exist non-trivial Dirac-harmonic maps for hypersphere in a hyperbolic space in this form.

In this paper, we will first give the following negative answer.

Theorem 1.1. *Let M^n be a compact positive scalar curved spinor manifold immersed in a non-positively constantly curved manifold N . Then there is no non-vanishing harmonic spinor ψ along this immersion with $\psi^T = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$, therefore, there is no non-trivial Dirac-harmonic map (ϕ, ψ) from M into N with $\psi^T = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$.*

For definitions of harmonic spinor and Dirac-harmonic map see Section 2 and Section 3. In the two-dimensional case we have the following:

Proposition 1.2. *Let $\phi : M \hookrightarrow N$ be a surface in a Riemannian manifold of constant curvature c with flat normal bundle. Then there is no non-trivial Dirac-harmonic map (ϕ, ψ) from M into N with $\psi^T = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$.*

Then we generalize CHEN–JOST–LI–WANG’ construction [4] as following:

Proposition 1.3. *Let M be a Riemann surface and N a Riemannian manifold. Let $\psi_{\phi, \Psi}$ be defined by $\psi_{\phi, \Psi} = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$ from a nonconstant conformal map $\phi : M \rightarrow N$ and a spinor $\Psi \in \Gamma(\Sigma M)$. If $(\phi, \psi_{\phi, \Psi})$ is a Dirac-harmonic map then ϕ is a branched minimal immersion and Ψ is a twistor spinor.*

See Section 2 for definition of twistor spinor. Using Proposition 1.3, we obtain the following structure theorem of Dirac-harmonic maps from Riemann surfaces:

Theorem 1.4. *Let (ϕ, ψ) is a non-constant Dirac-harmonic map from a compact Riemann surface M_g of genus g to the sphere M_0 with $|\deg \phi| > g - 1$. Then ϕ is $\pm$ holomorphic, and ψ could be written in the form*

$$\psi = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$$

where ϵ_α ($\alpha = 1, 2$) is a local orthonormal basis of M_g , and Ψ is a twistor spinor.

It is worth mentioning the condition that $\deg \phi > g - 1$ is sharp. If $g \geq 1$ and $0 \leq d \leq g - 1$, Lemaire has constructed Riemann surface M_g and harmonic non $\pm$ holomorphic maps $\phi : M_g \rightarrow M_0$ of degree d . Thus $(\phi, 0)$ is a Dirac-harmonic map from M_g into M_0 .

2. Dirac-harmonic maps

In this section, we recall the basic definitions and introduce our notation. Let (N, h) be a Riemannian manifold of dimension n' . This will be our target manifold. Likewise, let (M, g) , our domain manifold, be an n -dimensional Riemannian manifold with fixed spin structure. By ΣM , we denote its spinor bundle, on which we have a Hermitian metric $\langle \cdot, \cdot \rangle$ induced by the Riemannian metric $g(\cdot, \cdot)$ of M . Let ϕ be a smooth map from (M, g) to (N, h) and $\phi^{-1}TN$ the pull-back bundle of TN by ϕ . On the twisted bundle $\Sigma M \otimes \phi^{-1}TN$ there is a metric (still denoted by $\langle \cdot, \cdot \rangle$) induced from the metrics on ΣM and $\phi^{-1}TN$. There is also a natural connection $\tilde{\nabla}$ on $\Sigma M \otimes \phi^{-1}TN$ induced from those on ΣM and $\phi^{-1}TN$ (which in turn come from the Levi-Civita connections of (M, g) and (N, h) , resp.).

We have the Clifford product $X \cdot \Phi$ of $X \in \Gamma(TM)$, $\Phi \in \Gamma(\Sigma M)$. This Clifford product satisfies the skew-symmetry relation

$$\langle X \cdot \Phi, \Psi \rangle = -\langle \Phi, X \cdot \Psi \rangle \quad (2.1)$$

as well as the Clifford relations

$$X \cdot Y \cdot \Phi + Y \cdot X \cdot \Phi = -2g(X, Y)\Phi$$

for $X, Y \in \Gamma(TM)$, $\Phi, \Psi \in \Gamma(\Sigma M)$.

We are now prepared to introduce an operator that couples the geometries of M and N via the map ϕ . Let ψ be a section of the bundle $\Sigma M \otimes \phi^{-1}TN$. The *Dirac operator along the map ϕ* is defined as

$$\not{D}\psi := \epsilon_\alpha \cdot \tilde{\nabla}_{\epsilon_\alpha} \psi$$

where ϵ_α is a local orthonormal basis of M . For background material about the spinor bundle and the Dirac operator, we refer to [9], [12].

We consider the space

$$\chi := \{(\phi, \psi) | \phi \in C^\infty(M, N) \text{ and } \psi \in C^\infty(\Sigma M \otimes \phi^{-1}TN)\}$$

of mappings and sections along those mappings. On χ , we have the functional

$$L(\phi, \psi) := \frac{1}{2} \int_M [|d\phi|^2 + \langle \psi, \not{D}\psi \rangle] * 1_M.$$

This functional couples the two fields ϕ and ψ because the operator $\not{D}$ depends on the map ϕ . The Euler-Lagrange equations of $L(\phi, \psi)$ then also couple the two fields; they are:

$$\tau(\phi) = \mathcal{R}(\phi, \psi) \quad (2.2)$$

and

$$\mathcal{D}\psi = 0 \quad (2.3)$$

where $\tau(\phi)$ is the tension field of the map ϕ (the natural version of the Laplace operator for maps between manifolds) and the curvature term $\mathcal{R}(\phi, \psi)$ is defined by

$$\mathcal{R}(\phi, \psi) = \frac{1}{2} R^i_{jkl} \langle \psi^k, \nabla \phi^j \cdot \psi^l \rangle \frac{\partial}{\partial y^i},$$

where

$$\begin{aligned} \psi &= \psi^i \otimes \frac{\partial}{\partial y^i}, & (d\phi)^\# &= \nabla \phi^i \otimes \frac{\partial}{\partial y^i}, \\ R^{\phi^{-1}TN} \left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^l} \right) \frac{\partial}{\partial y^j} &= R^i_{jkl} \frac{\partial}{\partial y^i} \end{aligned}$$

where $\# : T^*M \otimes \phi^{-1}TN \rightarrow TM \otimes \phi^{-1}TN$ is the standard (“musical”) isomorphism obtained from the Riemannian metric g .

Solutions (ϕ, ψ) to (2.2) and (2.3) are called *Dirac-harmonic maps* from M into N [4].

We now start with some differential geometric identities: Let ϵ_α be a local orthonormal basis of M . By using the Clifford relations we have

$$\epsilon_\alpha \cdot \epsilon_\beta \cdot \Psi = (-1)^{\delta_{\alpha\beta}+1} \epsilon_\beta \cdot \epsilon_\alpha \cdot \Psi = \begin{cases} -\Psi, & \alpha = \beta \\ -\epsilon_\beta \cdot \epsilon_\alpha \cdot \Psi, & \alpha \neq \beta \end{cases} \quad (2.4)$$

for $\Psi \in \Gamma(\Sigma M)$.

Lemma 2.1. $\mathcal{R}(\phi, \psi) \in \Gamma(\phi^{-1}TN)$; in particular, it is real.

PROOF. For any (not necessarily orthonormal) frame $\{\epsilon_i\}$ on $\phi^{-1}TN$, we put

$$\psi = \psi^a \otimes \epsilon_a, \quad (2.5)$$

$$(d\phi)^\# = \nabla \phi^a \otimes \epsilon_a, \quad R^{\phi^{-1}TN}(\epsilon_a, \epsilon_b)\epsilon_c = R^d_{abc}\epsilon_d \quad (2.6)$$

where $\# : T^*M \otimes \phi^{-1}TN \rightarrow TM \otimes \phi^{-1}TN$ is the musical isomorphism as before. Take

$$\epsilon_a = u_a^i \frac{\partial}{\partial y^i},$$

then

$$\psi^i = u_a^i \psi^a, \quad \nabla \phi^i = u_a^i \nabla \phi^a, \quad u_a^j u_b^k u_c^l R^i_{jkl} = R^d_{abc} u_d^i.$$

A simple calculation gives following

$$R^i{}_{jkl} \langle \psi^k, \nabla \phi^j \cdot \psi^l \rangle \frac{\partial}{\partial y^i} = R^a{}_{bcd}(\phi(x)) \langle \psi^c, \nabla \phi^b \cdot \psi^d \rangle \epsilon_a(\phi(x)). \quad (2.7)$$

It follows that the definition of $\mathcal{R}(\phi, \psi)$ is independent of the choice of frame. It is then well-defined vector field on $\phi^{-1}TN$. On the other hand, from the skew-symmetry of $R^i{}_{jkl}$ with respect to the induces k and l , we have

$$\begin{aligned} \overline{\frac{1}{2} R^i{}_{jkl} \langle \psi^k, \nabla \phi^j \cdot \psi^l \rangle} &= \frac{1}{2} R^i{}_{jkl} \langle \nabla \phi^j \cdot \psi^l, \psi^k \rangle = \frac{1}{2} R^i{}_{jlk} \langle \nabla \phi^j \cdot \psi^k, \psi^l \rangle \\ &= -\frac{1}{2} R^i{}_{jkl} \langle \nabla \phi^j \cdot \psi^k, \psi^l \rangle = \frac{1}{2} R^i{}_{jkl} \langle \psi^k, \nabla \phi^j \cdot \psi^l \rangle. \end{aligned}$$

It follows that $\mathcal{R}(\phi, \psi) \in \Gamma(\phi^{-1}TN)$. $\square$

A spinor (field) $\Psi \in \Gamma(\Sigma M)$ is called a *twistor spinor* if Ψ belongs to the kernel of the twistor operator, equivalently,

$$\nabla_X \Psi + \frac{1}{n} X \cdot \not\partial \Psi = 0, \quad \forall X \in \Gamma(TM)$$

where we recall that n is the dimension of the Riemannian manifold M , ΣM is the associated spinor bundle of M and $\not\partial$ is the usual Dirac operator (cf. [1], [8], [11]).

In fact the concept of a twistor spinor (in particular, a Killing spinor) is motivated by theories from physics, like general relativity, 11-dimensional (resp. 10-dimensional) supergravity theory, supersymmetry (see, for example [2], [3], [5]).

We establish the following Lemma 2.2 required in the proof of Proposition 1.3.

Lemma 2.2. *Let $\Phi \in \Gamma(\Sigma M)$. Then Φ is a twistor spinor if and only if*

$$\epsilon_1 \cdot \nabla_{\epsilon_1} \Phi = \cdots = \epsilon_n \cdot \nabla_{\epsilon_n} \Phi \quad (2.8)$$

for some orthonormal frame field ϵ_α of M .

PROOF. Let us assume that (2.8) holds for some orthonormal frame field of ϵ_α . Hence we may set

$$\Psi := \epsilon_\alpha \cdot \nabla_{\epsilon_\alpha} \Phi. \quad (2.9)$$

Then

$$\not\partial \Phi = \sum_{\alpha} \epsilon_\alpha \cdot \nabla_{\epsilon_\alpha} \Phi = \sum_{\alpha} \Psi = n \Psi$$

where $n = \dim M$. Together with (2.4) and (2.9) we have

$$\nabla_{\epsilon_\beta} \Phi = -\epsilon_\beta \cdot \epsilon_\beta \cdot \nabla_{\epsilon_\beta} \Phi = -\epsilon_\beta \cdot \Psi = -\epsilon_\beta \cdot \left(\frac{1}{n} \not\partial \Phi \right) = -\frac{1}{n} \epsilon_\beta \cdot \not\partial \Phi.$$

It follows that

$$\begin{aligned}\nabla_X \Phi + \frac{1}{n} X \cdot \not\partial \Phi &= \nabla_{X^\alpha \epsilon_\alpha} \Phi + \frac{1}{n} (X^\alpha \epsilon_\alpha \cdot \not\partial \Phi) = X^\alpha \nabla_{\epsilon_\alpha} \Phi + \frac{1}{n} X^\alpha (\epsilon_\alpha \cdot \not\partial \Phi) \\ &= X^\alpha \left(\nabla_{\epsilon_\alpha} \Phi + \frac{1}{n} \epsilon_\alpha \cdot \not\partial \Phi \right) = 0\end{aligned}$$

for arbitrary $X = X^\alpha \epsilon_\alpha \in \Gamma(TM)$. Thus we see that Φ is a twistor spinor.

Conversely, if Φ is a twistor spinor, then the spinor field

$$X \cdot \nabla_X \Phi$$

does not depend on the unit vector field X [1, page 23, Theorem 2]. $\square$

3. Dirac-harmonic maps along an isometric immersion

In this section, we are going to give some sufficient conditions for a Dirac-harmonic map along an isometric immersion to be trivial.

Let $\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$ be a spinor field along $\phi : M \rightarrow N$. We call ψ to be *harmonic* if $\not{D}\psi = 0$ [4].

Let $\phi : M \hookrightarrow N$ be an isometric immersion. This means that the Riemannian metric on M induced from the ambient space N coincides with the original one on M . We identify M with its immersed image in N . For each $x \in M$ the tangent space $T_x N$ can be decomposed into a direct sum of $T_x M$ and its orthogonal complement $T_x^\perp M$. Such a decomposition is differentiable. Thus, we have an orthogonal decomposition of the tangent bundle TN along M

$$TN|_M = \phi^{-1}TN = TM \oplus T^\perp M.$$

Let $(\dots)^T$ denote the orthogonal projection into the subbundle $\Sigma M \otimes TM$ from the twsited bundle $\Sigma M \otimes \phi^{-1}TN$.

For a global section $\mathcal{R}(\phi, \psi)$ on $\phi^{-1}TN$ (see Lemma 2.1), we have

$$\mathcal{R}(\phi, \psi) = \mathcal{R}^T(\phi, \psi) + \mathcal{R}^N(\phi, \psi)$$

where

$$\mathcal{R}^T(\phi, \psi) \in \Gamma(TM), \quad \mathcal{R}^N(\phi, \psi) \in \Gamma(T^\perp M).$$

Similarly, for $\not{D}\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$, we have

$$\not{D}\psi = \not{D}^T \psi + \not{D}^N \psi$$

where

$$\not{D}^T \psi \in \Gamma(\Sigma M \otimes TM), \quad \not{D}^N \psi \in \Gamma(\Sigma M \otimes T^\perp M).$$

The mean curvature vector of M in N is

$$H = \frac{1}{n} \tau(\phi) \in \Gamma(T^\perp M)$$

where $\tau(\phi)$ is the tension field of the map ϕ . Hence we have the following:

Lemma 3.1. *Let $\phi : M \hookrightarrow N$ be an isometric immersion with the mean curvature vector ξ and $\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$. Then (ϕ, ψ) is a Dirac-harmonic map from M into N if and only if*

- (i) $\mathcal{R}^T(\phi, \psi) = 0$;
- (ii) $\mathcal{R}^N(\phi, \psi) = n\xi$ where $n = \dim M$;
- (iii) $\not{D}^T \psi = 0$;
- (iv) $\not{D}^N \psi = 0$.

We shall be using the following ranges of indices:

$$1 \leq \alpha, \beta, \dots \leq n, \quad n+1 \leq s, t, \dots \leq n', \quad 1 \leq i, j, \dots \leq n'.$$

Choose a local orthonormal frame field $\{\epsilon_i\}$ of $\phi^{-1}TN$ such that $\{\epsilon_\alpha\}$ lies in the tangent bundle TM and $\{\epsilon_s\}$ in the normal bundle $T^\perp M$ of M . We put

$$(d\phi)^\# = \nabla \phi^i \otimes \epsilon_i \quad (3.1)$$

where $\# : T^*M \otimes \phi^{-1}TN \rightarrow TM \otimes \phi^{-1}TN$ is the musical isomorphism. By using (3.1) we have

$$\nabla \phi^i = \sum \delta_\alpha^i \epsilon_\alpha. \quad (3.2)$$

Now we assume that $N = N(c)$ is a Riemannian manifold of constant curvature c . Then the components of the Riemannian curvature tensor of N satisfy

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \quad (3.3)$$

PROOF OF THEOREM 1.1. Let $\psi \in \Gamma(\Sigma M \otimes \phi^{-1}TN)$ be a spinor field along the isometric immersion ϕ and $\psi = \psi^i \epsilon_i$. From (3.2) and (3.3) we obtain

$$\begin{aligned} R_{ijkl} \langle \nabla \phi^k \cdot \psi^i, \nabla \phi^l \cdot \psi^j \rangle &= c [\langle \nabla \phi^i \cdot \psi^i, \nabla \phi^j \cdot \psi^j \rangle - \langle \nabla \phi^j \cdot \psi^i, \nabla \phi^i \cdot \psi^j \rangle] \\ &= c [\langle \epsilon_\alpha \cdot \psi^\alpha, \epsilon_\beta \cdot \psi^\beta \rangle - \langle \epsilon_\beta \cdot \psi^\alpha, \epsilon_\alpha \cdot \psi^\beta \rangle] \\ &= c \sum_{\alpha \neq \beta} [\langle \epsilon_\alpha \cdot \psi^\alpha, \epsilon_\beta \cdot \psi^\beta \rangle - \langle \epsilon_\beta \cdot \psi^\alpha, \epsilon_\alpha \cdot \psi^\beta \rangle]. \end{aligned} \quad (3.4)$$

By using the skew-symmetry relation of the Clifford product and the Clifford relation we have

$$\begin{aligned} \sum_{\alpha \neq \beta} \langle \epsilon_\beta \cdot \psi^\alpha, \epsilon_\alpha \cdot \psi^\beta \rangle &= - \sum_{\alpha \neq \beta} \langle \psi^\alpha, \epsilon_\beta \cdot \epsilon_\alpha \cdot \psi^\beta \rangle = \sum_{\alpha \neq \beta} \langle \psi^\alpha, \epsilon_\alpha \cdot \epsilon_\beta \cdot \psi^\beta \rangle \\ &= - \sum_{\alpha \neq \beta} \langle \epsilon_\alpha \cdot \psi^\alpha, \epsilon_\beta \cdot \psi^\beta \rangle. \end{aligned}$$

Plugging this into (3.4) yields

$$R_{ijkl} \langle \nabla \phi^k \cdot \psi^i, \nabla \phi^l \cdot \psi^j \rangle = 2c \sum_{\alpha \neq \beta} \langle \epsilon_\alpha \cdot \psi^\alpha, \epsilon_\beta \cdot \psi^\beta \rangle.$$

Now we assume

$$\psi^T = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha).$$

where $\Psi \in \Gamma(\Sigma M)$ is a spinor (field). It follows that $\psi^\alpha = \epsilon_\alpha \cdot \Psi$, and therefore

$$\begin{aligned} R_{ijkl} \langle \nabla \phi^k \cdot \psi^i, \nabla \phi^l \cdot \psi^j \rangle &= 2c \sum_{\alpha \neq \beta} \langle \epsilon_\alpha \cdot \epsilon_\alpha \cdot \Psi, \epsilon_\beta \cdot \epsilon_\beta \cdot \Psi \rangle \\ &= 2c \sum_{\alpha \neq \beta} \langle \Psi, \Psi \rangle = 2(n-1)nc|\Psi|^2. \end{aligned} \quad (3.5)$$

Assume that ψ is a harmonic spinor field along the isometric immersion ϕ . From Proposition 3.4 in [4], we have the following Bochner-type formula

$$\frac{1}{2} \Delta |\psi|^2 = |\tilde{\nabla} \psi|^2 + \frac{1}{4} R |\psi|^2 - \frac{1}{2} R_{ijkl} \langle \nabla \phi^k \cdot \psi^i, \nabla \phi^l \cdot \psi^j \rangle \quad (3.6)$$

where R is the scalar curvature of M . Substituting (3.5) into (3.6) yields

$$\frac{1}{2} \Delta |\psi|^2 = |\tilde{\nabla} \psi|^2 + \frac{1}{4} R |\psi|^2 - 2(n-1)nc|\Psi|^2. \quad (3.7)$$

Therefore, under the assumption $R > 0$ and $c \leq 0$, (3.7) shows that $|\psi|^2$ is subharmonic on M . By the Hopf maximum principle, we see that this function must be a constant and the right hand side of (3.7) must be zero. In particular $|\psi| = 0$. $\square$

PROOF OF PROPOSITION 1.2. Plugging (3.2) into (2.7) yields

$$\mathcal{R}(\phi, \psi) = \frac{1}{2} R^i{}_{\alpha kl}(x) \langle \psi^k, \epsilon_\alpha \cdot \psi^l \rangle \epsilon_i(x). \quad (3.8)$$

From which together with (3.3) we obtain

$$\begin{aligned}\mathcal{R}(\phi, \psi) &= c(\delta^i_k \delta_{\alpha l} - \delta^i_l \delta_{\alpha k}) \operatorname{Re}\langle \psi^k, \epsilon_\alpha \cdot \psi^l \rangle \epsilon_i \\ &= c [\operatorname{Re}\langle \psi^i, \epsilon_\alpha \cdot \psi^\alpha \rangle - \operatorname{Re}\langle \psi^\alpha, \epsilon_\alpha \cdot \psi^i \rangle] \epsilon_i = 2c \operatorname{Re}\langle \psi^i, \epsilon_\alpha \cdot \psi^\alpha \rangle \epsilon_i.\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{R}^N(\phi, \psi) &= 2c \operatorname{Re}\langle \psi^s, \epsilon_\alpha \cdot \psi^\alpha \rangle \epsilon_s = 2c \operatorname{Re}\langle \psi^s, \epsilon_\alpha \cdot \epsilon_\alpha \cdot \Psi \rangle \epsilon_s \\ &= -2nc \operatorname{Re}\langle \psi^s, \Psi \rangle \epsilon_s.\end{aligned}$$

Together with (ii) of Lemma 3.1, we obtain

$$-2c \operatorname{Re}\langle \psi^{n+1}, \Psi \rangle = \xi \quad (3.9)$$

where ξ is the mean curvature of ϕ . Choose a local orthonormal frame field $\{\epsilon_\alpha\}$ near $x \in M$ with $\nabla_{\epsilon_\alpha} \epsilon_\beta|_x = 0$. By (2.5) we have

$$\begin{aligned}\mathcal{D}\psi &= \mathcal{D}(\psi^i \otimes \epsilon_i) = \epsilon_\alpha \cdot \tilde{\nabla}_{\epsilon_\alpha}(\psi^i \otimes \epsilon_i) \\ &= \epsilon_\alpha \cdot [(\nabla_{\epsilon_\alpha} \psi^i) \otimes \epsilon_i + \psi^i \otimes \nabla_{\epsilon_\alpha} \epsilon_i] \\ &= (\epsilon_\alpha \cdot \nabla_{\epsilon_\alpha} \psi^i) \otimes \epsilon_i + \epsilon_\alpha \cdot [\psi^i \otimes \nabla_{\epsilon_\alpha} \epsilon_i] \\ &= \mathcal{D}\psi^i \otimes \epsilon_i + \epsilon_\alpha \cdot \psi^s \otimes \nabla_{\epsilon_\alpha} \epsilon_s\end{aligned} \quad (3.10)$$

at x .

Let A_ν be the shape operator and $\nabla_X^\perp$ the normal connection of M in N where X denotes a tangent vector of M and ν a normal vector to M . Then

$$\nabla_{\epsilon_\alpha} \epsilon_s = -A_{\epsilon_s} \epsilon_\alpha + \nabla_{\epsilon_\alpha}^\perp \epsilon_s. \quad (3.11)$$

Let B be the second fundamental form of M in N . Then B satisfies the Weingarten equation

$$\langle B(X, Y), \nu \rangle = \langle A_\nu(X), Y \rangle \quad (3.12)$$

where $X, Y \in \Gamma(TM)$. By using (3.11) and (3.12) we have

$$\nabla_{\epsilon_\alpha} \epsilon_s = -\langle B(\epsilon_\alpha, \epsilon_\beta), \epsilon_s \rangle \epsilon_\beta + \nabla_{\epsilon_\alpha}^\perp \epsilon_s. \quad (3.13)$$

By plugging (3.13) into (3.10) we obtain

$$\mathcal{D}\psi = \mathcal{D}\psi^i \otimes \epsilon_i - \langle B(\epsilon_\alpha, \epsilon_\beta), \epsilon_s \rangle \epsilon_\alpha \cdot \psi^s \otimes \epsilon_\beta + \epsilon_\alpha \cdot \psi^s \otimes \nabla_{\epsilon_\alpha}^\perp \epsilon_s. \quad (3.14)$$

Let ψ^T be defined by

$$\psi^T = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha).$$

Choose a local orthonormal frame field $\{\epsilon_\alpha\}$ near $x \in M$ with $\nabla_{\epsilon_\alpha} \epsilon_\beta|_x = 0$.

$$\begin{aligned} \not\partial \psi^\alpha &= \not\partial(\epsilon_\alpha \cdot \Psi) = \epsilon_\beta \cdot \nabla_{\epsilon_\beta}(\epsilon_\alpha \cdot \Psi) = \epsilon_\beta [(\nabla_{\epsilon_\beta} \epsilon_\alpha) \cdot \Psi + \epsilon_\alpha \cdot \nabla_{\epsilon_\beta} \Psi] \\ &= \epsilon_\beta \cdot \epsilon_\alpha \cdot \nabla_{\epsilon_\beta} \Psi = -\nabla_{\epsilon_\alpha} \Psi - \sum_{\beta \neq \alpha} \epsilon_\alpha \cdot \epsilon_\beta \cdot \nabla_{\epsilon_\beta} \Psi = -2\nabla_{\epsilon_\alpha} \Psi - \epsilon_\alpha \cdot \not\partial \Psi. \end{aligned} \quad (3.15)$$

Substituting (3.15) into (3.14) and taking the tangent projection yield

$$\not{D}^T \psi = -[2\nabla_{\epsilon_\beta} \Psi + \epsilon_\beta \cdot \not\partial \Psi + \langle B(\epsilon_\alpha, \epsilon_\beta), \epsilon_s \rangle \epsilon_\alpha \cdot \psi^s] \otimes \epsilon_\beta. \quad (3.16)$$

It is easy to see that

$$\langle B(\epsilon_\alpha, \epsilon_\beta), \epsilon_s \rangle \epsilon_\alpha \cdot \psi^s \otimes \epsilon_\beta$$

does not depend on the choice of $\{\epsilon_\alpha\}$. Since the normal bundle of M is flat, we choose $\{\epsilon_\alpha\}$ such that

$$\langle B(\epsilon_\alpha, \epsilon_\beta), \epsilon_s \rangle = \lambda_\alpha^s \delta_{\alpha\beta}. \quad (3.17)$$

Therefore we have

$$\sum_\beta \lambda_\beta^{n+1} = n\xi, \text{ and } \sum_\beta \lambda_\beta^s = 0 \text{ for } s \neq n+1. \quad (3.18)$$

Plugging (3.17) into (3.16) yields

$$\not{D}^T \psi = -\sum_\beta \left[2\nabla_{\epsilon_\beta} \Psi + \epsilon_\beta \cdot \not\partial \Psi + \sum_{\alpha, A} \lambda_\beta^A \epsilon_\beta \cdot \psi^A \right] \otimes \epsilon_\beta. \quad (3.19)$$

Thus $\not{D}^T \psi = 0$ if and only if

$$2\nabla_{\epsilon_\beta} \Psi + \epsilon_\beta \cdot \not\partial \Psi + \sum_A \lambda_\beta^A \epsilon_\beta \cdot \psi^A = 0 \quad (3.20)$$

for all β . By (2.4), (3.20) holds if and only if

$$2\epsilon_\beta \cdot \nabla_{\epsilon_\beta} \Psi - \not\partial \Psi = \sum_s \lambda_\beta^s \psi^s. \quad (3.21)$$

Summing on β and using (3.18) we have

$$(2-n)\not\partial \Psi = n\xi \psi^{n+1}.$$

Note that $n = \dim M = 2$. It follows that

$$\xi \psi^{n+1} = 0. \quad (3.22)$$

Suppose that $\xi(x) \neq 0$ for some $x \in M$, then (3.22) implies that $\psi^{n+1}(x) = 0$. Plugging this into (3.9) yields $\xi(x) = 0$ which is a contradiction and therefore $\xi \equiv 0$. $\square$

Corollary 3.2. *Let $\phi : M \hookrightarrow N$ be an $n(\geq 3)$ -dimensional submanifold in a Riemannian manifold of constant curvature c with flat normal bundle and let (ϕ, ψ) be a Dirac-harmonic map where*

$$\psi^T = \sum_{\alpha} \epsilon_{\alpha} \cdot \Psi \otimes \phi_*(\epsilon_{\alpha})$$

for some $\Psi \in \Gamma(\Sigma M)$. Then ϕ is minimal if and only if Ψ is harmonic.

4. Dirac-harmonic maps from a Riemann surface

In this section, we extend Chen–Jost–Li–Wang’ result and give a structure theorem of Dirac-harmonic maps from a Riemann surface.

PROOF OF PROPOSITION 1.3. We claim that

$$\mathcal{R}(\phi, \psi_{\phi, \Psi}) \equiv 0, \quad \mathcal{D}\psi_{\phi, \Psi} = -\Psi \otimes \tau(\phi) - 2 \left(\nabla_{\epsilon_{\alpha}} \Psi + \frac{1}{2} \epsilon_{\alpha} \cdot \not\partial \Psi \right) \otimes \phi_*(\epsilon_{\alpha}) \quad (4.1)$$

where ϵ_{α} ($\alpha = 1, 2$), as always, is a local orthonormal basis of M .

In fact, we define local vector fields $\nabla \phi^i$ on M by

$$\nabla \phi^i := (d\phi)^{\sharp}(dy^i)$$

where $\{dy^i\}$ is the natural local dual basis on N . By using (1.1), we have

$$\psi^i := \psi_{\phi, \Psi}(dy^i) = \nabla \phi^i \cdot \Psi.$$

Set $d\phi = \phi_{\alpha}^i \theta^{\alpha} \otimes \frac{\partial}{\partial y^i}$ where θ^{α} is the dual basis for ϵ_{α} . Then $\nabla \phi^i = \sum \phi_{\alpha}^i \epsilon_{\alpha}$ and

$$\langle \psi^k, \nabla \phi^j \cdot \psi^l \rangle = \phi_{\alpha}^k \phi_{\beta}^j \phi_{\gamma}^l \langle \epsilon_{\alpha} \cdot \Psi, \epsilon_{\beta} \cdot \epsilon_{\gamma} \cdot \Psi \rangle.$$

Note that $\text{Re} \langle \epsilon_{\alpha} \cdot \Psi, \epsilon_{\beta} \cdot \epsilon_{\gamma} \cdot \Psi \rangle = 0$ [10, Lemma 3.1]. We conclude that $R^i_{jkl} \langle \psi^k, \nabla \phi^j \cdot \psi^l \rangle$ is purely imaginary. On the other hand, from the proof of Lemma 2.1, $R^i_{jkl} \langle \psi^k, \nabla \phi^j \cdot \psi^l \rangle$ must be real, and hence

$$\mathcal{R}(\phi, \psi_{\phi, \Psi}) \equiv \frac{1}{2} R^i_{jkl} \langle \psi^k, \nabla \phi^j \cdot \psi^l \rangle \frac{\partial}{\partial y^i} \equiv 0.$$

By using (2.4) we have

$$\nabla_{\epsilon_{\alpha}} \Psi + \frac{1}{2} \epsilon_{\alpha} \cdot \not\partial \Psi = \nabla_{\epsilon_{\alpha}} \Psi + \frac{1}{2} \epsilon_{\alpha} \cdot [\Sigma \epsilon_{\beta} \cdot \nabla_{\epsilon_{\beta}} \Psi]$$

$$= \begin{cases} \frac{1}{2}(\nabla_{\epsilon_1} \Psi + \epsilon_1 \cdot \epsilon_2 \cdot \nabla_{\epsilon_2} \Psi), & \alpha = 1 \\ \frac{1}{2}(\nabla_{\epsilon_2} \Psi - \epsilon_1 \cdot \epsilon_2 \cdot \nabla_{\epsilon_1} \Psi), & \alpha = 2 \end{cases} \quad (4.2)$$

We choose a local orthonormal frame field ϵ_α such that $\nabla_{\epsilon_\alpha} \epsilon_\beta = 0$ at $x \in M$. Then

$$\begin{aligned} \mathcal{D}\psi_{\phi, \Psi} &= \epsilon_\beta \cdot \tilde{\nabla}_{\epsilon_\beta} \psi_{\phi, \Psi} = \epsilon_\beta \cdot \tilde{\nabla}_{\epsilon_\beta} (\epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)) \\ &= \epsilon_\beta \cdot [\nabla_{\epsilon_\beta} (\epsilon_\alpha \cdot \Psi) \otimes \phi_*(\epsilon_\alpha) + \epsilon_\alpha \cdot \Psi \otimes \nabla_{\epsilon_\beta} (\phi_*(\epsilon_\alpha))] \\ &= \epsilon_\beta \cdot [((\nabla_{\epsilon_\beta} (\epsilon_\alpha) \cdot \Psi + \epsilon_\alpha \cdot \nabla_{\epsilon_\beta} \Psi) \otimes \phi_*(\epsilon_\alpha) + \epsilon_\alpha \cdot \Psi \otimes \nabla_{\epsilon_\beta} (\phi_*(\epsilon_\alpha)))] \\ &= \epsilon_\beta \cdot \epsilon_\alpha \cdot \{\nabla_{\epsilon_\beta} \Psi \otimes \phi_*(\epsilon_\alpha) + \Psi \otimes \nabla_{\epsilon_\beta} (\phi_*(\epsilon_\alpha))\} \\ &= (\sum_{\alpha=\beta} + \sum_{\alpha \neq \beta}) \epsilon_\beta \cdot \epsilon_\alpha \cdot \{\nabla_{\epsilon_\beta} \Psi \otimes \phi_*(\epsilon_\alpha) + \Psi \otimes \nabla_{\epsilon_\beta} (\phi_*(\epsilon_\alpha))\} \\ &= (I) + (II) \end{aligned} \quad (4.3)$$

where

$$\begin{aligned} (I) &= \epsilon_\alpha \cdot \epsilon_\alpha \cdot \{\nabla_{\epsilon_\alpha} \Psi \otimes \phi_*(\epsilon_\alpha) + \Psi \otimes \nabla_{\epsilon_\alpha} (\phi_*(\epsilon_\alpha))\} \\ &= -\{\nabla_{\epsilon_\alpha} \Psi \otimes \phi_*(\epsilon_\alpha) + \Psi \otimes [\nabla_{\epsilon_\alpha} (\phi_*(\epsilon_\alpha)) - \phi_*(\nabla_{\epsilon_\alpha} (\phi_*(\epsilon_\alpha))]\} \\ &= -\{\nabla_{\epsilon_\alpha} \Psi \otimes \phi_*(\epsilon_\alpha) + \Psi \otimes \tau(\phi)\} \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} (II) &= \epsilon_1 \cdot \epsilon_2 \cdot \{\nabla_{\epsilon_1} \Psi \otimes \phi_*(\epsilon_2) + \Psi \otimes \nabla_{\epsilon_1} (\phi_*(\epsilon_2))\} \\ &\quad + \epsilon_2 \cdot \epsilon_1 \cdot \{\nabla_{\epsilon_2} \Psi \otimes \phi_*(\epsilon_1) + \Psi \otimes \nabla_{\epsilon_2} (\phi_*(\epsilon_1))\} \\ &= \epsilon_1 \cdot \epsilon_2 \cdot \{\nabla_{\epsilon_1} \Psi \otimes \phi_*(\epsilon_2) - \nabla_{\epsilon_2} \Psi \otimes \phi_*(\epsilon_1) + \Psi \otimes \nabla_{\epsilon_1} (\phi_*(\epsilon_2)) \\ &\quad - \Psi \otimes \nabla_{\epsilon_2} (\phi_*(\epsilon_1))\} \\ &= \epsilon_1 \cdot \epsilon_2 \cdot \{\nabla_{\epsilon_1} \Psi \otimes \phi_*(\epsilon_2) - \nabla_{\epsilon_2} \Psi \otimes \phi_*(\epsilon_1)\} \end{aligned} \quad (4.5)$$

here we have used the following

$$\nabla_{\epsilon_1} (\phi_*(\epsilon_2)) = (\nabla_{\epsilon_1} \phi_*)(\epsilon_2) = (\nabla_{\epsilon_2} \phi_*)(\epsilon_1) = \nabla_{\epsilon_2} (\phi_*(\epsilon_1)).$$

Substituting (4.4) and (4.5) into (4.3) yields

$$\begin{aligned} \mathcal{D}\psi_{\phi, \Psi} &= -\{\nabla_{\epsilon_\alpha} \Psi \otimes \phi_*(\epsilon_\alpha) + \Psi \otimes \tau(\phi)\} \\ &\quad + \epsilon_1 \cdot \epsilon_2 \cdot \{\nabla_{\epsilon_1} \Psi \otimes \phi_*(\epsilon_2) - \nabla_{\epsilon_2} \Psi \otimes \phi_*(\epsilon_1)\} \\ &= -\Psi \otimes \tau(\phi) - (\nabla_{\epsilon_1} \Psi + \epsilon_1 \cdot \epsilon_2 \cdot \nabla_{\epsilon_2} \Psi) \otimes \phi_*(\epsilon_1) \\ &\quad + (\epsilon_1 \cdot \epsilon_2 \cdot \nabla_{\epsilon_1} \Psi - \nabla_{\epsilon_2} \Psi) \otimes \phi_*(\epsilon_2). \end{aligned} \quad (4.6)$$

Plugging (4.2) into (4.6) yields the second equation of (4.1).

By using the Clifford relation, ones obtain

$$\nabla_{\epsilon_1} \Psi + \frac{1}{2} \epsilon_1 \cdot \not\partial \Psi = \frac{1}{2} (\nabla_{\epsilon_1} \Psi + \epsilon_1 \cdot \epsilon_2 \cdot \nabla_{\epsilon_2} \Psi) = \frac{1}{2} \epsilon_1 \cdot \Phi \quad (4.7)$$

where

$$\Phi := -\epsilon_1 \cdot \nabla_{\epsilon_1} \Psi + \epsilon_2 \cdot \nabla_{\epsilon_2} \Psi.$$

We recall that $\Phi = 0$ if and only if Ψ is a twistor spinor, equivalently, Ψ belongs to the kernel of the twistor operator (cf. Lemma 2.2). Similarly, we have

$$\nabla_{\epsilon_2} \Psi + \frac{1}{2} \epsilon_2 \cdot \not\partial \Psi = \frac{1}{2} (\nabla_{\epsilon_2} \Psi + \epsilon_2 \cdot \epsilon_1 \cdot \nabla_{\epsilon_1} \Psi) = -\frac{1}{2} \epsilon_2 \cdot \Phi. \quad (4.8)$$

Plugging (4.7) and (4.8) into (4.1) yields

$$\not\partial \psi_{\phi, \Psi} = -\Psi \otimes \tau(\phi) + \frac{1}{2} \epsilon_1 \cdot \Phi \otimes \phi_*(\epsilon_1) - \frac{1}{2} \epsilon_2 \cdot \Phi \otimes \phi_*(\epsilon_2). \quad (4.9)$$

Note that $(\phi, \psi_{\phi, \Psi})$ is a Dirac-harmonic map, i.e.

$$\tau(\phi) = \mathcal{R}(\phi, \psi_{\phi, \Psi}), \quad (4.10)$$

$$\not\partial \psi_{\phi, \Psi} = 0. \quad (4.11)$$

(4.1) and (4.10) imply that

$$\tau(\phi) = 0. \quad (4.12)$$

Hence ϕ is a harmonic map, equivalently, it is a branched minimal immersion. Substituting (4.12) into (4.9) and using (4.11) yield

$$\epsilon_1 \cdot \Phi \otimes \phi_*(\epsilon_1) - \epsilon_2 \cdot \Phi \otimes \phi_*(\epsilon_2) = 0. \quad (4.13)$$

Since $\phi : (M, g) \rightarrow (N, h)$ is conformal, we can assume that $\phi^* h = e^\lambda g$. It follows that

$$h(\phi_*(\epsilon_\alpha), \phi_*(\epsilon_\beta)) = \delta_{\alpha\beta} e^\lambda. \quad (4.14)$$

Note that ϕ is non-constant, there exists an α such that $\phi_*(\epsilon_\alpha) \neq 0$. Without loss of generality, we assume $\phi_*(\epsilon_1) \neq 0$. From (4.13) and (4.14) we have $\epsilon_1 \cdot \Phi = 0$. It follows that

$$\Phi = -\epsilon_1 \cdot (\epsilon_1 \cdot \Phi) = 0.$$

Thus Ψ is a twistor spinor. $\square$

Remark. Note that the Dirac-harmonicity of $(\phi, \psi_{\phi, \Psi})$ implies the harmonicity of ϕ and any harmonic map from a sphere is conformal. Hence Proposition 1.3 is a natural generalization of Proposition 2.2 of [4].

PROOF OF THEOREM 1.4. We only consider the case that $g > 0$ and $\deg \phi > g - 1$ where g is the genus of compact Riemann surface M . Let (ϕ, ψ) is a Dirac-harmonic map from a compact Riemann surface M_g of genus g to the sphere M_0 and ϕ is non-constant. By using Theorem 1.1 in [15], ϕ is a harmonic map. Note that M_0 is homeomorphic to $S^2 = \mathbf{CP}^1$ and ϕ is a non-constant map. Hence ϕ is linearly full into $\mathbf{CP}^1$. By using Liao's result ϕ is isotropic [13, Corollary 1]. Recall that isotropic harmonic maps are generated from holomorphic maps by a process of taking derivatives. Therefore ϕ is $\pm$ holomorphic for $n = 1$. Consider the Fubini-Study metric on $\mathbf{CP}^1$ with the constant holomorphic sectional curvature 4. The degree of ϕ can be computed as follows [7], [14]

$$\deg(\phi) = \frac{1}{\pi} [E'(\phi) - E''(\phi)]$$

where $E'(\phi)$ (resp. $E''(\phi)$) is the holomorphic (resp. anti-holomorphic) energy of ϕ . If ϕ is anti-holomorphic, then $E'(\phi) = 0$. It follows that

$$0 \leq g - 1 < \deg(\phi) = -\frac{E''(\phi)}{\pi}$$

Thus $E''(\phi) \leq 0$. Hence ϕ is also holomorphic. We conclude that ϕ is constant which is a contradiction.

The twisted bundle $\Sigma M_g \otimes \phi^{-1}TM_0$ can be divided into the following

$$\begin{aligned} \Sigma M_g \otimes \phi^{-1}TM_0 &= \Sigma M_g \otimes (\phi^{-1}TM_0)^{\mathbb{C}} \\ &= (\Sigma^+ M_g \otimes \phi^{-1}T'M_0) \oplus (\Sigma^+ M_g \otimes \phi^{-1}T''M_0) \\ &\quad \oplus (\Sigma^- M_g \otimes \phi^{-1}T'M_0) \oplus (\Sigma^- M_g \otimes \phi^{-1}T''M_0) \end{aligned} \quad (4.15)$$

where

$$\Sigma^{\pm} M_g := \{ \Psi \in \Sigma M_g \mid \sqrt{-1} \epsilon_1 \cdot \epsilon_2 \cdot \Psi = \pm \Psi \} \quad (4.16)$$

for some orthonormal frame field ϵ_{α} of M_g and $T'M_0$ (resp. $T''M_0$) denote the tangent bundle of M_0 of type $(1, 0)$ (resp. $(0, 1)$). Denote by π^+ (resp. π^-) the projection of the twisted bundle $\Sigma M_g \otimes \phi^{-1}TM_0$ onto the subbundle $\Sigma^+ M_g \otimes \phi^{-1}T''M_0$ (resp. $\Sigma^- M_g \otimes \phi^{-1}T'M_0$). Let m denote the sum of the multiplicities of the zeros of the function $|\pi^+(\psi)|$. If $|\pi^+(\psi)|$ is not identically zero, then (cf. [15, Theorem 4.2])

$$m = g - 1 - 2 \deg(\phi).$$

Note that

$$g - 1 - 2 \deg(\phi) < g - 1 - 2(g - 1) = 1 - g \leq 0.$$

It follows that $m \leq -1$ which is a contradiction, therefore

$$|\pi^+(\psi)| \equiv 0. \quad (4.17)$$

Similarly we have

$$|\pi^-(\psi)| \equiv 0. \quad (4.18)$$

For non-constant holomorphic map ϕ

$$\phi^{-1}T'M_0 = \text{Span}\{\phi_*(\epsilon_1) - \sqrt{-1}\phi_*(\epsilon_2)\},$$

$$\phi^{-1}T''M_0 = \text{Span}\{\phi_*(\epsilon_1) + \sqrt{-1}\phi_*(\epsilon_2)\}.$$

We write

$$\Sigma^\pm M_g = \text{Span}\{\psi^\pm\}$$

where

$$\psi^- = \epsilon_1 \cdot \psi^+. \quad (4.19)$$

By using (4.15), (4.17) and (4.18), we have

$$\psi = f\psi^+ \otimes [\phi_*(\epsilon_1) - \sqrt{-1}\phi_*(\epsilon_2)] + g\psi^- \otimes [\phi_*(\epsilon_1) + \sqrt{-1}\phi_*(\epsilon_2)]. \quad (4.20)$$

From (4.16), (4.19) and the Clifford relation ones obtain

$$\psi^+ = -\epsilon_1 \cdot \psi^- = -\sqrt{-1}\epsilon_2 \cdot \psi^-, \quad (4.21)$$

$$\psi^- = \epsilon_1 \cdot \psi^+ = -\sqrt{-1}\epsilon_2 \cdot \psi^+. \quad (4.22)$$

Plugging (4.21) and (4.22) into (4.20) yields

$$\psi = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$$

where $\Psi = g\psi^+ - f\psi^-$. Note that arbitrary isotropic harmonic map is conformal. From Proposition 1.3, Ψ is a twistor spinor. $\square$

In particular, we have the following

Corollary 4.1. *Let (ϕ, ψ) is a non-constant Dirac-harmonic map from a torus T^2 to a sphere with non-zero degree. Then ϕ is $\pm$ holomorphic, and ψ could be written in the form*

$$\psi = \Sigma_\alpha \epsilon_\alpha \cdot \Psi \otimes \phi_*(\epsilon_\alpha)$$

where ϵ_α ($\alpha = 1, 2$) is a local orthonormal basis of T^2 , and Ψ is a twistor spinor.

We have several special case of Theorem 1.4.

- (1) When $g = 0$, our corollary have been given by YANG LING [15];
- (2) When $\psi = 0$, our result is reduced to Liao's isotropy work [13, Corollary 1].

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