Publ. Math. Debrecen
76/1-2 (2010), 227–250

On the resolution of equations $Ax^n - By^n = C$ in integers x, y and $n \geq 3$, II

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To Professors A. Pethő and J. Pintz on their 60th birthdays

Abstract. In Part I (cf. [13]) of this paper, the title equation was solved in $x, y, n \in \mathbb{Z}$ with $|xy| > 1$, $n \geq 3$ for a collection of positive integers A, B, C under certain bounds. In the present paper we extend these results to much larger ranges of A, B, C. We give among other things all the solutions for $A = C = 1$, $B < 235$ (cf. Theorem 1), and for $C = 1$, $A, B \leq 50$, with six explicitly given exceptions (A, B, n) (cf. Theorem 3). The equations under consideration are solved by combining powerful techniques, including Frey curves and associated modular forms, lower bounds for linear forms in logarithms, the hypergeometric method of Thue and Siegel, local methods, classical cyclotomy and computational approaches to Thue equations of low degree. Along the way, we derive a new result on the solvability of binomial Thue equations (cf. Theorem 6) which is crucial in the proof of our Theorems 1 and 2. Some important applications of our theorems will be given in a forthcoming paper.

Mathematics Subject Classification: Primary: 11D45, 11D61; Secondary: 11D41, 11J82, 11J86.
Key words and phrases: Diophantine equations, binomial Thue equations, exponential equations, resolution of equations.
Research was supported in part by the Hungarian Academy of Sciences, by the OTKA grants T67580 and K75566 and by János Bolyai Fellowship.

1. Introduction

Many problems in number theory can be reduced to Diophantine equations of the form

$$|Ax^n - By^n| = C, \tag{1}$$

where x, y, n are unknown positive integers with $|xy| > 1$, $\gcd(x, y) = 1$ and $n \geq 3$, and A, B and C are positive integers with

$$1 \leq A < B \quad \text{and} \quad \gcd(A, B) = 1, \tag{2}$$

such that A, B, C are bounded or are only divisible by finitely many fixed primes. Here $|x|$, $|y|$, n and, in the second case, even A, B, C can be effectively bounded; see [12] and [14]. This bound is, however, too large for the practical use, to determine the solutions.

Before the appearance of Part I (cf. [13]) of the present paper equation (1) was solved for some values of A, B, C with $C = 1$, including the cases when $B = A + 1$ (cf. [1]) or $AB = p^\alpha q^\beta$ with at most two distinct prime factors $p, q < 30$ (cf. [3], [6], [9], [15]). For further references concerning earlier special results and applications see [13].

In [13], Győry and Pintér studied the solutions of equation (1) for bounded positive integer coefficients A, B and C. They first derived, for concrete values of $A, B, C \leq 100$, a relatively small upper bound for n, provided that (1) has no solutions with $|xy| \leq 1$ (cf. Lemma 2). Then they explicitly solved equation (1) in integers x, y and n with $|xy| > 1$, $n \geq 3$ for a collection of coefficients A, B, C. Under the assumptions (2) and $\max\{A, B, C\} \leq 10$ they gave all integer solutions (x, y, n) with $|xy| > 1$, $n \geq 3$ and with

$$B \pm A \neq C \quad \text{if } C \geq 2. \tag{3}$$

For $C = 1$, assuming (2) and $\max\{A, B\} \leq 20$, they determined all solutions to the equation

$$|Ax^n - By^n| = 1 \tag{4}$$

in integers x, y, n with $|xy| > 1$ and $n \geq 3$. Finally, in the case $A = C = 1$, $B \leq 70$, they gave all solutions to the equation

$$|x^n - By^n| = 1 \tag{5}$$

in integers x, y, n with $|xy| > 1$ and $n \geq 3$. We note that these statements of [13] cannot be deduced from the results of [1], [3], [6], [9] and [15].

The purpose of the present paper is to extend the above-mentioned results of [13] for much larger values of A, B, C (cf. Theorems 1 to 5). As in Part I ([13]), our proofs require directly or indirectly almost all techniques in modern Diophantine analysis, including local methods, some classical results in cyclotomic fields, lower bounds for linear forms in logarithms of algebraic numbers, the hypergeometric method, computational methods for finding the solutions to Thue equations of small degree and results on ternary equations based on Galois representations and modular forms. The main novelty in our proofs is a new result of ours (Theorem 6) concerning the solvability of binomial Thue equations of the form (5). The use of our Theorem 6 is crucial in proving Theorems 1 and 2. It is important to note that in our Theorems 1 to 5 we arrived at the limit of the applicability of the currently available methods.

2. Results

For equation (5) we prove the following results.

Theorem 1. *If $1 < B \leq 400$, then all integer solutions (x, y, n) of equation (5) with $|xy| > 1, n \geq 3$ and with $(B, n) \notin \{(235, 23), (282, 23), (295, 29), (329, 23), (354, 29)\}$ are given by*

$$
\begin{aligned}
&n = 3,\ (B, x, y) = (7, \pm(2, 1)), (9, \pm(2, 1)), (17, \pm(18, 7)), (19, \pm(8, 3)),\\
&\quad (20, \pm(19, 7)), (26, \pm(3, 1)), (63, \pm(4, 1)), (91, \pm(9, 2)), (124, \pm(5, 1)),\\
&\quad (126, \pm(5, 1)), (182, \pm(17, 3)), (215, \pm(6, 1)), (217, \pm(6, 1)),\\
&\quad (254, \pm(19, 3)), (342, \pm(7, 1)), (344, \pm(7, 1)),\\
&n = 4,\ (B, x, y) = (5, \pm 3, \pm 2), (15, \pm 2, \pm 1), (17, \pm 2, \pm 1), (39, \pm 5, \pm 2),\\
&\quad (80, \pm 3, \pm 1), (150, \pm 7, \pm 2), (255, \pm 4, \pm 1),\\
&n = 5,\ (B, x, y) = (31, \pm(2, 1)), (242, \pm(3, 1)), (244, \pm(3, 1)),\\
&n = 6,\ (B, x, y) = (63, \pm 2, \pm 1),\\
&n = 7,\ (B, x, y) = (127, \pm(2, 1)), (129, \pm(2, 1)),\\
&n = 8,\ (B, x, y) = (255, \pm 2, \pm 1).
\end{aligned}
$$

This is a considerable extension of Theorem 4 of [13]. In the proofs of our Theorems 1 and 2 the method of modular forms and Theorem 6 play a very important role. In Theorem 1, and in Theorems 2 to 4 below, there are some exceptions (B, n) resp. (A, B, n) for which our methods do not work. This is

partly due to the fact that the necessary data concerning the arising modular forms of too high level are not at our disposal.

In the next theorem we restrict ourselves to the case when B is odd. Then 2 must divide xy which considerably extends the applicability of our method of proof.

Theorem 2. (i) *If $400 < B < 800$ is odd, then all integer solutions (x, y, n) of equation (5) with $|xy| > 1$, $n \geq 3$ and with the possible exceptions (B, n) listed in Table 1 below are given by*

$$n = 3,\ (B, x, y) = (511, \pm(8, 1)), (513, \pm(8, 1)), (635, \pm(361, 42)), (651, \pm(26, 3)),$$
$$n = 9,\ (B, x, y) = (511, \pm(2, 1)), (513, \pm(2, 1)).$$

Table 1

(B, n)	(B, n)	(B, n)	(B, n)	(B, n)
$(413, 29)$	$(519, 43)$	$(649, 29)$	$(695, 23)$	$(757, 379)$
$(415, 41)$	$(535, 53)$	$(669, 37)$	$(699, 29)$	$(767, 29)$
$(417, 23)$	$(537, 89)$	$(681, 113)$	$(717, 17)$	$(789, 131)$
$(447, 37)$	$(573, 19)$	$(683, 31)$	$(721, 17)$	$(799, 23)$
$(501, 83)$	$(581, 41)$	$(685, 17)$	$(745, 37)$	
$(517, 23)$	$(611, 23)$	$(687, 19)$	$(749, 53)$	

(ii) *Let $800 < B < 2000$ be odd. If $n < 13$, then all integer solutions (x, y, n) of equation (5) with $|xy| > 1$, $n \geq 3$ are given by*

$$\begin{aligned} n = 3,\ & (B, x, y) = (813, \pm(28, 3)),\ (999, \pm(10, 1)),\ (1001, \pm(10, 1)),\\ & (1521, \pm(23, 2)), (1657, \pm(71, 6)), (1727, \pm(12, 1)), (1729, \pm(12, 1)),\\ & (1801, \pm(73, 6)), (1953, \pm(25, 2))\\ n = 5,\ & (B, x, y) = (1023, \pm(4, 1)), (1025, \pm(4, 1)),\\ n = 10,\ & (B, x, y) = (1023, \pm 2, \pm 1), (1025, \pm 2, \pm 1). \end{aligned}$$

If $n > 100$ is a prime, then equation (5) has no solutions in integers (x, y, n) with $|xy| > 1, n \geq 3$ and with the possible exceptions (B, n) listed in Table 2 below.

Table 2

(B, n)	(B, n)	(B, n)
$(1041, 173)$	$(1509, 251)$	$(1795, 179)$
$(1077, 179)$	$(1527, 127)$	$(1821, 101)$
$(1135, 113)$	$(1589, 113)$	$(1841, 131)$
$(1149, 191)$	$(1671, 139)$	$(1857, 103)$
$(1315, 131)$	$(1689, 281)$	$(1915, 191)$
$(1401, 233)$	$(1735, 173)$	$(1929, 107)$
$(1437, 239)$	$(1761, 293)$	$(1959, 163)$

In (ii), solving equation (5) with our present methods, for $13 \le n \le 100$ we obtained so many exceptions that we disregard that case.

For equation (4), we have the following.

Theorem 3. *Under the assumptions* (2) *and* $\max\{A, B\} \le 50$, *all integer solutions* (x, y, n) *to equation* (4) *with* $|xy| > 1$, $n \ge 3$ *and with* $(A, B, n) \notin \{(21, 38, 17), (26, 41, 17), (22, 43, 17), (17, 46, 17), (31, 46, 17), (21, 38, 19)\}$ *are given by*

$$
\begin{aligned}
n = 3,\ & (A, B, x, y) = (1, 7, \pm(2, 1)), (1, 9, \pm(2, 1)), (1, 17, \pm(18, 7)), \\
& (1, 19, \pm(8, 3)), (1, 20, \pm(19, 7)), (1, 26, \pm(3, 1)), (2, 15, \pm(2, 1)), \\
& (2, 17, \pm(2, 1)), (3, 10, \pm(3, 2)), (5, 13, \pm(11, 8)), (5, 17, \pm(3, 2)), \\
& (8, 17, \pm(9, 7)), (8, 19, \pm(4, 3)), (11, 19, \pm(6, 5)) \\
n = 4,\ & (A, B, x, y) = (1, 5, \pm 3, \pm 2), (1, 15, \pm 2, \pm 1), (1, 17, \pm 2, \pm 1), \\
& (1, 39, \pm 5, \pm 2).
\end{aligned}
$$

The next theorem can be regarded as an extension of Theorem 3 to the case $\max\{A, B\} \le 100$. For $n = 17$ and 19, there are, however, a lot of exceptions (A, B, n) when none of our methods applies. Hence we consider only the situation when n is a prime greater than 19.

Theorem 4. *Let* A, B *be integers with* $\max\{A, B\} \le 100$ *and* (2), *and let* n *be a prime.*

(i) *If* $n > 41$, *then equation* (4) *has no integer solutions* (x, y, n) *with* $|xy| > 1$.

(ii) *If* $19 < n \le 41$, *then equation* (4) *has no integer solutions* (x, y, n) *with* $|xy| > 1$, *apart from the possible exceptions* $(A, B, n) = (35, 58, 29), (8, 75, 31), (11, 76, 31), (23, 78, 31), (31, 58, 31), (39, 71, 31)$ *and* $(17, 82, 41)$.

We conjecture that for $\max\{A, B\} \leq 100$, equation (4) possesses only the solutions listed in Theorem 3.

Finally, we consider the case when C is not necessarily 1.

Theorem 5. *Let A, B, C be integers with $\max\{A, B, C\} \leq 30$ and with (2), (3), and let n be a prime.*

(i) *If $n > 31$, then equation (1) has no integer solutions (x, y, n) with $|xy| > 1$.*

(ii) *If $19 < n \leq 31$, then equation (1) has no integer solutions (x, y, n) with $|xy| > 1$, apart from the possible exceptions $(A, B, C, n) = (1, 19, 26, 31)$, $(1, 26, 19, 31), (2, 15, 14, 31), (2, 23, 6, 31), (6, 23, 2, 31)$ and $(13, 21, 30, 31)$.*

In [5] and [13], some special cases of our Theorems 1 and 3 were used to solve, for certain values of k and C, the equations $1^k + 2^k + \ldots + x^k = y^n$ and $x(x+1) = Cy^n$. Here x, y and n are unknown positive integers with $n \geq 2$. As an application of our theorems we shall considerably generalize these results of [5] and [13] in a separate paper.

We remark that our results and their proofs provide the theoretical background of a possible implementation of a binomial Thue equation solver subroutine in certain computer algebraic systems like MAGMA [8] or SAGE [24]. For a computational approach of modular forms we refer to [23].

3. Auxiliary results

To prove our theorems we need several lemmas. Set

$$M = \max\{A, B, 3\} \text{ and } \lambda = \log\left(1 + \frac{\log M}{|\log(A/B)|}\right).$$

Lemma 1. *Suppose that (x, y, n) is an integer solution to (1) with*

$$x > |y| > 0, \quad 3\,\log(1.5|C/B|) \leq 7400\,\frac{\log M}{\lambda} \text{ and } \frac{\log 2C}{\log 2} \leq 8\,\log M.$$

Then we have

$$n \leq \min\left(7400\,\frac{\log M}{\lambda},\ 3106\,\log M\right).$$

Proof. A similar result was proved by Mignotte [19] with a weaker upper bound for n. Mignotte's estimate has been improved in [22] by iterated application of Baker's theory of logarithmic forms. □

Combining Lemma 1 with local arguments GYŐRY and PINTÉR ([13], Theorem 1) obtained considerably sharper upper bounds for n whenever $|xy| > 1$. We now formulate this result of [13]. Lemmas 1 and 2 will be used to bound the exponent n in our equations.

Lemma 2. *Suppose that* (2) *holds and*

$$C \notin \{A, B, B \pm A\}\,. \tag{6}$$

For the pairs (M_1, n_1), (M_2, n_2) *given in Table 3, and for every integer solution* (x, y, n) *of* (1) *with* $n \geq 3$ *prime, we have*

(i) $n \leq n_1$ *if* $\max\{A, B, C\} \leq M_1$, *and*

(ii) $n \leq n_2$ *if* $C = 1$ *and* $\max\{A, B\} \leq M_2$.

Table 3

M_1	n_1	M_2	n_2
100	71	200	79
35	43	100	53
20	37	50	31
10	19	20	19

PROOF. See GYŐRY and PINTÉR [13]. The proof depends on Lemma 1 and on a short MAGMA program which is based on the following version of the local method. For each quadruple (A, B, C, n) one can search for a local obstruction by considering (1) modulo a prime of the form $p = 2kn + 1$, coprime to A, B and C, with $k \in \mathbb{N}$. For such a prime, there are at most $(2k+1)^2$ possible residue classes for $Ax^n - By^n$. If none of these contains C, then equation (1) is impossible modulo p. If one cannot find such a prime with $k \leq 150$, then one can test the solvability of the equation modulo n^2. We note that using the method of the proof, Table 3 can be extended to larger values of M_1 and M_2 as well. □

The following Lemma 3 summarizes some recent results obtained by KRAUS [17], and BENNETT, VATSAL and YAZDANI [7] on ternary equations of the form

$$Ax^n + By^n = Cz^m \quad \text{with} \quad m \in \{3, n\}\,, \tag{7}$$

where A, B, C are given nonzero integers, $n \geq 3$ and x, y, z are unknown integers. Approaches to solve such equations, analogous to that employed by WILES [26] to prove Fermat's Last Theorem, are based on the connection between putative integer solutions (x, y, z) of ternary equations, Frey curves and certain modular

forms. In this direction significant contributions were also made among others by Frey, Serre, Darmon, Merel, Ribet, Bennett and Skinner.

For a given prime q and non-zero integer u, set

$$\mathrm{Rad}_q(u) := \prod_{\substack{p \mid u \\ p \neq q}} p,$$

where the product is taken over all positive primes p different from q and dividing u, and write $\mathrm{ord}_q(u)$ for the largest integer k with $q^k | u$. Suppose that for given A, B, C and $n \geq 3$, we have a solution (x, y, z) to (7) in nonzero integers.

If $m = 3$ (see [7]) we assume, without loss of generality, that $3 \nmid Ax$ and $By^n \not\equiv 2 \pmod 3$. Further, suppose that C is cube-free, A and B are nth-power free and that equation (7) does not correspond to one of the following identities:

$$1 \cdot 2^5 + 27 \cdot (-1)^5 = 5 \cdot 1^3 \quad \text{or} \quad 1 \cdot 2^7 + 3 \cdot (-1)^7 = 1 \cdot 5^3.$$

We consider the elliptic curve

$$E : Y^2 + 3CzXY + By^nY = X^3,$$

and set

$$N_n(E) = \mathrm{Rad}_3(AB)\,\mathrm{Rad}_3(C)^2 \varepsilon_3,$$

where

$$\varepsilon_3 := \begin{cases} 3^2 & \text{if } 9 \mid (2 + C^2By^n - 3Cz), \\ 3^3 & \text{if } 3 \| (2 + C^2By^n - 3Cz), \\ 3^4 & \text{if } \mathrm{ord}_3(By^n) = 1, \\ 3^3 & \text{if } \mathrm{ord}_3(By^n) = 2, \\ 1 & \text{if } \mathrm{ord}_3(B) = 3, \\ 3 & \text{if } \mathrm{ord}_3(By^n) > 3 \text{ and } \mathrm{ord}_3(B) \neq 3, \\ 3^5 & \text{if } 3 \mid C. \end{cases}$$

If $m = n$ (see [17]), then we may assume without loss of generality that $Ax^n \equiv -1 \pmod 4$ and $By^n \equiv 0 \pmod 2$. The corresponding Frey curve is

$$E : Y^2 = X(X - Ax^n)(X + By^n).$$

Put

$$N_n(E) = \mathrm{Rad}_2(ABC)\varepsilon_n,$$

where

$$\varepsilon_n := \begin{cases} 1 & \text{if } \operatorname{ord}_2(ABC) = 4, \\ 2 & \text{if } \operatorname{ord}_2(ABC) = 0 \text{ or } ord_2(ABC) \geq 5, \\ 2 & \text{if } 1 \leq \operatorname{ord}_2(B) \leq 3 \text{ and } xyz \text{ even}, \\ 8 & \text{if } \operatorname{ord}_2(ABC) = 2 \text{ or } 3 \text{ and } xyz \text{ odd}, \\ 32 & \text{if } \operatorname{ord}_2(ABC) = 1 \text{ and } xyz \text{ odd}. \end{cases}$$

We note that both for $m = 3$ and for $m = n$, the numbers $N_n(E)$ are closely related to the conductors of the above curves (cf. [7] and [17]).

Lemma 3. *Suppose that A, B, C, x, y and z are nonzero integers with Ax, By and Cz pairwise coprime, $xy \neq \pm 1$, satisfying equation (7) with prime $n \geq 5$ and $n \nmid ABC$. Then there exists a cuspidal newform $f = \sum_{r=1}^{\infty} c_r q^r$ ($q := e^{2\pi iz}$) of weight 2, trivial Nebentypus character and level $N_n(E)$ for $N_n(E)$ given as above. Moreover, if we write K_f for the field of definition of the Fourier coefficients c_r of the form f and suppose that p is a prime coprime to $nN_n(E)$, then*

$$\operatorname{Norm}_{K_f/\mathbb{Q}}(c_p - a_p) \equiv 0 \pmod{n}$$

with $a_p = \pm(p+1)$ (if $p \mid xy$) or $a_p \in S_{p,m}$ (if $p \nmid xy$), where

$$S_{p,3} = \{u : |u| < 2\sqrt{p},\ u \equiv p+1 \pmod 3\}$$

and

$$S_{p,n} = \{u : |u| < 2\sqrt{p},\ u \equiv p+1 \pmod 4\}.$$

Proof. This is a combination of some deep results of [7] and [17]. (For a survey on this topic, see [2].) □

Combining several powerful techniques, Bennett [1] obtained the following results.

Lemma 4. *If A, B and n are nonzero integers and $n \geq 3$, then equation (4) has at most one solution in positive integers x, y.*

Proof. See Theorem 1.1 in [1]. We shall use this lemma in the special case $B = A + 1$. Then $x = y = 1$ is a solution to (4), hence no further solution exists. □

Lemma 5. *Let $b > a$ be positive, relatively prime integers and suppose that*

$$17 \leq n \leq 43 \text{ is prime}, \quad m = \left[\frac{n+1}{3}\right],$$

and define $c_1(n), d(n)$ via

n	$c_1(n)$	$d(n)$	n	$c_1(n)$	$d(n)$
17	8.93	13.06	31	17.92	30.55
19	9.40	15.46	37	21.92	32.51
23	13.03	17.66	41	25.83	36.08
29	17.39	29.95	43	26.62	33.95

If we have

$$\left(\sqrt[m]{b}-\sqrt[m]{a}\right)^m e^{c_1(n)}<1$$

then, if x and $y>0$ are integers, we may conclude that

$$\left|\left(\frac{b}{a}\right)^{\frac{1}{n}}-\frac{x}{y}\right|>\left(3.15\cdot 10^{24}(m-1)^2n^{m-1}e^{c_1(n)+d(n)}\left(\sqrt[m]{b}+\sqrt[m]{a}\right)^m\right)^{-1}y^{-\lambda},$$

where

$$\lambda=(m-1)\left(1-\frac{\log\left(\left(\sqrt[m]{b}+\sqrt[m]{a}\right)^m e^{c_1(n)+1/20}\right)}{\log\left(\left(\sqrt[m]{b}-\sqrt[m]{a}\right)^m e^{c_1(n)}\right)}\right).$$

Proof. This is a special case of Theorem 7.1 in [1] which is stated and proved for primes $17\le n\le 347$. □

Recently, Bennett [4] improved this result by giving a sharper lower bound for primes $37\le n\le 73$. However, in our applications we cannot benefit from this improvement.

We recall that for a finite set of primes S, an integer u is an S-unit if all its prime factors lie in S. The following result is due to Bennett, Győry, Mignotte and Pintér [6] for $2\le p,\ q\le 13$, and to Győry and Pintér [15] for $2\le p,\ q<30$.

Lemma 6. *Let $S=\{p,q\}$ for p and q primes with $2\le p,\ q<30$. If A, B, x, y and n are positive integers with A, B S-units, $A<B$ and $n\ge 3$, then the only solutions to equation* (4) *are those with*

$$n\ge 3,\ A\in\{1,2,3,4,7,8,16\},\quad x=y=1$$

and

$$\begin{aligned}
&n=3,\ (A,x)=(1,2),(1,3),(1,4),(1,9),(1,19),(1,23),(3,2),(5,11),\\
&n=4,\ (A,x)=(1,2),(1,3),(1,5),(3,2),\\
&n=5,\ (A,x)=(1,2),(1,3),\\
&n=6,\ (A,x)=(1,2).
\end{aligned}$$

Proof. This is Theorem 1 in [15]; see also Theorem 1.1 in [6]. □

The following two lemmas are special cases of two theorems of Bugeaud, Mignotte and Siksek [9] and will be used in the proofs of our Theorems 3 and 4.

Lemma 7. *Suppose $3 \le q < 100$ is a prime. The equation*

$$q^u x^n - 2^v y^n = \pm 1$$

has no solutions in integers x, y, u, v, n with $x, y > 0$, $|xy| > 1$, $u, v \ge 0$ and $n > 5$.

Proof. Cf. Theorem 1.1 in [9]. □

Lemma 8. *Suppose $3 \le p < q \le 31$ are primes. The equation*

$$p^u x^n - q^v y^n = \pm 1$$

has no solutions in integers x, y, u, v, n with $x, y > 0$, $u, v \ge 0$ and $n > 5$.

Proof. Cf. Theorem 1.2 in [9]. □

We note that in contrast with Lemma 6, the Lemmas 7 and 8 cannot be applied to equations of the form (4) when $A = 1$ and B has two distinct prime factors. Further, in case $A = 1$ equation (4) cannot be solved by the methods used in [6], [15] and [9] when B is divisible by more than two distinct primes.

We now consider the equation

$$x^n + y^n = Bz^n, \tag{8}$$

where $n > 3$ is a prime, B is a nonzero integer and x, y, z are coprime nonzero rational integers. Let $\phi(B)$ denote Euler's function.

Lemma 9. *Suppose that n is coprime to $B\phi(B)$, $B^{n-1} \not\equiv 2^{n-1} \pmod{n^2}$ and (8) has a solution in pairwise relatively prime nonzero integers x, y and z. Then either* (i) *$n \mid z$ or* (ii) *$n \mid xy$, Bz is odd and $r^{n-1} \equiv 1 \pmod{n^2}$ for each divisor r of B.*

Proof. This lemma was proved in [5] (see also [11]). □

Assume that in (8) $n \mid B$ but $n \nmid z$. Let $n, p_1, \dots, p_r$ denote the distinct prime factors of B. For $r \ge 1$, denote by $f_1, \dots, f_r$ the smallest positive integers for which

$$p_i^{f_i} \equiv 1 \pmod{n}, \quad i = 1, \dots, r,$$

and set $\operatorname{ord}_n(B) = N$.

Remark. If $N = 1$, then (8) has no solution x, y, z with $n \nmid z$. Indeed, in the opposite case (8) implies $n \mid x + y$ whence $n \mid \frac{x^n+y^n}{x+y}$, a contradiction.

Let $\zeta = e^{2\pi/n}$. We recall that a prime n is called regular if n does not divide the class number of the cyclotomic field $\mathbb{Q}(\zeta)$. The next assertion is due to Maillet [18].

Lemma 10. *Suppose that the prime n is regular. If $N \geq 1$, $N \equiv 0$ or $1 \pmod{n}$ and, for $r \geq 1$,*

$$\sum_{i=1}^{r} \frac{1}{f_i} \leq \frac{n-3}{n-1}, \tag{9}$$

then (8) has no solutions in coprime nonzero rational integers x, y, z not divisible by n.

Proof. See [18]. □

Denote by h_n^+ the class number of the maximal real subfield $\mathbb{Q}\left(\zeta + \zeta^{-1}\right)$ of $\mathbb{Q}(\zeta)$. The following result has been recently proved by Mihăilescu [20].

Lemma 11. *Let $n \geq 17$ be a prime. If the equation*

$$\frac{x^n - 1}{x - 1} = n^e \cdot w^n, \quad e \in \{0, 1\}$$

has an integer solution (x, w) with $x \equiv 0$, 1 or $-1 \pmod{n}$, then

$$n \mid h_n^+. \tag{10}$$

Remark. It follows from the results of Buhler, Crandall, Ernvall, Metsänkylä and Shokrollahi [10] that condition (10) implies that $n > 12 \cdot 10^6$.

The following result, which will be proved by means of Lemmas 1, 9 and 11, will be crucial in solving equation (5) in many cases.

Theorem 6. *Suppose that in equation (5) n is a prime and that each of the following conditions holds:*

(i) $n \geq 17$,

(ii) $B \leq \exp\{3000\}$,

(iii) $n \nmid B\phi(B)$,

(iv) $B^{n-1} \not\equiv 2^{n-1} \pmod{n^2}$,

(v) $r^{n-1} \not\equiv 1 \pmod{n^2}$ *for some divisor r of B.*

Then equation (5) *has no solutions in integers* (x, y, n) *with* $|xy| > 1$.

PROOF. Suppose that in equation (5) n and B satisfy the conditions of Theorem 6 and that we have an integer solution (x, y, n) to (5) with $|xy| > 1$. It is clear that then the equation

$$x^n - 1 = By^n \tag{11}$$

also has an integer solution (x, y, n) with $|xy| > 1$. We can apply Lemma 1 with the choice $A = C = 1$ to obtain that $n \leq 3106 \log B$. Together with condition (ii) this yields $n \leq 9.318 \cdot 10^6$. Furthermore, in view of (iii), (iv) and (v), Lemma 9 implies that $n \mid y$. Thus we have $n \mid x - 1$ and hence $n \mid \frac{x^n-1}{x-1}$. It is known that

$$\gcd\left(\frac{x^n - 1}{x - 1}, x - 1\right) \mid n.$$

Further, each prime factor of $\frac{x^n-1}{x-1}$ is either n or $\equiv 1 \pmod{n}$ and $n^2 \nmid \frac{x^n-1}{x-1}$. Since by assumption $n \nmid \phi(B)$, we infer from (11) that

$$\frac{x^n - 1}{x - 1} = nw^n \tag{12}$$

with some nonzero integer w. Now, since $n \geq 17$ by (i), one can apply Lemma 11 to equation (12) which implies that $n \mid h_n^+$. But as is remarked after Lemma 11, it then follows that $n > 12 \cdot 10^6$ which is a contradiction. Thus Theorem 6 is proved. □

4. Proofs of Theorems 1–5

In our proofs several ideas will be utilized from Part I. Here we shall detail only those arguments which were not used in Part I.

To prove our Theorems 1, 3 and Theorem 2 (i), it will be enough to solve the corresponding equations for $n = 4$ and for odd primes n. From the values of the solutions x, y so obtained one can easily determine all solutions (x, y, n) with composite $n \geq 3$.

PROOF OF THEOREM 1. In view of Theorem 4 of [13] it suffices to deal with the case when $71 \leq B \leq 400$. For $n \leq 13$ we resolved the corresponding Thue equations using PARI [25] or MAGMA.

In case of $n \geq 17$, we obtained an upper bound n_0 on n for each B by means of Lemma 1. Then combining Theorem 6 with the modular approach (Lemma 3)

with signature (n, n, n) we could exclude the solvability of most of the equations under consideration with $17 \leq n \leq n_0$.

To illustrate our method we give an example. Set $B = 119$. Then Lemma 1 implies that $n \leq n_0 = 14843$. With an easy MAGMA program we checked that each of the conditions of Theorem 6 is fulfilled for each such n, except $n = 17$. Thus Theorem 6 implies that the equation $x^n - 119y^n = \pm 1$ has no solutions with $|xy| > 1$, unless possibly when $n = 17$. Then we considered the equation $x^{17} - 119y^{17} = \pm 1$ as a ternary equation and applied Lemma 3 with signature (n, n, n). The level of the corresponding newforms is 238. There are 6 newforms of level 238. If (x, y) is a solution of the equation then one can show by local arguments that $103 \mid xy$. We recall that K_f denotes the field generated by the Fourier coefficients c_r of a modular form f. In the case $|xy| > 1$, Lemma 3 implies that

$$103 \mid \mathrm{Norm}_{K_f/\mathbb{Q}}(c_{103} - 104) \cdot \mathrm{Norm}_{K_f/\mathbb{Q}}(c_{103} + 104)$$

for some cuspidal newform f of level 238. However, an easy calculation shows that the above relation is impossible for each newform under consideration, hence no nontrivial x, y solutions exist.

After these computations, it remained to consider equation (5) in the following cases

$$(B, n) \in \{(141, 23), (177, 29), (235, 23), (249, 41), (268, 17), (274, 29), (282, 23), \\ (295, 29), (309, 17), (321, 53), (329, 23), (354, 29)\}.$$

For $(B, n) = (268, 17)$ and $(309, 17)$, we resolved the corresponding Thue equations using PARI.

For $(B, n) \in \{(141, 23), (177, 29), (249, 41), (274, 29), (321, 53)\}$ we applied Lemma 3 with signature $(n, n, 3)$ and used MAGMA to get a contradiction in each case. For instance, when $(B, n) = (249, 41)$, one can see that $2 \mid xy$ for each solution x, y of the equation $x^{41} - 249y^{41} = \pm 1$. In view of Lemma 3 it is enough to check the relation

$$2 \mid \mathrm{Norm}_{K_f/\mathbb{Q}}(c_2 - 3) \cdot \mathrm{Norm}_{K_f/\mathbb{Q}}(c_2 + 3), \tag{13}$$

for each newform f of level $N \in \{249, 6723\}$, where c_2 denotes the second Fourier coefficient of f. There are 5 and 22 newforms at levels 249 and 6723 respectively. It is easy to check that condition (13) does not hold for any of those newforms, hence there are no nontrivial solutions x, y.

Finally, in the exceptional cases listed in the theorem we were unable to solve the corresponding Thue equations. This completes the proof of Theorem 1. □

PROOF OF THEOREM 2. (i) For $n \leq 13$, $(B, n) \neq (649, 13)$, we resolved the corresponding Thue equations of the form (5) one by one using PARI or MAGMA. When $(B, n) = (649, 13)$, PARI cannot handle the corresponding Thue equation. We then applied Lemmas 9 and 10 in the following way. It is easy to check that $(13, 649 \cdot \phi(649)) = 1$, $649^{12} \not\equiv 2^{12} \pmod{13^2}$ and $649^{12} \not\equiv 1 \pmod{13^2}$. Thus Lemma 9 gives that in (5) 13 must divide y. Then we can rewrite our equation $x^{13} - 649y^{13} = 1$ as $x^{13} - 649 \cdot 13^N y_1^{13} = 1$ where $13 \mid N$ and $13 \nmid y_1$. With the notation of Lemma 10 we have $r = 2$, $p_1 = 11$, $p_2 = 59$, $f_1 = f_2 = 12$ and since $2/12 < 10/12$, Lemma 10 yields a contradiction.

Consider now the case $n \geq 17$. As in the proof of Theorem 1, we obtained an upper bound n_0 on n for each B using Lemma 1. Then we combined again Theorem 6 with the modular approach (Lemma 3) with signature (n, n, n). We used this sieve for each of the equations of the form (5) with odd $400 < B < 800$ and with primes $17 \leq n \leq n_0$. We considered the pairs

$$(B, n) \in \{(411, 17), (423, 23), (509, 17), (531, 29), (747, 41)\}.$$

For $(B, n) = (411, 17)$ and $(509, 17)$, the corresponding Thue equations of the form (5) can be solved using PARI. When $(B, n) = (423, 23), (531, 29)$ or $(747, 41)$, we applied Lemma 3 with signature $(n, n, 3)$ to prove that the corresponding Thue equations of the form (5) have no integer solutions x, y with $|xy| > 1$.

Unfortunately, for the remaining pairs (B, n) which are listed in Table 1 we failed to resolve the corresponding Thue equations. This completes the proof of part (i) of Theorem 2.

(ii) For $n < 13$, $(B, n) \neq (1799, 11)$, we resolved the corresponding Thue equations of the form (5) one by one using PARI. When $(B, n) = (1799, 11)$, PARI cannot handle the occuring Thue equation, hence we applied Lemma 3 with signature (n, n, n) to prove that no nontrivial solutions exist.

In the sequel we assume that $n > 100$. For each odd B under consideration we deduced first an upper bound n_0 on n using Lemma 1. Then we applied again the sieve consisting of Theorem 6 and the modular technique (Lemma 3) with signature (n, n, n) for each of the equations of the form (5) with odd $800 < B < 2000$ and with primes $101 \leq n \leq n_0$. We obtained that all equations of the form (5) under consideration can have integer solutions (x, y, n) with $|xy| \leq 1$ only, except possibly for the pairs (B, n) listed in Table 2. This completes the proof of part (ii) of Theorem 2. □

PROOF OF THEOREM 3. Our Theorem 1 provides all solutions of equation (4) with $A = 1$ and $B \leq 50$. Further GYŐRY and PINTÉR [13] gave, under the

assumption (2), all solutions to equation (4) for $\max\{A,B\} \leq 20$. In view of these results we may assume that $A > 1$ and $\max\{A,B\} \geq 21$. If $B - A = 1$ then $x = y = 1$ is a solution of (4), and Lemma 4 gives that equation (4) has no solution with $|xy| > 1$. Hence we may also assume that $B - A > 1$. Then Lemma 2, (ii) yields that $n \leq 31$.

We used the local method described in the proof of Lemma 2 to prove that under the assumptions of Theorem 3 equation (4) has no solutions (x, y, n) with $|xy| > 1$, $n \geq 3$, except for the triples (A, B, n) contained in Table A1 of the Appendix.

Using PARI, we resolved the corresponding Thue equations (4) for $n \leq 19$ wherever it was possible. We note that this subroutine of PARI that we used is based on theoretical work of Hanrot [16], and it works without assuming the GRH if the right-hand side of the Thue equation is 1 or if the conditional class group is trivial.

If $(A, B, n) \in \{(2, 37, 19), (4, 23, 13), (8, 43, 31), (11, 32, 19), (17, 32, 17)\}$ or $(A, B, n) \in \{(7, 23, 13), (13, 23, 13), (17, 29, 17), (23, 25, 13), (23, 29, 13), (23, 29, 19), (23, 49, 19), (31, 49, 19)\}$, then the corresponding Thue equations are impossible by Lemmas 7 or 8, respectively.

In the case when (A, B, n) is one of the triples listed in Table A2 of the Appendix, we applied Lemma 5 to show that equation (4) has no solutions. For example, when $(A, B, n) = (19, 26, 31)$, we applied Lemma 5 with $b = 26$, $a = 19$. Then one can check that the condition $\left(\sqrt[m]{26} - \sqrt[m]{19}\right)^m e^{c_1(31)} < 1$ is fulfilled for $m = \left[\frac{31+1}{3}\right] = 10$ and $c_1(31) = 17.92$. Thus Lemma 5 yields that

$$\left|\left(\frac{26}{19}\right)^{\frac{1}{31}} - \frac{x}{y}\right| > \frac{1}{1.7259 \cdot 10^{65}} y^{-27.5338}.$$

On the other hand, the equation $|19x^{31} - 26y^{31}| = 1$ implies that

$$\left|\left(\frac{26}{19}\right)^{\frac{1}{31}} - \frac{x}{y}\right| < \frac{1}{y^{31}},$$

i.e. we have $y < 5.5677 \cdot 10^{21}$. Then in each case we used an algorithm developed by Pethő [21] for finding the small solutions of Thue equations to resolve our corresponding equation.

When $(A, B, n) = (27, 37, 19)$ or $(27, 47, 19)$, we considered the corresponding equations as ternary equations with signature $(n, n, 3)$ and we applied Lemma 3 as in the proof of Theorem 1 to solve our equations.

For (A, B, n) contained in Table A3, we considered the corresponding equations as ternary equations with signature (n, n, n). Using again Lemma 3, an easy MAGMA program proved that the equations under consideration have no nontrivial solutions.

In the exceptional cases excluded in the theorem, we were unable to prove with the above-mentioned methods that the corresponding Thue equations have no nontrivial integer solutions . □

Proof of Theorem 4. As in the proof of Theorem 3, it suffices to consider the case when $A > 1$ and $B - A > 1$. Then, in view of Lemma 2 we may assume that $19 < n \leq 53$. By Theorem 3 we may further assume that $\max\{A, B\} \geq 51$. Using the local method, we obtained that for most of the triples (A, B, n) under consideration, the corresponding equation (4) has no solutions. Those triples (A, B, n) for which the local method does not work are listed in Table A4.

In the cases corresponding to the triples of Table A5 we applied Lemma 5 and the above-mentioned algorithm of [21] to show the impossibility of equation (4).

For $(A, B, n) = (27, 91, 31)$ one can see that $2 \mid xy$ for all solutions. Then we applied Lemma 3 with signature $(n, n, 3)$ to infer that if x, y is a solution to equation (4) with $|xy| > 1$ then (13) holds for the Fourier coefficient c_2 of some newform f of level 91. There are 4 newforms of level 91 and using MAGMA we arrived at a contradiction with (13) in each case.

When $(A, B, n) \in \{(6, 67, 31), (31, 73, 31), (31, 77, 53), (31, 89, 31), (37, 88, 31), (40, 79, 31), (44, 83, 31), (52, 83, 31), (64, 99, 31)\}$ we applied Lemma 3 with signature (n, n, n). Here, for the computation of the corresponding Fourier coefficients of the arising newforms, we used again MAGMA.

Unfortunately, in the remaining 7 cases mentioned in the theorem, we could not find any way to solve the corresponding equations. □

Proof of Theorem 5. Let A, B, C be positive integers with $\max\{A, B, C\} \leq 30$ which satisfy conditions (2) and (3) and let (x, y, n) be a fixed solution of the corresponding equation (1). By Theorem 3 it suffices to consider the case when $C > 1$.

1) First assume that $(A - C)(B - C) \neq 0$. In this case Lemma 2 yields $19 < n \leq 43$. Using the local method, we showed that for most of the quadruples (A, B, C, n) under consideration, the corresponding equation (1) has no solutions. Those quadruples for which the local method does not work can be found in Table A6 of the Appendix.

For (A, B, C, n) listed in Table A7 we applied Lemma 3 with signature (n, n, n), and using MAGMA we arrived at a contradiction in each case.

When $(A, B, C, n) = (1, 15, 21, 31)$ or $(1, 21, 15, 31)$, we applied Lemma 3 with signature $(n, n, 3)$. We note that here $2 \mid xy$ for every solution x, y of both Thue equations. Computing again in MAGMA we checked the impossibility of relation (13) for all arising newforms f.

To exclude the cases $(A, B, C, n) \in \{$(11,14,17, 37),(13, 21, 30, 31),(14,15,2, 31), $(14, 17, 11, 37), (15, 19, 21, 31), (17, 24, 21, 31), (18, 19, 22, 31), (19, 21, 15, 31),$ $(21, 29, 26, 31)\}$, we combined Lemma 5 as above with PETHŐ's algorithm [21] to get a contradiction.

In the remaining 6 exceptional cases that are listed in the theorem, we were unable to solve the corresponding Thue equations. This completes the first part of the proof.

2) Next consider the case when $(A - C)(B - C) = 0$. In this case equation (1) leads to an equation of the form

$$|x_1^n - B_1 y_1^n| = 1 \text{ in integers } x_1, y_1, \tag{14}$$

where B_1 is a positive integer not having prime factors greater than 29. If in (1) AB has at most 2 prime factors, then Lemma 6 applies to the new equation (14) and gives the possible solutions. The remaining cases for (A, B) in (1) are listed in Table A8 of the Appendix. For the pairs (A, B) occurring in Table A8 we infer that $B_1 = A \cdot B^{n-1}$ or $B \cdot A^{n-1}$.

Since equation (14) always have the trivial solution $(x, y) = (1, 0)$, the local method cannot be used for showing the unsolvability of such an equation. However, since $n > 19$, we can apply Theorem 6 to equation (14). In this way we could exclude on one hand each case when $A = C$ except the ones corresponding to the triples (A, B, n) of Table A9. On the other hand, when $B = C$ we could exclude each case but the ones corresponding to the triples (A, B, n) occurring in Table A10.

When $(A, B, C, n) = (29, 30, 29, 29)$ or $(29, 30, 29, 67)$, we applied as above Lemma 5 combined with Pethő's algorithm.

For the rest of the equations corresponding to the triples in Tables A9 and A10 we applied Lemma 3 using MAGMA for the computations. For $(A, B, C, n) = (8, 21, 21, 31)$, we considered equation (1) as a ternary equation with signature $(n, n, 3)$ and arrived at the desired contradiction. In the remaining cases we applied Lemma 3 with signature (n, n, n) to prove that there is no nontrivial integer solutions of the corresponding equations. This completes the proof of Theorem 5. □

Appendix

Table A1

(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)
$(2,23,13)$	$(7,47,19)$	$(15,23,13)$	$(20,49,19)$	$(23,50,13)$	$(35,47,11)$
$(2,37,19)$	$(8,43,17)$	$(15,26,13)$	$(21,29,13)$	$(24,41,17)$	$(36,49,17)$
$(3,26,13)$	$(8,43,31)$	$(15,26,17)$	$(21,38,17)$	$(25,36,31)$	$(37,46,17)$
$(3,35,19)$	$(8,45,11)$	$(15,32,11)$	$(21,38,19)$	$(26,41,17)$	$(37,46,19)$
$(3,37,19)$	$(9,31,11)$	$(15,38,13)$	$(21,44,17)$	$(27,34,19)$	$(38,41,19)$
$(3,43,19)$	$(9,38,11)$	$(17,29,17)$	$(22,39,11)$	$(27,37,19)$	$(38,47,11)$
$(3,50,11)$	$(9,40,19)$	$(17,32,17)$	$(22,43,17)$	$(27,47,19)$	$(38,49,17)$
$(4,23,13)$	$(10,33,13)$	$(17,37,13)$	$(23,25,13)$	$(28,43,19)$	$(39,44,13)$
$(5,22,31)$	$(10,37,11)$	$(17,46,17)$	$(23,29,13)$	$(29,33,17)$	$(39,44,17)$
$(5,27,11)$	$(11,32,19)$	$(18,29,17)$	$(23,29,19)$	$(29,37,19)$	$(39,46,17)$
$(5,39,13)$	$(11,34,11)$	$(18,41,13)$	$(23,34,13)$	$(29,41,11)$	$(39,50,13)$
$(5,42,11)$	$(12,23,13)$	$(18,47,17)$	$(23,35,13)$	$(29,47,11)$	$(40,47,11)$
$(5,46,17)$	$(13,23,13)$	$(19,22,11)$	$(23,37,13)$	$(31,37,11)$	$(41,43,19)$
$(7,23,13)$	$(13,36,13)$	$(19,24,19)$	$(23,37,29)$	$(31,46,17)$	$(43,46,23)$
$(7,29,11)$	$(13,37,11)$	$(19,26,31)$	$(23,38,13)$	$(31,49,19)$	$(44,49,11)$
$(7,33,17)$	$(13,41,13)$	$(19,37,19)$	$(23,39,13)$	$(33,47,11)$	$(44,49,19)$
$(7,37,19)$	$(13,42,13)$	$(19,41,13)$	$(23,47,13)$	$(33,47,13)$	
$(7,41,31)$	$(14,23,31)$	$(19,49,11)$	$(23,48,13)$	$(34,49,13)$	
$(7,47,11)$	$(15,22,11)$	$(20,27,11)$	$(23,49,19)$	$(35,44,19)$	

Table A2

(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)
$(19,24,19)$	$(29,33,17)$	$(37,46,17)$	$(39,44,17)$	$(43,46,23)$
$(19,26,31)$	$(29,37,19)$	$(37,46,19)$	$(39,46,17)$	$(44,49,19)$
$(27,34,19)$	$(35,44,19)$	$(38,41,19)$	$(41,43,19)$	

Table A3

(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)
$(5,22,31)$	$(7,41,31)$	$(15,26,17)$	$(21,44,17)$	$(28,43,19)$	$(38,49,17)$
$(5,46,17)$	$(7,47,19)$	$(18,47,17)$	$(23,37,29)$	$(33,47,11)$	
$(7,33,17)$	$(8,43,17)$	$(19,37,19)$	$(24,41,17)$	$(33,47,13)$	
$(7,37,19)$	$(14,23,31)$	$(20,49,19)$	$(25,36,31)$	$(36,49,17)$	

Table A4

(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)
$(6,67,31)$	$(27,91,31)$	$(35,58,29)$	$(44,83,31)$	$(64,99,31)$	$(79,84,41)$
$(8,75,31)$	$(31,58,31)$	$(37,88,31)$	$(45,59,31)$	$(67,82,41)$	$(82,91,31)$
$(11,76,31)$	$(31,73,31)$	$(39,71,31)$	$(52,83,31)$	$(68,95,43)$	$(93,95,31)$
$(17,82,41)$	$(31,77,53)$	$(40,79,31)$	$(55,82,41)$	$(69,91,31)$	$(95,98,37)$
$(23,78,31)$	$(31,89,31)$	$(44,53,31)$	$(61,79,23)$	$(79,82,41)$	

Table A5

(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)
$(44,53,31)$	$(55,82,41)$	$(67,82,41)$	$(69,91,31)$	$(79,84,41)$	$(93,95,31)$
$(45,59,31)$	$(61,79,23)$	$(68,95,43)$	$(79,82,41)$	$(82,91,31)$	$(95,98,37)$

Table A6

(A,B,C,n)	(A,B,C,n)	(A,B,C,n)	(A,B,C,n)	(A,B,C,n)
$(1,4,28,31)$	$(2,23,4,31)$	$(7,24,4,31)$	$(13,15,9,31)$	$(17,24,18,29)$
$(1,5,22,31)$	$(2,23,6,31)$	$(7,25,13,31)$	$(13,20,11,31)$	$(17,24,21,31)$
$(1,8,6,31)$	$(3,4,24,31)$	$(8,19,23,31)$	$(13,21,30,31)$	$(17,27,21,31)$
$(1,14,23,31)$	$(3,7,16,31)$	$(8,27,16,31)$	$(13,30,4,31)$	$(17,29,9,31)$
$(1,15,21,31)$	$(4,7,24,31)$	$(9,13,26,31)$	$(14,15,2,31)$	$(18,19,22,31)$
$(1,19,26,31)$	$(4,13,10,31)$	$(9,17,29,31)$	$(14,17,11,37)$	$(19,21,15,31)$
$(1,21,15,31)$	$(4,13,30,31)$	$(9,29,17,31)$	$(15,19,21,31)$	$(19,22,26,31)$
$(1,23,14,31)$	$(4,23,2,31)$	$(10,13,4,31)$	$(16,19,27,31)$	$(19,23,8,31)$
$(1,26,19,31)$	$(4,27,16,31)$	$(11,14,17,37)$	$(16,27,4,31)$	$(19,26,22,31)$
$(1,28,4,31)$	$(5,27,2,31)$	$(11,15,30,31)$	$(16,27,8,31)$	$(19,27,16,31)$
$(2,5,27,31)$	$(6,23,2,31)$	$(11,20,13,31)$	$(16,27,19,31)$	$(21,29,26,31)$
$(2,9,13,31)$	$(7,10,18,31)$	$(11,28,30,23)$	$(16,27,20,43)$	$(25,28,5,31)$
$(2,13,9,31)$	$(7,13,25,31)$	$(11,30,15,31)$	$(17,20,11,37)$	
$(2,15,14,31)$	$(7,16,3,31)$	$(12,13,26,31)$	$(17,24,6,31)$	

Table A7

(A,B,C,n)	(A,B,C,n)	(A,B,C,n)	(A,B,C,n)	(A,B,C,n)
$(1,4,28,31)$	$(3,7,16,31)$	$(7,25,13,31)$	$(12,13,26,31)$	$(17,24,18,29)$
$(1,5,22,31)$	$(4,7,24,31)$	$(8,19,23,31)$	$(13,15,9,31)$	$(17,27,21,31)$
$(1,8,6,31)$	$(4,13,10,31)$	$(8,27,16,31)$	$(13,20,11,31)$	$(17,29,9,31)$
$(1,14,23,31)$	$(4,13,30,31)$	$(9,13,26,31)$	$(13,30,4,31)$	$(19,22,26,31)$
$(1,23,14,31)$	$(4,23,2,31)$	$(9,17,29,31)$	$(16,19,27,31)$	$(19,23,8,31)$
$(1,28,4,31)$	$(4,27,16,31)$	$(9,29,17,31)$	$(16,27,4,31)$	$(19,26,22,31)$
$(2,5,27,31)$	$(5,27,2,31)$	$(10,13,4,31)$	$(16,27,8,31)$	$(19,27,16,31)$
$(2,9,13,31)$	$(7,10,18,31)$	$(11,15,30,31)$	$(16,27,19,31)$	$(25,28,5,31)$
$(2,13,9,31)$	$(7,13,25,31)$	$(11,20,13,31)$	$(16,27,20,43)$	
$(2,23,4,31)$	$(7,16,3,31)$	$(11,28,30,23)$	$(17,20,11,37)$	
$(3,4,24,31)$	$(7,24,4,31)$	$(11,30,15,31)$	$(17,24,6,31)$	

Table A8

(A,B)	(A,B)	(A,B)	(A,B)	(A,B)	(A,B)	(A,B)	(A,B)
$(2,15)$	$(5,28)$	$(8,15)$	$(11,15)$	$(13,21)$	$(15,28)$	$(19,24)$	$(23,26)$
$(2,21)$	$(6,7)$	$(8,21)$	$(11,18)$	$(13,22)$	$(15,29)$	$(19,26)$	$(23,28)$
$(3,10)$	$(6,11)$	$(9,10)$	$(11,20)$	$(13,24)$	$(16,21)$	$(19,28)$	$(23,30)$
$(3,14)$	$(6,13)$	$(9,14)$	$(11,21)$	$(13,28)$	$(17,18)$	$(19,30)$	$(24,25)$
$(3,20)$	$(6,17)$	$(9,20)$	$(11,24)$	$(13,30)$	$(17,20)$	$(20,21)$	$(24,29)$
$(3,22)$	$(6,19)$	$(9,22)$	$(11,26)$	$(14,15)$	$(17,21)$	$(20,23)$	$(25,26)$

(A,B)	(A,B)	(A,B)	(A,B)	(A,B)	(A,B)	(A,B)	(A,B)
$(3,26)$	$(6,23)$	$(9,26)$	$(11,28)$	$(14,17)$	$(17,22)$	$(20,27)$	$(25,28)$
$(3,28)$	$(6,25)$	$(9,28)$	$(11,30)$	$(14,19)$	$(17,24)$	$(20,29)$	$(26,27)$
$(4,15)$	$(6,29)$	$(10,11)$	$(12,13)$	$(14,23)$	$(17,26)$	$(21,22)$	$(26,29)$
$(4,21)$	$(7,10)$	$(10,13)$	$(12,17)$	$(14,25)$	$(17,28)$	$(21,23)$	$(27,28)$
$(5,6)$	$(7,12)$	$(10,17)$	$(12,19)$	$(14,27)$	$(17,30)$	$(21,25)$	$(28,29)$
$(5,12)$	$(7,15)$	$(10,19)$	$(12,23)$	$(14,29)$	$(18,19)$	$(21,26)$	$(29,30)$
$(5,14)$	$(7,18)$	$(10,21)$	$(12,25)$	$(15,16)$	$(18,23)$	$(21,29)$	
$(5,18)$	$(7,20)$	$(10,23)$	$(12,29)$	$(15,17)$	$(18,25)$	$(22,23)$	
$(5,21)$	$(7,22)$	$(10,27)$	$(13,14)$	$(15,19)$	$(18,29)$	$(22,25)$	
$(5,22)$	$(7,24)$	$(10,29)$	$(13,15)$	$(15,22)$	$(19,20)$	$(22,27)$	
$(5,24)$	$(7,26)$	$(11,12)$	$(13,18)$	$(15,23)$	$(19,21)$	$(22,29)$	
$(5,26)$	$(7,30)$	$(11,14)$	$(13,20)$	$(15,26)$	$(19,22)$	$(23,24)$	

Table A9

(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)
$(2,15,2081)$	$(6,25,811)$	$(10,23,269)$	$(14,23,23)$	$(18,23,23)$	$(22,27,5393)$
$(2,21,31)$	$(6,29,29)$	$(10,27,43)$	$(14,23,47)$	$(18,25,3457)$	$(22,29,29)$
$(3,10,383)$	$(6,29,67)$	$(10,27,229)$	$(14,23,4513)$	$(18,29,29)$	$(23,24,23)$
$(3,20,61)$	$(7,10,593)$	$(10,27,263)$	$(14,27,2437)$	$(18,29,151)$	$(23,24,2731)$
$(3,22,359)$	$(7,15,5749)$	$(10,29,29)$	$(14,29,29)$	$(18,29,157)$	$(23,26,23)$
$(3,22,4813)$	$(7,18,113)$	$(10,29,283)$	$(14,29,617)$	$(18,29,173)$	$(23,28,23)$
$(3,26,269)$	$(7,20,41)$	$(10,29,8387)$	$(14,29,677)$	$(18,29,191)$	$(23,28,509)$
$(4,21,131)$	$(7,20,97)$	$(11,21,2711)$	$(14,29,2273)$	$(18,29,5261)$	$(23,28,599)$
$(5,6,383)$	$(7,20,653)$	$(11,24,107)$	$(15,17,2617)$	$(19,24,829)$	$(23,28,5197)$
$(5,12,3457)$	$(7,30,31)$	$(11,24,4637)$	$(15,19,281)$	$(19,24,1663)$	$(23,30,23)$
$(5,14,593)$	$(7,30,47)$	$(11,26,47)$	$(15,19,2999)$	$(19,26,83)$	$(24,29,29)$
$(5,21,89)$	$(7,30,73)$	$(11,26,79)$	$(15,23,23)$	$(19,26,8329)$	$(24,29,601)$
$(5,21,719)$	$(7,30,491)$	$(11,26,2053)$	$(15,23,293)$	$(19,28,3499)$	$(25,26,29)$
$(5,21,2857)$	$(7,30,1987)$	$(12,17,2273)$	$(15,28,5749)$	$(19,30,2399)$	$(25,26,233)$
$(5,22,1531)$	$(9,14,113)$	$(12,23,23)$	$(15,29,29)$	$(20,21,71)$	$(25,28,61)$
$(5,26,89)$	$(9,20,67)$	$(12,23,43)$	$(15,29,73)$	$(20,21,137)$	$(26,27,103)$
$(5,26,3607)$	$(9,20,887)$	$(12,23,179)$	$(15,29,101)$	$(20,21,2339)$	$(26,29,29)$
$(5,26,6619)$	$(9,20,9257)$	$(12,23,1637)$	$(15,29,6217)$	$(20,23,23)$	$(26,29,2287)$
$(5,28,43)$	$(9,26,727)$	$(12,25,353)$	$(16,21,173)$	$(20,29,29)$	$(27,28,149)$
$(6,11,107)$	$(9,28,439)$	$(12,29,29)$	$(17,20,401)$	$(21,23,23)$	$(27,28,1291)$
$(6,11,4637)$	$(10,11,1279)$	$(12,29,6833)$	$(17,21,71)$	$(21,23,41)$	$(28,29,29)$
$(6,17,1231)$	$(10,13,61)$	$(13,15,2297)$	$(17,21,251)$	$(21,23,73)$	$(29,30,29)$
$(6,17,1493)$	$(10,13,157)$	$(13,18,727)$	$(17,21,2851)$	$(21,25,31)$	$(29,30,67)$
$(6,19,829)$	$(10,17,31)$	$(13,21,89)$	$(17,24,1231)$	$(21,26,331)$	
$(6,19,1663)$	$(10,17,71)$	$(13,22,47)$	$(17,24,1493)$	$(21,29,29)$	
$(6,23,23)$	$(10,19,269)$	$(13,22,79)$	$(17,26,23)$	$(21,29,1601)$	
$(6,23,2731)$	$(10,21,29)$	$(13,22,2053)$	$(17,26,1117)$	$(22,23,23)$	
$(6,25,23)$	$(10,23,23)$	$(14,15,61)$	$(17,28,257)$	$(22,27,4793)$	

Table A10

(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)	(A,B,n)
$(3,14,223)$	$(6,25,4253)$	$(10,23,6857)$	$(13,28,6211)$	$(17,28,9209)$	$(21,29,467)$
$(3,20,29)$	$(6,29,29)$	$(10,29,29)$	$(13,30,701)$	$(17,28,9623)$	$(22,23,23)$
$(3,20,2311)$	$(7,10,4889)$	$(10,29,367)$	$(14,15,31)$	$(17,30,179)$	$(22,25,137)$
$(3,22,2203)$	$(7,12,4253)$	$(11,12,31)$	$(14,15,47)$	$(18,19,947)$	$(22,27,47)$
$(3,22,8111)$	$(7,15,29)$	$(11,12,1321)$	$(14,15,73)$	$(18,23,23)$	$(22,27,89)$
$(3,26,137)$	$(7,15,109)$	$(11,14,281)$	$(14,15,491)$	$(18,25,383)$	$(22,29,23)$
$(3,28,23)$	$(7,18,1039)$	$(11,21,37)$	$(14,15,1987)$	$(18,29,29)$	$(22,29,29)$
$(4,15,163)$	$(7,18,2131)$	$(11,24,1289)$	$(14,17,733)$	$(18,29,109)$	$(23,24,23)$
$(5,14,193)$	$(7,20,37)$	$(11,28,271)$	$(14,23,23)$	$(19,20,101)$	$(23,26,23)$
$(5,18,1291)$	$(7,20,487)$	$(11,30,61)$	$(14,25,431)$	$(19,21,53)$	$(23,28,23)$
$(5,22,43)$	$(7,20,7699)$	$(12,13,61)$	$(14,27,2111)$	$(19,21,2861)$	$(23,28,53)$
$(5,22,97)$	$(7,22,2897)$	$(12,13,1889)$	$(14,29,29)$	$(19,22,5839)$	$(23,30,23)$
$(5,22,157)$	$(7,24,103)$	$(12,19,29)$	$(15,17,1879)$	$(19,24,47)$	$(23,30,41)$
$(5,24,47)$	$(7,24,797)$	$(12,19,163)$	$(15,19,41)$	$(19,26,163)$	$(23,30,47)$
$(5,24,113)$	$(7,30,571)$	$(12,19,193)$	$(15,19,233)$	$(19,28,23)$	$(23,30,139)$
$(5,24,1481)$	$(8,15,2081)$	$(12,23,23)$	$(15,19,5297)$	$(19,28,659)$	$(24,25,23)$
$(5,26,31)$	$(8,21,31)$	$(12,23,199)$	$(15,22,53)$	$(19,28,7079)$	$(24,25,811)$
$(5,26,101)$	$(9,10,1733)$	$(12,23,5867)$	$(15,22,487)$	$(19,30,89)$	$(24,29,29)$
$(5,28,83)$	$(9,14,29)$	$(12,25,3967)$	$(15,22,5431)$	$(19,30,1163)$	$(24,29,67)$
$(5,28,89)$	$(9,14,41)$	$(12,29,29)$	$(15,23,23)$	$(19,30,8599)$	$(25,26,41)$
$(5,28,163)$	$(9,20,41)$	$(12,29,179)$	$(15,23,421)$	$(20,21,89)$	$(25,26,347)$
$(6,11,359)$	$(9,22,227)$	$(12,29,2837)$	$(15,28,151)$	$(20,21,719)$	$(25,28,53)$
$(6,11,4813)$	$(9,28,61)$	$(13,14,199)$	$(15,28,1229)$	$(20,21,2857)$	$(25,28,59)$
$(6,13,269)$	$(10,11,1531)$	$(13,15,53)$	$(15,28,1291)$	$(20,23,23)$	$(26,27,47)$
$(6,17,157)$	$(10,13,89)$	$(13,15,743)$	$(15,29,29)$	$(20,23,659)$	$(26,27,2243)$
$(6,19,23)$	$(10,13,3607)$	$(13,18,37)$	$(15,29,71)$	$(20,29,29)$	$(26,29,29)$
$(6,19,263)$	$(10,13,6619)$	$(13,18,8563)$	$(16,21,131)$	$(21,22,151)$	$(27,28,43)$
$(6,23,23)$	$(10,17,1657)$	$(13,20,7603)$	$(17,20,107)$	$(21,22,1013)$	$(28,29,29)$
$(6,23,79)$	$(10,17,2237)$	$(13,21,563)$	$(17,20,151)$	$(21,23,23)$	$(28,29,73)$
$(6,23,151)$	$(10,19,47)$	$(13,22,263)$	$(17,20,241)$	$(21,23,5419)$	$(29,30,29)$
$(6,23,673)$	$(10,19,601)$	$(13,24,43)$	$(17,21,47)$	$(21,25,103)$	
$(6,25,197)$	$(10,19,821)$	$(13,28,89)$	$(17,26,173)$	$(21,26,2347)$	
$(6,25,313)$	$(10,23,23)$	$(13,28,569)$	$(17,28,3931)$	$(21,29,29)$	

Acknowledgement. We are grateful to the referee for his useful comments.

References

[1] M. A. Bennett, Rational approximation to algebraic numbers of small height: the Diophantine equation $|ax^n - by^n| = 1$, *J. Reine Angew. Math.* **535** (2001), 1–49.

[2] M. A. Bennett, Recipes for ternary Diophantine equations of signature (p,p,k), *RIMS Kokyuroku (Kyoto)* **1319** (2003), 51–55.

[3] M. A. Bennett, Products of consecutive integers, *Bull. London Math. Soc.* **36** (2004), 683–694.

[4] M. A. Bennett, The Diophantine equation $(x^k - 1)(y^k - 1) = (z^k - 1)^t$, *Indag. Math. (N.S.)* **18**, no. 4 (2007), 507–525.

[5] M. A. Bennett, K. Győry and Á. Pintér, On the Diophantine equation $1^k + 2^k + \ldots + x^k = y^n$, *Compos. Math.* **140** (2004), 1417–1431.

[6] M. A. Bennett, K. Győry, M. Mignotte and Á. Pintér, Binomial Thue equations and polynomial powers, *Compos. Math.* **142**, no. 5 (2006), 1103–1121.

[7] M. A. Bennett, M. Vatsal and S. Yazdani, Ternary Diophantine equations of signature $(p, p, 3)$, *Compositio Math.* **140** (2004), 1399–1416.

[8] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system. I. The user language, *J. Symbolic Comput.* **24** (1997), 235–265.

[9] Y. Bugeaud, M. Mignotte and S. Siksek, A multi-frey approach to some multi-parameter families of Diophantine equations, *Canad. J. Math.* **60** (2008), 491–519.

[10] J. Buhler, R. Crandall, R. Ernvall, T. Metsänkylä and M. A. Shokrollahi, Irregular primes and cyclotomic invariants to 12 million, *J. Symbolic Comput.* **31** (2001), 89–96.

[11] K. Győry, Über die diophantische gleichung $x^p + y^p = cz^p$, *Publ. Math. Debrecen* **13** (1966), 301–305.

[12] K. Győry, I. Pink and Á. Pintér, Power values of polynomials and binomial Thue–Mahler equations, *Publ. Math. Debrecen* **65** (2004), 341–362.

[13] K. Győry and Á. Pintér, On the resolution of equations $\mathrm{A}x^n - \mathrm{B}y^n = \mathrm{C}$ in integers x, y and $n \geq 3$, I, *Publ. Math. Debrecen* **70** (2007), 483–501.

[14] K. Győry and Á. Pintér, Polynomial powers and a common generalization of binomial Thue–Mahler equations and S-unit equations, Diophantine Equations, *Narosa Publ. House, India*, 2008, 103–119.

[15] K. Győry and Á. Pintér, Binomial Thue equations, ternary equations and power values of polynomials, *Fundamental and Applied Mathematics* (*to appear*).

[16] G. Hanrot, Solving Thue equations without the full unit group, *Math. Comp.* **69**, no. 229 (2000), 395–405.

[17] A. Kraus, Majorations effectives pour l'équation de Fermat généralisée, *Canad. J. Math.* **49** (1997), 1139–1161.

[18] E. Maillet, Sur les équations indéterminés de la forme $x^\lambda + y^\lambda = cz^\lambda$, *Acta Math.* **21** (1901), 247–256.

[19] M. Mignotte, A note on the equation $ax^n - by^n = c$, *Acta Arith.* **75** (1996), 287–295.

[20] P. Mihăilescu, Class number conditions for the diagonal case of the equation of Nagell and Ljunggren, Diophantine Approximation, *Springer-Verlag*, 2008, 245–273.

[21] A. Pethő, On the resolution of Thue inequalities, *J. Symbolic Comput.* **4** (1987), 103–109.

[22] Á. Pintér, On the power values of power sums, *J. Number Theory* **125** (2007), 412–423.

[23] W. Stein, Modular forms, a computational approach, *Graduate Studies in Mathematics* **79**, With an appendix by Paul E. Gunnells, American Mathematical Society, Providence, 2007.

[24] W. Stein *et al.*, Sage Mathematics Software (Version 3.4), The Sage Development Team, 2009 `http://www.sagemath.org/`.

[25] The PARI Group, Bordeaux, *PARI/GP, version* `2.1.5`, 2004, available from `http://pari.math.u-bordeaux.fr/`.

[26] A. Wiles, Modular elliptic curves and Fermat's last theorem, *Annals of Math.* (2) **141** (1995), 443–551.

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(Received October 28, 2009; revised November 28, 2009)