

On approximation properties of bi-parametric parabolic type potentials

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Abstract. In this paper we investigate the approximation properties of bi-parametric parabolic potentials type operators $A_\beta^\alpha f$ and $\mathcal{A}_\beta^\alpha f$ as the parameter $\alpha > 0$ tends to zero. For $\beta = 2$ these potentials coincide with the classical Jones-Sampson type parabolic potentials $H^\alpha f$ and $\mathcal{H}^\alpha f$, respectively.

1. Introduction

Bi-parametric parabolic type potentials are defined as follows (see [1]):

$$\begin{aligned}(A_\beta^\alpha f)(x, t) &= \frac{1}{\Gamma(\frac{\alpha}{\beta})} \int_0^\infty \int_{\mathbb{R}^n} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) f(x-y, t-\tau) dy d\tau \\ &\equiv (h_\alpha * f)(x, t)\end{aligned}\tag{1.1}$$

and

$$\begin{aligned}(\mathcal{A}_\beta^\alpha f)(x, t) &= \frac{1}{\Gamma(\frac{\alpha}{\beta})} \int_0^\infty \int_{\mathbb{R}^n} e^{-\tau} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) f(x-y, t-\tau) dy d\tau \\ &\equiv (\widetilde{h}_\alpha * f)(x, t).\end{aligned}\tag{1.2}$$

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Here $\alpha > 0$, $\beta > 0$; $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}^1$; $f \in L_p(\mathbb{R}^{n+1})$, $1 \leq p \leq \infty$;

$$h_\alpha(y, \tau) = \frac{1}{\Gamma(\alpha/\beta)} \tau_+^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau), \quad \widetilde{h}_\alpha(y, \tau) = \frac{1}{\Gamma(\alpha/\beta)} e^{-\tau} \tau_+^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau);$$

$\tau_+^\theta = \tau^\theta$ if $\tau > 0$ and $\tau_+ = 0$, otherwise.

The kernel $w^{(\beta)}(y, \tau)$ is defined as the Fourier transform of $\exp(-\tau|\xi|^\beta)$

$$w^{(\beta)}(y, \tau) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{iy \cdot \xi - \tau|\xi|^\beta} d\xi, \quad y \cdot \xi = y_1\xi_1 + \cdots + y_n\xi_n, \quad \tau > 0. \quad (1.3)$$

The kernel $w^{(\beta)}(y, \tau)$ coincides with the classical Poisson kernel and Gauss–Weierstrass kernel for $\beta = 1$ and $\beta = 2$, respectively.

Also, note that for $\beta = 2$, the operators (1.1) and (1.2) become the classical Jones–Sampson parabolic potentials $H^\alpha f$ and $\mathcal{H}^\alpha f$, respectively (see, [3], [6], [7], [9], [12], [14]).

In this paper it is investigated some approximation properties of the families of $A_\beta^\alpha f$ and $\mathcal{A}_\beta^\alpha f$ as the parameter $\alpha > 0$ tends to zero. It should be noted that the approximation properties of parabolic potentials $H^\alpha f$ and $\mathcal{H}^\alpha f$ were investigated in [15]. The classical Riesz and Bessel kernels as approximations of the identity have been studied by T. KUROKAWA [10]. See, also [4], [8] in which the relevant problems concerning to Riesz and Bessel type potentials have been studied in more general contexts.

We will need the following statements:

Lemma 1.1 ([1], [2], [5]; cf. [11, p. 44] for $n = 1$). *Let $w^{(\beta)}(y, \tau)$ be defined as (1.3). Then*

a) *For $y \in \mathbb{R}^n$ and $\tau > 0$,*

$$w^{(\beta)}(y, \tau) = \tau^{-n/\beta} w^{(\beta)}(\tau^{-1/\beta} y, 1); \quad (1.4)$$

b) *$w^{(\beta)}(y, \tau)$ is radial with respect to $y \in \mathbb{R}^n$ and positive provided that $0 < \beta \leq 2$;*

c) *If $\beta > 0$ is an even integer, then $w^{(\beta)}(y, 1)$ is infinitely smooth and rapidly decreasing. If $\beta \neq 2, 4, 6, \dots$ then $w^{(\beta)}(y, 1)$ has the following behavior when $|y| \rightarrow \infty$:*

$$w^{(\beta)}(y, 1) = c_\beta |y|^{-n-\beta} (1 + o(1)); \quad (1.5)$$

d) *For all $\tau > 0$ and $\beta > 0$,*

$$\int_{\mathbb{R}^n} w^{(\beta)}(y, \tau) dy = 1. \quad (1.6)$$

For $f \in L_p(\mathbb{R}^{n+1})$ we set

$$\|f\|_p = \left(\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |f(x, t)|^p dx dt \right)^{1/p}, \quad 1 \leq p < \infty, \quad \|f\|_{\infty} = \sup_{\mathbb{R}^{n+1}} |f(x, t)|.$$

We also need the following classes of smooth functions:

$$\Lambda_{\mu} = \{f \in L_{\infty}(\mathbb{R}^{n+1}) : \|f(x - y, t - \tau) - f(x, t)\|_{\infty} \leq \mu(|y|^{\beta} + \tau)\} \quad (1.7)$$

where $\mu(r)$, $r \geq 0$ is a function of type of modulus of continuity. In the case of $\mu(r) = cr^{\gamma}$, $0 < \gamma \leq 1$, the class (1.7) will be called as bi-parametric Lipschitz class, depending on the parameters γ and β .

Throughout the paper, the letters $c, c_1, c_2, \dots$ and $c(\delta, \beta)$, $c_1(\delta, \beta)$, $c_2(\delta, \beta)$, are used for constants ($c_i(\delta, \beta)$, $i = 1, 2, \dots$ depends on the parameters δ and β). We write “ $\varphi(\alpha) = O(\psi(\alpha))$, $\alpha \rightarrow 0^+$ ”, if $|\varphi(\alpha)| \leq c|\psi(\alpha)|$ as $\alpha \rightarrow 0^+$.

2. Formulation and proofs of the main results

Theorem 2.1. *Let $f \in L_p(\mathbb{R}^{n+1})$, $1 \leq p \leq \infty$, $0 < \beta \leq 2$, and bi-parametric family of operators A_{β}^{α} be defined as (1.1).*

a) *If the limit*

$$\lim_{(y, \tau) \rightarrow (x, t)} f(y, \tau) = L, \quad -\infty \leq L \leq \infty,$$

exists at a point $(x, t) \in \mathbb{R}^{n+1}$, then $\lim_{\alpha \rightarrow 0^+} (A_{\beta}^{\alpha} f)(x, t) = L$. In particular, if f is continuous at a point (x, t) , then

$$\lim_{\alpha \rightarrow 0^+} (A_{\beta}^{\alpha} f)(x, t) = f(x, t).$$

b) *Let $f \in L_p \cap C_0$, where $C_0 \equiv C_0(\mathbb{R}^{n+1})$ is the class of continuous functions f , for which $\lim_{|x| \rightarrow \infty} f(x) = 0$. Then the convergence $\lim_{\alpha \rightarrow 0^+} A_{\beta}^{\alpha} f = f$ is uniform on $\mathbb{R}^{n+1}$. If $f \in L_p \cap C$, the convergence is uniform on any compact $K \subset \mathbb{R}^{n+1}$.*

PROOF. a) Firstly, it should be pointed out that the statement of the theorem is true also for complex-valued functions because of the linearity of operators A_{β}^{α} . Now, let $-\infty < L < \infty$. The positivity of $w^{(\beta)}(y, \tau)$ for $0 < \beta \leq 2$ and the equalities

$$\int_{\mathbb{R}^n} w^{(\beta)}(y, \tau) dy = 1 \quad \text{and} \quad \int_0^{\infty} \tau^{\frac{\alpha}{\beta}-1} e^{-\tau} d\tau = \Gamma(\alpha/\beta),$$

yield that

$$\begin{aligned}
|(A_\beta^\alpha f)(x, t) - L| &\leq \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y| < \delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) |f(x-y, t-\tau) - Le^{-\tau}| dy \, d\tau \\
&+ \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y| \geq \delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) |f(x-y, t-\tau) - Le^{-\tau}| dy \, d\tau \\
&+ \frac{1}{\Gamma(\alpha/\beta)} \int_\delta^\infty \int_{\mathbb{R}^n} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) |f(x-y, t-\tau) - Le^{-\tau}| dy \, d\tau \\
&\equiv i_1(\alpha) + i_2(\alpha) + i_3(\alpha).
\end{aligned} \tag{2.1}$$

The choice of parameter $\delta > 0$ is at our disposal. Given $\varepsilon > 0$, we choose $\delta > 0$ such that

$$|f(x-y, t-\tau) - L| < \varepsilon \quad \text{and} \quad |1 - e^{-\tau}| < \varepsilon \tag{2.2}$$

for all $|y| < \delta^{1/\beta}$ and $0 < \tau < \delta$. Then we have

$$\begin{aligned}
i_1(\alpha) &\leq \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y| < \delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) |f(x-y, t-\tau) - L| dy \, d\tau \\
&+ \frac{|L|}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y| < \delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} (1 - e^{-\tau}) w^{(\beta)}(y, \tau) dy \, d\tau \\
&\leq \varepsilon \frac{(1 + |L|)}{\Gamma(\alpha/\beta)} \int_0^\delta \tau^{\frac{\alpha}{\beta}-1} d\tau \int_{\mathbb{R}^n} w^{(\beta)}(y, \tau) dy \\
&\stackrel{(1.6)}{=} \frac{\varepsilon(1 + |L|)}{\frac{\alpha}{\beta} \Gamma(\frac{\alpha}{\beta})} \delta^{\alpha/\beta} = \varepsilon \frac{(1 + |L|) \delta^{\alpha/\beta}}{\Gamma(1 + \frac{\alpha}{\beta})}.
\end{aligned} \tag{2.3}$$

Further,

$$\begin{aligned}
i_2(\alpha) &\leq \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y| > \delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) |f(x-y, t-\tau)| dy \, d\tau \\
&+ \frac{|L|}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y| > \delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} e^{-\tau} w^{(\beta)}(y, \tau) dy \, d\tau \equiv i_2'(\alpha) + i_2''(\alpha).
\end{aligned} \tag{2.4}$$

The application of Hölder's inequality gives

$$\begin{aligned}
i'_2(\alpha) &\leq \frac{\|f\|_p}{\Gamma(\alpha/\beta)} \left(\int_0^\delta \int_{|y| > \delta^{1/\beta}} [\tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau)]^{p'} dy d\tau \right)^{1/p'} \\
&\stackrel{(1.4)}{=} \frac{\|f\|_p}{\Gamma(\alpha/\beta)} \left(\int_0^\delta \int_{|y| > \delta^{1/\beta}} [\tau^{\frac{\alpha}{\beta}-1} \tau^{-n/\beta} w^{(\beta)}(\tau^{-1/\beta} y, 1)]^{p'} dy d\tau \right)^{1/p'} \\
&\quad (\text{we set } y = \tau^{\frac{1}{\beta}} z, \quad dy = \tau^{\frac{n}{\beta}} dz) \\
&= \frac{\|f\|_p}{\Gamma(\alpha/\beta)} \left(\int_0^\delta \int_{|z| > (\frac{\delta}{\tau})^{1/\beta}} \tau^{(\frac{\alpha}{\beta}-1-\frac{n}{\beta})p' + \frac{n}{\beta}} (w^{(\beta)}(z, 1))^{p'} dz d\tau \right)^{1/p'} \\
&\stackrel{(1.5)}{\leq} \frac{c_1(\delta, \beta) \|f\|_p}{\Gamma(\alpha/\beta)} \left(\int_0^\delta \tau^{(\frac{\alpha}{\beta}-1-\frac{n}{\beta})p' + \frac{n}{\beta}} d\tau \int_{|z| > (\frac{\delta}{\tau})^{1/\beta}} |z|^{-(n+\beta)p'} dz \right)^{1/p'} \\
&= \frac{c_2(\delta, \beta) \|f\|_p}{\Gamma(\alpha/\beta)} \left(\int_0^\delta \tau^{(\frac{\alpha}{\beta}-1-\frac{n}{\beta})p' + \frac{n}{\beta}} \tau^{(\frac{n}{\beta}+1)p' - \frac{n}{\beta}} d\tau \right)^{1/p'} \\
&= \frac{c_2(\delta, \beta) \|f\|_p}{\Gamma(\alpha/\beta)} \left(\int_0^\delta \tau^{\frac{\alpha}{\beta} p'} d\tau \right)^{1/p'} \leq \frac{c_3(\delta, \beta) \|f\|_p}{\Gamma(\alpha/\beta)} \leq c_4(\delta, \beta) \|f\|_p \alpha, \quad (2.5)
\end{aligned}$$

as $\alpha \rightarrow 0^+$. Similarly,

$$\begin{aligned}
i''_2(\alpha) &\leq \frac{|L|}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y| > \delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) dy d\tau \\
&= \frac{|L|}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|z| > (\frac{\delta}{\tau})^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(z, 1) dz d\tau \stackrel{(1.5)}{\leq} c_5(\delta, \beta) |L| \alpha. \quad (2.6)
\end{aligned}$$

Now, using (2.5) and (2.6) in (2.4) we have

$$i_2(\alpha) \leq c_6(\delta, \beta) (\|f\|_p + |L|) \alpha, \quad \text{as } \alpha \rightarrow 0^+. \quad (2.7)$$

Let us estimate $i_3(\alpha)$. By using Hölder inequality, we have

$$\begin{aligned} i_3(\alpha) &\leq \frac{|L|}{\Gamma(\alpha/\beta)} \int_{\delta}^{\infty} e^{-\tau} \tau^{\frac{\alpha}{\beta}-1} d\tau \int_{\mathbb{R}^n} w^{(\beta)}(y, \tau) dy \\ &\quad + \frac{\|f\|_p}{\Gamma(\alpha/\beta)} \left(\int_{\delta}^{\infty} \int_{\mathbb{R}^n} (\tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau))^{p'} dy d\tau \right)^{1/p'} \equiv i'_3(\alpha) + i''_3(\alpha). \end{aligned} \quad (2.8)$$

By (1.6) it yields that

$$i'_3(\alpha) \leq c_6(\delta, \beta) |L| \alpha \quad (2.9)$$

Using (1.4) and changing variables as $y = \tau^{\frac{1}{\beta}} z$, and then taking into account the estimate (1.5), we have

$$i''_3(\alpha) \leq \frac{c_7(\delta, \beta) \|f\|_p}{\Gamma(\alpha/\beta)} \left(\int_{\delta}^{\infty} \tau^{(\alpha-1-\frac{n}{\beta})p' + \frac{n}{\beta}} d\tau \right)^{1/p'} \leq c_8(\delta, \beta) \|f\|_p \alpha \quad (2.10)$$

for $\alpha < \frac{1}{p}(\frac{n}{\beta} + 1)$

Therefore,

$$i_3(\alpha) \leq c_9(\delta, \beta) (\|f\|_p + |L|) \alpha, \quad \text{as } \alpha \rightarrow 0. \quad (2.11)$$

Now, substituting (2.3), (2.7) and (2.11) in (2.1), we obtain

$$|(A_{\beta}^{\alpha} f)(x, t) - L| \leq \varepsilon \frac{(1 + |L|) \delta^{\alpha/\beta}}{\Gamma(1 + \frac{\alpha}{\beta})} + c(\delta, \beta) (\|f\|_p + 1) \alpha, \quad (\alpha \ll 1). \quad (2.12)$$

Since $\varepsilon > 0$ is arbitrary, the last estimate yields that

$$\lim_{\alpha \rightarrow 0} |(A_{\beta}^{\alpha} f)(x, t) - L| = 0.$$

Let now $\lim_{(y, \tau) \rightarrow (x, t)} f(y, \tau) = +\infty$ (the case of $L = -\infty$ is investigated in a similar way). For a given $M > 0$ there exists $\delta > 0$ such that $f(x - y, t - \tau) > M$, for $|y| < \delta^{1/\beta}$ and $0 < \tau < \delta$.

We have

$$\begin{aligned} (A_{\beta}^{\alpha} f)(x, t) &= \frac{1}{\Gamma(\alpha/\beta)} \int_0^{\delta} \int_{|y| < \delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) f(x - y, t - \tau) dy d\tau \\ &\quad + \frac{1}{\Gamma(\alpha/\beta)} \int_0^{\delta} \int_{|y| > \delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) f(x - y, t - \tau) dy d\tau \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\Gamma(\alpha/\beta)} \int_{\delta}^{\infty} \int_{\mathbb{R}^n} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) f(x-y, t-\tau) dy d\tau \\
& \equiv j_1(\alpha) + j_2(\alpha) + j_3(\alpha).
\end{aligned}$$

Further,

$$j_1(\alpha) \geq \frac{M}{\Gamma(\alpha/\beta)} \int_0^{\delta} \int_{|y| < \delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) dy d\tau$$

we use (1.4) and set $y = \tau^{\frac{1}{\beta}} z$

$$\begin{aligned}
& = \frac{M}{\Gamma(\alpha/\beta)} \int_0^{\delta} \int_{|z| < (\frac{\delta}{\tau})^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(z, 1) dz d\tau \geq \frac{M}{\Gamma(\alpha/\beta)} \int_0^{\delta} \tau^{\frac{\alpha}{\beta}-1} d\tau \int_{|z| \leq 1} w^{(\beta)}(z, 1) dz \\
& = c \frac{M}{\frac{\alpha}{\beta} \Gamma(\frac{\alpha}{\beta})} \delta^{\alpha/\beta} = c \frac{\delta^{\alpha/\beta}}{\Gamma(1 + \frac{\alpha}{\beta})} M. \tag{2.13}
\end{aligned}$$

It is not difficult to see that (cf. (2.7) and (2.11))

$$|j_2(\alpha)| \leq c_6(\delta, \beta)(\|f\|_p + 1)\alpha, \quad |j_3(\alpha)| \leq c_9(\delta, \beta)(\|f\|_p + 1)\alpha. \tag{2.14}$$

Using (2.13) and (2.14) we have

$$(A_{\beta}^{\alpha} f)(x, t) \geq c \frac{\delta^{\alpha/\beta}}{\Gamma(1 + \frac{\alpha}{\beta})} M - c_6(\delta, \beta)(\|f\|_p + 1)\alpha - c_9(\delta, \beta)(\|f\|_p + 1)\alpha,$$

and therefore,

$$\liminf_{\alpha \rightarrow 0^+} (A_{\beta}^{\alpha} f)(x, t) \geq cM.$$

Since $M > 0$ is arbitrary, it follows $\lim_{\alpha \rightarrow 0^+} (A_{\beta}^{\alpha} f)(x, t) = \infty$.

b) Let $f \in L_p \cap C_0$. Given $\varepsilon > 0$, we choose a parameter $\delta > 0$ such that

$$\sup_{(x,t) \in \mathbb{R}^{n+1}} |f(x-y, t-\tau) - f(x, t)| < \varepsilon \quad \text{and} \quad (1 - e^{-\tau}) < \varepsilon \tag{2.15}$$

for all $|y| < \delta^{\frac{1}{\beta}}$ and $0 < \tau < \delta$.

Now, setting $L = f(x, t)$ in (2.1), and using (2.3), (2.7), (2.11) and (2.15), we have

$$\|A_{\beta}^{\alpha} f - f\|_{\infty} \leq \varepsilon(1 + \|f\|_{\infty}) \frac{\delta^{\alpha/\beta}}{\Gamma(1 + \frac{\alpha}{\beta})} + c(\delta, \beta)(1 + \|f\|_{\infty})\alpha.$$

The latter estimate implies $\lim_{\alpha \rightarrow 0^+} \|A_{\beta}^{\alpha} f - f\|_{\infty} = 0$. $\square$

Remark 2.2. By using the estimate

$$\|\mathcal{A}_\beta^\alpha f\|_p \leq c\|f\|_p, \quad 1 \leq p \leq \infty \quad (2.16)$$

and equality (cf. (2.1))

$$\begin{aligned} & \mathcal{A}_\beta^\alpha f(x, t) - L \\ &= \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y| < \delta^{1/\beta}} e^{-\tau} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) (f(x-y, t-\tau) - L) dy d\tau \\ &+ \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y| > \delta^{1/\beta}} e^{-\tau} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) (f(x-y, t-\tau) - L) dy d\tau \\ &+ \frac{1}{\Gamma(\alpha/\beta)} \int_\delta^\infty \int_{\mathbb{R}^n} e^{-\tau} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) (f(x-y, t-\tau) - L) dy d\tau, \end{aligned} \quad (2.17)$$

in complete analogy to Theorem 2.1 the following theorem can be proved.

Theorem 2.3. Let $f \in L_p(\mathbb{R}^{n+1})$, $1 \leq p < \infty$, and the operator A_β^α be defined by (1.2). Then,

- a) If the limit $\lim_{(y,\tau) \rightarrow (x,t)} f(y, \tau) = L$ exists, at a point $(x, t) \in \mathbb{R}^{n+1}$, then $\lim_{\alpha \rightarrow 0^+} (A_\beta^\alpha f)(x, t) = L$. In particular, if f is continuous at a point (x, t) , then $\lim_{\alpha \rightarrow 0^+} (A_\beta^\alpha f)(x, t) = f(x, t)$.
- b) Let $f \in L_p \cap C_0$. Then the convergence $\lim_{\alpha \rightarrow 0^+} A_\beta^\alpha f = f$ is uniform on $\mathbb{R}^{n+1}$. If $f \in L_p \cap C$, the convergence is uniform on any compact $K \subset \mathbb{R}^{n+1}$.
- c) If $f \in L_p(\mathbb{R}^{n+1})$, $1 \leq p < \infty$, then $\lim_{\alpha \rightarrow 0^+} \|\mathcal{A}_\beta^\alpha f - f\|_p = 0$.

The next theorem gives an estimation of an error of approximation of functions $f \in \Lambda_\mu$ by the families $A_\beta^\alpha f$ and $\mathcal{A}_\beta^\alpha f$ as $\alpha \rightarrow 0^+$.

Theorem 2.4. Let the operator P_β^α be either A_β^α or $\mathcal{A}_\beta^\alpha$, $\alpha > 0$.

- a) Suppose that $f \in L_p \cap \Lambda_\mu$, where $1 \leq p < \infty$ and $\mu(s)$, $s \geq 0$, is a function of type of modulus of continuity which satisfies the condition

$$\int_0^1 \frac{\mu(\tau)}{\tau} \ln \frac{1}{\tau} d\tau < \infty. \quad (2.18)$$

If $0 < \beta \leq 2$, then

$$\|P_\beta^\alpha f - f\|_\infty = O(1)\alpha \quad \text{as } \alpha \rightarrow 0^+.$$

In particular, for Lipschitz functions (i.e., for $f \in \Lambda_\mu \cap L_p$, where $\mu(s) = cs^\gamma$, $0 < \gamma \leq 1$) we have

$$\|P_\beta^\alpha f - f\|_\infty = O(1)\alpha \quad \text{as } \alpha \rightarrow 0^+.$$

b) Let $f \in L_p \cap \Lambda_\mu$, where $\mu(s) = cs^\lambda |\log s|^\gamma$, $0 < \lambda < 1$ and $\gamma \geq 0$. Then

$$\|P_\beta^\alpha f - f\|_\infty = O(1)\alpha \quad \text{as } \alpha \rightarrow 0^+.$$

PROOF. We only prove the case when $P_\beta^\alpha = A_\beta^\alpha$. The case of $P_\beta^\alpha = \mathcal{A}_\beta^\alpha$ may be proved by the same way.

a) Let $f \in L_p \cap \Lambda_\mu$, where Λ_μ is defined by (1.7). Setting $L = f(x, t)$ in (2.1), we get

$$|(A_\beta^\alpha f)(x, t) - f(x, t)| \leq i_1(\alpha) + i_2(\alpha) + i_3(\alpha). \quad (2.19)$$

In a complete analogy with (2.7) and (2.11), it follows that

$$i_2(\alpha) \leq c(\delta, \beta)(\|f\|_p + \|f\|_\infty)\alpha, \quad i_3(\alpha) \leq c(\delta, \beta)(\|f\|_p + \|f\|_\infty)\alpha. \quad (2.20)$$

Let us estimate $i_1(\alpha)$. We have

$$\begin{aligned} i_1(\alpha) &\leq \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y| < \delta^{\frac{1}{\beta}}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) |f(x-y, t-\tau) - e^{-\tau} f(x, t)| dy d\tau \\ &\leq \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y| < \delta^{\frac{1}{\beta}}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) \|f(x-y, t-\tau) - f(x, t)\|_\infty dy d\tau \\ &\quad + \frac{\|f\|_\infty}{\Gamma(\alpha/\beta)} \int_0^\delta \tau^{\frac{\alpha}{\beta}-1} (1 - e^{-\tau}) \left(\int_{|y| < \delta^{\frac{1}{\beta}}} w^{(\beta)}(y, \tau) dy \right) d\tau \\ &\equiv i_1'(\alpha) + i_2''(\alpha). \end{aligned} \quad (2.21)$$

Since,

$$\int_{|y| < \delta^{\frac{1}{\beta}}} w^{(\beta)}(y, \tau) dy \leq \int_{\mathbb{R}^n} w^{(\beta)}(y, \tau) dy = 1,$$

it follows that

$$i_1''(\alpha) \leq c(\delta, \beta) \frac{\|f\|_\infty}{\Gamma(\alpha/\beta)} = c_1(\delta, \beta) \|f\|_\infty \alpha. \quad (2.22)$$

On the other hand,

$$\begin{aligned}
i'_1(\alpha) &\leq \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y| < \delta^{\frac{1}{\beta}}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) \mu(|y|^\beta + \tau) dy d\tau \\
&= \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y| < \delta^{\frac{1}{\beta}}} \tau^{\frac{\alpha}{\beta}-1} \tau^{-n/\beta} w^{(\beta)}(\tau^{-\frac{1}{\beta}} y, 1) \mu(|y|^\beta + \tau) dy d\tau \\
&\quad (\text{we set } y = \tau^{\frac{1}{\beta}} z, \quad dy = \tau^{\frac{n}{\beta}} dz) \\
&= \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|z| < (\frac{\delta}{\tau})^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(z, 1) \mu(\tau(|z|^\beta + 1)) dz d\tau.
\end{aligned}$$

Since $\mu(s)$ is a function of type of modulus of continuity, we have

$$\mu(\tau(|z|^\beta + 1)) \leq (|z|^\beta + 2)\mu(\tau).$$

Therefore, for $0 < \delta < 1$ we have

$$i'_1(\alpha) \leq \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \tau^{\frac{\alpha}{\beta}-1} \mu(\tau) \left(\int_{|z| < (\frac{\delta}{\tau})^{1/\beta}} w^{(\beta)}(z, 1) (|z|^\beta + 2) dz \right) d\tau.$$

Since

$$\int_{|z| \leq 1} w^{(\beta)}(z, 1) (|z|^\beta + 2) dz \equiv c_1(\delta, \beta) < \infty,$$

it follows that

$$\begin{aligned}
&\int_{|z| < (\frac{\delta}{\tau})^{1/\beta}} w^{(\beta)}(z, 1) (|z|^\beta + 2) dz = \int_{|z| \leq 1} w^{(\beta)}(z, 1) (|z|^\beta + 2) dz \\
&+ \int_{1 < |z| < (\frac{\delta}{\tau})^{1/\beta}} w^{(\beta)}(z, 1) (|z|^\beta + 2) dz = c_1(\delta, \beta) + \int_{1 < |z| < (\frac{\delta}{\tau})^{1/\beta}} w^{(\beta)}(z, 1) (|z|^\beta + 2) dz,
\end{aligned}$$

and therefore,

$$\begin{aligned}
i'_1(\alpha) &\leq \frac{c_1(\delta, \beta)}{\Gamma(\alpha/\beta)} \int_0^\delta \tau^{\frac{\alpha}{\beta}-1} \mu(\tau) d\tau \\
&+ \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \tau^{\frac{\alpha}{\beta}-1} \mu(\tau) \left(\int_{1 < |z| < (\frac{\delta}{\tau})^{1/\beta}} w^{(\beta)}(z, 1) (|z|^\beta + 2) dz \right) d\tau. \quad (2.23)
\end{aligned}$$

Further,

$$\begin{aligned} \int_{1 < |z| < (\frac{\delta}{\tau})^{1/\beta}} w^{(\beta)}(z, 1)(|z|^\beta + 2)dz &\stackrel{(1.5)}{\leq} c_2(\delta, \beta) \int_{1 < |z| < (\frac{\delta}{\tau})^{1/\beta}} |z|^{-n-\beta}(|z|^\beta + 2)dz \\ &\leq c_3(\delta, \beta) \int_{1 < |z| < (\frac{\delta}{\tau})^{1/\beta}} |z|^{-n}dz = c_4(\delta, \beta) \ln \left(\frac{\delta}{\tau} \right), \quad (0 < \tau < \delta). \end{aligned}$$

Using this in (2.23), for $\alpha \rightarrow 0^+$ we have

$$\begin{aligned} i'_1(\alpha) &\leq \frac{c_1(\delta, \beta)}{\Gamma(\alpha/\beta)} \int_0^\delta \tau^{\frac{\alpha}{\beta}-1} \mu(\tau) d\tau + \frac{c_4(\delta, \beta)}{\Gamma(\alpha/\beta)} \int_0^\delta \tau^{\frac{\alpha}{\beta}-1} \mu(\tau) \ln \frac{\delta}{\tau} d\tau \\ &\leq \frac{c_5(\delta, \beta)}{\Gamma(\alpha/\beta)} \int_0^\delta \frac{\mu(\tau)}{\tau} \ln \frac{\delta}{\tau} d\tau \stackrel{(2.18)}{\leq} c_6(\delta, \beta) \alpha. \end{aligned} \quad (2.24)$$

The estimates (2.21), (2.22) and (2.24) yield that

$$i_1(\alpha) \leq c_7(\delta, \beta)(\|f\|_\infty + 1)\alpha. \quad (2.25)$$

Finally, from (2.20) and (2.25) it follows that

$$\|A_\beta^\alpha f - f\|_\infty \leq c_8(\delta, \beta)(\|f\|_p + \|f\|_\infty + 1)\alpha, \quad \text{as } \alpha \rightarrow 0^+.$$

b) By taking into account (2.21), we have

$$\begin{aligned} i'_1(\alpha) &\equiv \frac{1}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y| < \delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) \|f(x-y, t-\tau) - f(x, t)\|_\infty dy d\tau \\ &\leq \frac{c}{\Gamma(\alpha/\beta)} \int_0^\delta \int_{|y| < \delta^{1/\beta}} \tau^{\frac{\alpha}{\beta}-1} w^{(\beta)}(y, \tau) (|y|^\beta + \tau)^\lambda |\log(|y|^\beta + \tau)|^\gamma dy d\tau \\ &\quad \text{(we assume } 0 < \delta < \frac{1}{e} \text{ and set } y = \tau^{\frac{1}{\beta}} z, \quad dy = \tau^{\frac{n}{\beta}} dz) \\ &\leq \frac{c}{\Gamma(\alpha/\beta)} \int_0^\delta \tau^{\frac{\alpha}{\beta}-1+\lambda} |\log \tau|^\gamma \int_{\mathbb{R}^n} w^{(\beta)}(z, 1)(|z|^\beta + 1)^\lambda \left(1 + \frac{\log(|z|^\beta + 1)}{|\log \tau|}\right)^\gamma dz d\tau. \end{aligned}$$

Since $0 < \tau < \delta < \frac{1}{e}$, we have $\frac{1}{|\log \tau|} < 1$ and therefore, by (1.5)

$$\begin{aligned} i'_1(\alpha) &\leq \frac{c_1(\delta, \beta)}{\Gamma(\alpha/\beta)} \int_0^\delta \tau^{\frac{\alpha}{\beta}-1+\lambda} |\log \tau|^\gamma d\tau \\ &\quad \times \int_{|z| \geq 1} |z|^{-n-\beta} (|z|^\beta + 1)^\lambda (1 + \log(|z|^\beta + 1))^\gamma dz \\ &\leq \frac{c_2(\delta, \beta)}{\Gamma(\alpha/\beta)} \int_0^\delta \tau^{-1+\lambda} |\log \tau|^\gamma d\tau \int_{|z| \geq 1} |z|^{-n-\beta+\beta\lambda} (\log |z|)^\gamma dz. \end{aligned} \quad (2.26)$$

Since $\lambda > 0$, the first integral is finite for all $\gamma \geq 0$.

Let us estimate the second integral. We have

$$\int_{|z| \geq 1} |z|^{-n-\beta+\beta\lambda} (\log |z|)^\gamma dz = c_3(\delta, \beta) \int_1^\infty r^{\beta(\lambda-1)-1} (\log r)^\gamma dr.$$

Since $0 < \lambda < 1$, the latter integral is finite for all $\gamma \geq 0$. Now it follows from (2.26) that

$$i'_1(\alpha) \leq c_4(\delta, \beta)\alpha, \quad \text{as } \alpha \rightarrow 0^+. \quad (2.27)$$

Finally, the desired result follows from (2.19), (2.20), (2.21) and (2.27). $\square$

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