

The influence of SNS-permutability of some subgroups on the structure of finite groups

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Abstract. The following concept is introduced: a subgroup H of the group G is said to be SNS-permutable (Subnormal-Sylow-permutable) in G if there is a subnormal subgroup B of G such that $HB = G$ and H permutes with every Sylow subgroup of B . Groups with certain SNS-permutable subgroups of prime power order are studied.

1. Introduction

All groups considered in this paper will be finite; the notation and terminology used in this paper are standard, as in [8]–[10] or [16]. Given a group G , two subgroups H and K of G are said to permute if $HK = KH$, that is, HK is a subgroup of G . A subgroup H of G is said to be S-permutable in G if H permutes with every Sylow subgroup of G . This concept was introduced by KEGEL and DESKINS in 1962 and has been investigated by many authors, for example, see [1]–[7], [11]–[15], [17]–[25]. In 1998, BALLESTER-BOLINCHES and PEDRAZA-AGUILERA extended this concept to S-quasinormally embedded subgroups. A subgroup H of G is S-quasinormally embedded in G if for every Sylow subgroup P of H , there is a S-quasinormal subgroup K in G such that P is also a Sylow subgroup of K . Recently, in [21], SKIBA introduced the concept of weakly

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S-permutable subgroup. In [12]–[13], LI, SHEN, and other other authors gave the following definition:

Definition 1.1. Let G be a group. A subgroup H of G is said to be an SS-quasinormal subgroup (supplement-Sylow-quasinormal subgroup) of G if there is a supplement B of H in G such that H permutes with every Sylow subgroup of B .

In this paper, we consider another generalization of S-permutable subgroup and give the following definition:

Definition 1.2. Let G be a group. A subgroup H of G is said to be an SNS-permutable subgroup (Subnormal-Sylow-permutable subgroup) of G if there is a subnormal subgroup B such that $HB = G$ and H permutes with every Sylow subgroup of B .

Obviously, every S-permutable subgroup of G is SNS-permutable and every SNS-permutable subgroup is SS-quasinormal. In general, an SNS-permutable subgroup need not be S-permutable. For instance, S_3 is an SNS-permutable subgroup of the symmetric group S_4 , but S_3 is not S-permutable. Moreover, an SS-quasinormal subgroup need not be SNS-permutable. For instance, S_4 is an SS-quasinormal subgroup of $PSL(2, 7)$, but S_4 is not SNS-permutable in $PSL(2, 7)$.

Recall that a formation is a class $\mathcal{F}$ of groups satisfying the following conditions: (i) if $G \in \mathcal{F}$ and $N \trianglelefteq G$, then $G/N \in \mathcal{F}$, and (ii) if $N_1, N_2 \trianglelefteq G$ are such that $G/N_1, G/N_2 \in \mathcal{F}$, then $G/(N_1 \cap N_2) \in \mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$.

We study the influence of the SNS-permutable subgroups on the structure of group G . The main results are as follows:

Theorem 1.1. *Let p be the smallest prime dividing the order of a group G and P a Sylow p -subgroup of G . If P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) not having a supersolvable supplement in G are SNS-permutable in G , then G is p -nilpotent.*

Theorem 1.2. *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are SNS-permutable in G . Then $G \in \mathcal{F}$.*

2. Preliminaries

Our first result is very useful in proofs using induction arguments. Its proof is a routine checking.

Lemma 2.1. *Suppose that H is SNS-permutable in a group G , $K \leq G$ and N a normal subgroup of G . We have:*

- (i) *If $H \leq K$, then H is SNS-permutable in K ;*
- (ii) *HN/N is SNS-permutable in G/N ;*
- (iii) *If $N \leq K$ and K/N is SNS-permutable in G/N , then K is SNS-permutable in G .*

Lemma 2.2. *Suppose that H is a p -subgroup for some prime p and H is not S -permutable in G . Assume that H is SNS-permutable in G . Then G has a normal subgroup M such that $|G : M| = p$ and $G = HM$.*

PROOF. By hypothesis G has a subnormal subgroup T such that $HT = G$ and $T \cap H < H$. Hence G has a proper normal subgroup K such that $T \leq K$. Since G/K is a p -group, G has a normal maximal subgroup M such that $HM = G$ and $|G : M| = p$. $\square$

Lemma 2.3. *Let H be a p -subgroup of G . Then the following statements are equivalent:*

- (i) *H is S -permutable in G ;*
- (ii) *$H \leq O_p(G)$ and H is SNS-permutable in G ;*
- (iii) *$H \leq O_p(G)$ and H is SS-quasinormal in G .*

PROOF. We only need to prove that (iii) implies (i). As $H \leq O_p(G)$, it is clear that H permutes with all Sylow p -subgroup of G . By the hypothesis, there is a subgroup $B \leq G$ such that $G = HB$ and $HX = XH$ for all $X \in \text{Syl}(B)$. In particular, if $X = Q \in \text{Syl}_q(B)$, $q \neq p$, then $HQ = QH$. Notice that Q is a Sylow q -subgroup of G . Assume T is another Sylow q -subgroup of G . Then $T = Q^g$ with $g \in G$. Moreover, $g = bh$ with $b \in B$; $h \in H$. Thus $T = Q^g = (Q^b)^h$. As Q^b is another Sylow q -subgroup of B , by the hypothesis, HQ^b is a subgroup of G and from here $H^h(Q^b)^h = HT$ is a subgroup of G . Consequently H permutes with all Sylow q -subgroups of G . Because this holds for all primes $q \neq p$, we have H is S -permutable in G . $\square$

Lemma 2.4. *Let N be an elementary abelian normal subgroup of a group G . Assume that N has a subgroup D such that $1 < |D| < |N|$ and every subgroup H*

of N satisfying $|H| = |D|$ is SNS-permutable in G . Then some maximal subgroup of N is normal in G .

PROOF. It follows from Lemma 2.11 of [21] and Lemma 2.3. $\square$

Lemma 2.5. *Let $\mathcal{F}$ be a saturated formation containing all nilpotent groups and let G be a group with solvable $\mathcal{F}$ -residual $P = G^{\mathcal{F}}$. Suppose that every maximal subgroup of G not containing P belongs to $\mathcal{F}$. Then P is a p -group for some prime p . In addition, if every cyclic subgroup of P with prime order or order 4 (if $p = 2$ and P is non-abelian) not having a supersolvable supplement in G is SNS-permutable in G , then $|P/\Phi(P)| = p$.*

PROOF. By Lemma 2.12 of [21] and Lemma 2.3. $\square$

Lemma 2.6 ([10]). *Let G be a group and M a subgroup of G .*

- (i) *If M is normal in G , then $F^*(M) \leq F^*(G)$.*
- (ii) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = \text{soc}(F(G)C_G(F(G)))/F(G)$.*
- (iii) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*
- (iv) *Suppose K is a subgroup of G contained in $Z(G)$, then $F^*(G/K) = F^*(G)/K$.*

3. Proofs of the main Theorems

PROOF OF THEOREM 1.1. Assume that the theorem is not true and let G be a counterexample of minimal order. We prove the theorem by the following steps.

- (1) $O_{p'}(G) = 1$.

In fact, if $O_{p'}(G) \neq 1$, then we consider the quotient group $G/O_{p'}(G)$. By Lemma 2.1, $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. Thus it follows that $G/O_{p'}(G)$ is p -nilpotent by the choice of G . Hence G is p -nilpotent, a contradiction.

- (2) $|D| > p$.

If $|D| = p$, then by Lemma 2.1, G is a minimal non- p -nilpotent group, so $G = [P]Q$, where P , Q are the Sylow p -subgroup and a Sylow q -subgroup of G , respectively. Set $\Phi = \Phi(P)$ and let X/Φ be a subgroup of P/Φ of order p , $x \in X \setminus \Phi$ and $L = \langle x \rangle$. Then L is order p or 4. By the hypotheses, L has a supersolvable supplement in G or is SNS-permutable in G . If L has a supersolvable supplement T in G , then $T \neq G$. So $|G/\Phi : T\Phi/\Phi| = p$. Hence $T\Phi/\Phi \trianglelefteq G/\Phi$

and $P/\Phi \cap T\Phi/\Phi = 1$, it follows that $|P/\Phi| = p$. Therefore P is cyclic and G is p -nilpotent, a contradiction. So L is SNS-permutable in G . By Lemma 2.3, L is S-permutable in G . Moreover, Lemma 2.5 implies that $|P/\Phi| = p$. Consequently, it follows that G is p -nilpotent.

(3) $|P : D| > p$.

If $|P : D| = p$, then by [14, Theorem 1.1], we have that G is p -nilpotent, a contradiction.

(4) All subgroups of P of order $|D|$ and $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) have supersolvable supplement subgroups in G or are S-permutable in G .

Let $H \leq P$ with $|H| = |D|$ or $2|D|$. Assume H has not a supersolvable supplement, therefore it is SNS-permutable in G and it is not S-permutable in G , by Lemma 2.2, there is a normal subgroup M of G such that $|G : M| = p$ and $G = HM$. By (3) and the minimality of G , M is p -nilpotent, and it follows that G is p -nilpotent, a contradiction.

(5) If $N \leq P$ and N is a minimal normal subgroup of G , then $|N| \leq |D|$.

Suppose $|N| > |D|$. Since $N \leq O_p(G)$, N is an elementary abelian group. If a subgroup H of N of order $|D|$ has a supersolvable supplement T in G , then $G = HT = NT$. Hence $N \cap T \leq G$. By minimality of N , we have that $N \cap T = 1$ or $N \cap T = N$. If $N \cap T = 1$, then $N = N \cap HT = H(N \cap T) = H$, a contradiction. Thus $N \cap T = N$ and $G = NT = T$, this is also a contradiction. Hence all subgroups of N of order $|D|$ are SNS-permutable. By Lemma 2.2, some maximal subgroup N_1 of N is normal in G . It follows from the minimality of N that $N_1 = 1$, thus $|N| = |D| = p$, a contradiction.

(6) If $N \leq P$ and N is a minimal normal subgroup of G , then G/N is p -nilpotent.

Suppose $|N| < |D|$. By Lemma 2.1 and the minimality of G , G/N is p -nilpotent. By (5), we have $|N| = |D|$. Let $N \leq K \leq P$ with $|K/N| = p$. By (2), N is non-cyclic, so K is also non-cyclic, it follows that K has a maximal subgroup $L \neq N$ and $K = LN$. If L has a supersolvable supplement in G , then K has a supersolvable supplement in G and G/N would be p -nilpotent. So we may assume that L is S-permutable in G , and then $K/N = LN/N$ is S-permutable in G/N . If P/N is abelian, then G/N satisfies the hypothesis. Next suppose that P/N is a non-abelian 2-group. Hence every subgroup of P of order $2|D|$ not having a supersolvable supplement in G is S-permutable in G . In this case one can show as above that every subgroup X of P containing N and such that $|X : N| = 4$

either has a supersolvable supplement in G or is S-permutable in G . Therefore G/N also satisfies the hypothesis.

(7) $O_p(G) = 1$.

If $O_p(G) \neq 1$, then we can find a minimal normal subgroup N of G contained in $O_p(G)$. By (6), there exists a unique minimal normal subgroup of G , N say (notice that p -nilpotent groups are a saturated formation). Moreover N is not contained in $\Phi(G)$. Therefore $N = O_p(G)$ and there is a maximal subgroup M of G such that $G = NM$, $M \cap N = 1$.

Then by (4) every subgroup H of P satisfying $|H| = |D|$ and not having a supersolvable supplement in G is S-permutable. Since every S-permutable subgroup of G is contained in $O_p(G) = N$, it follows that every subgroup H of P different from N satisfying $|H| = |D|$ has a supersolvable supplement in G . Therefore every maximal subgroup of P has a supersolvable supplement in G , which contradicts Lemma 2.2 of [21]. Thus we have (7).

(8) The final contradiction.

Let H be a subgroup of P of order $|D|$. If H is S-permutable, then $H \leq O_p(G) = 1$, a contradiction. Therefore all subgroups of P of order $|D|$ have supersolvable supplement in G and by Lemma 2.2 of [21], G is p -nilpotent, a contradiction. $\square$

Corollary 3.1. *Let G be a group. If, for every prime p dividing the order of G and $P \in \text{Syl}_p(G)$, P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) not having a supersolvable supplement in G are SNS-permutable in G , then G has the Sylow tower property of supersolvable type.*

Corollary 3.2. *Let p be the smallest prime dividing the order of a group G and P a Sylow p -subgroup of G . If P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) not having a supersolvable supplement in G are S-permutable in G , then G is p -nilpotent.*

Theorem 3.3. *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of E has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) not having a supersolvable supplement in G are SNS-permutable in G . Then $G \in \mathcal{F}$.*

PROOF. Suppose that the theorem is not true and let G be a counterexample of the smallest order. We have the following claims:

(1) $G/Q \in \mathcal{F}$, where Q is a Sylow q -subgroup of E and q is the largest prime dividing $|E|$.

By Lemma 2.1 and Corollary 3.1, E has the Sylow Tower property. Let q be the largest prime dividing $|E|$ and Q a Sylow q -subgroup of E . The fact that E possesses an order Sylow Tower property implies that Q is normal in E . Now Q is characteristic in E and $E \trianglelefteq G$, so $Q \trianglelefteq G$. Furthermore, $(G/Q)/(E/Q) \cong G/E \in \mathcal{F}$ and Lemma 2.1 shows that G/Q satisfies the conditions of the theorem, thus by the choice of G , $G/Q \in \mathcal{F}$.

(2) Every subgroup H of Q with order $|H| = |D|$ not having a supersolvable supplement in G is S-permutable in G .

By Lemma 2.3, we have (2).

(3) If $N \leq Q$ and N is minimal normal subgroup of G , then $G/N \in \mathcal{F}$.

If either $|N| < |D|$ or $|Q : D| = q$, it is clear. So let $|N| = |D|$ and $|Q : D| > q$. Let $N \leq K \leq Q$ where $|K/N| = q$. By Lemma 2.5, $|D| > q$, it follows that N is non-cyclic, so K is also non-cyclic. Hence K has a maximal subgroup $L \neq N$ and $K = LN$. If L has a supersolvable supplement in G then K has a supersolvable supplement in G and G/N would be supersolvable, therefore it would be an $\mathcal{F}$ -group. So L is S-permutable in G . Therefore $K/N = LN/N$ is S-permutable in G/N . Consequently, G/N satisfies the hypothesis, as desired.

(4) Final contradiction.

Let N be a minimal normal subgroup of G contained in Q . Applying (3) and the fact that $\mathcal{F}$ is a saturated formation, we obtain that N is the only minimal normal subgroup of G contained in Q and $\Phi(Q) = 1$. Moreover, $N \not\leq \Phi(G)$. Therefore, G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. On the other hand, $\Phi(Q) = 1$ implies that $Q \cap M$ is normalized by N and M , hence the uniqueness of N yields $N = Q$. But by Lemma 2.4 it is impossible, because Q is a minimal normal subgroup of G . This contradiction completes the proof of this theorem. $\square$

By Theorem 1.3 of [21] and Lemma 2.3, we have:

Corollary 3.4. *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G a group with a solvable normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F(E)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are SNS-permutable in G . Then $G \in \mathcal{F}$.*

Theorem 3.5. *Let G be a group with a normal subgroup E such that G/E is supersolvable, Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are SNS-permutable in G . Then G is supersolvable.*

PROOF. Suppose that the theorem is false and let G be a counterexample of smallest order, then we have:

(1) Every proper normal subgroup of G containing $F^*(E)$ is supersolvable.

If N is a proper normal subgroup of G containing $F^*(E)$, we have that $N/N \cap E \cong NE/E$ is supersolvable. By Lemma 2.6, $F^*(E) = F^*(F^*(E)) \leq F^*(E \cap N) \leq F^*(E)$, so $F^*(E \cap N) = F^*(E)$. By Lemma 2.1, $(N, N \cap E)$ satisfy the hypotheses of the theorem, thus the minimal choice of G implies that N is supersolvable.

(2) $E = G$, and $F^*(E) = F(G) < G$.

If $E < G$, then E is supersolvable by (1). In particular, E is solvable, so G is solvable and $F^*(E) = F(E)$. It follows that G is supersolvable by applying Corollary 3.4, a contradiction. If $F^*(G) = G$, then G is supersolvable by Theorem 3.3, a contradiction. Thus $F^*(G) < G$ and $F^*(G)$ is supersolvable by (1), it follows that $F^*(E) = F^*(G) = F(G)$ by Lemma 2.6.

(3) Final contradiction.

Let P be a Sylow p -subgroup of $F(G)$, for some prime p , and let P_1 be an arbitrary subgroup of P of order $|D|$. Then $P_1 \trianglelefteq P \trianglelefteq F(G) \trianglelefteq G$. By the hypotheses, P_1 is SNS-permutable in G . So P_1 is S-permutable in G by Lemma 2.3. Thus all subgroups of P of order $|D|$ are S-permutable in G . Applying Corollary 3.4, G is supersolvable, the final contradiction. $\square$

PROOF OF THEOREM 1.2. By Lemma 2.1, we have that all subgroups of any Sylow subgroup of order $|D|$ of $F^*(E)$ are SNS-permutable in E , so Theorem 3.5 implies that E is supersolvable. Hence $F^*(E) = F(E)$. Let P be a Sylow p -subgroup of $F(E)$, for some prime p , and let H be an arbitrary subgroup of order $|D|$ of P . Since P is normal in G , it follows that H is subnormal in G . By the hypotheses, H is SNS-permutable in G . So H is S-permutable in G by Lemma 2.3. Thus all subgroups of P of order $|D|$ are S-permutable in G . Applying Corollary 3.4, G belongs to $\mathcal{F}$. $\square$

In connection with Theorem 1.1 and 1.2 the following natural questions arise:

Remark. Whether Theorem 1.1 and Theorem 1.2 remain true if we replace SNS-permutable by SS-quasinormal or S-quasinormally embedded.

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References

- [1] J. MANUEL, A. ALEJANDRE and JOHN COSSEY BALLESTER-BOLINCHES, Permutable products of supersoluble groups, *J. Algebra* **276** (2004), 453–461.
- [2] M. ASAAD, On maximal subgroups of Sylow subgroups of finite groups, *Comm. Algebra* **26** (1998), 3647–3652.
- [3] M. ASAAD and A. A. HELIEL, On S-quasinormally embedded subgroups of finite groups, *J. Pure Appl. Algebra* **165** (2001), 129–135.
- [4] A. BALLESTER-BOLINCHES, $\mathcal{H}$ -normalizers and local definitions of saturated formations of finite groups, *Israel J. Math.* **67** (1989), 312–326.
- [5] A. BALLESTER-BOLINCHES and M.C. PEDRAZA-AGUILERA, On minimal subgroups of finite groups, *Acta Math. Hungar.* **73** (1996), 335–342.
- [6] A. BALLESTER-BOLINCHES and M. C. PEDRAZA-AGUILERA, Sufficient conditions for supersolubility of finite groups, *J. Pure Appl. Algebra* **127** (1998), 113–118.
- [7] W. E. DESKINS, On quasinormal subgroups of finite groups, *Math. Z.* **82** (1963), 125–132.
- [8] K. DOERK and T. O. HAWKES, Finite Soluble Groups, *de Gruyter, Berlin*, 1992.
- [9] D. GORENSTEIN, Finite Groups, *Harper & Row, Publishers, New York – London* 1968.
- [10] B. HUPPERT, Endliche Gruppen I, *Springer-Verlag, Berlin*, 1968.
- [11] O. KEGEL, Sylow-Gruppen und Subnormalteiler endlicher Gruppen, *Math. Z.* **78** (1962), 205–221.
- [12] SHIRONG LI, ZHENCAI SHEN, JIANJUN LIU and XIAOCHUN LIU, The influence of SS -quasinormality of some subgroups on the structure of finite groups, *J. Algebra* **319** (2008), 4275–4287.
- [13] SHIRONG LI, ZHENCAI SHEN and XIANGHONG KONG, On SS -quasinormal subgroups of finite groups, *Comm. Algebra* **36** (2008), 4436–4447.
- [14] YANGMING LI and YANMING WANG, On S -quasinormally embedded subgroups of finite group, *J. Algebra* **281** (2004), 109–123.
- [15] R. MAIER and P. SCHMID, The embedding of quasinormal subgroups in finite groups, *Math. Z.* **131** (1973), 269–272.
- [16] D. J. S. ROBINSON, A Course in the Theory of Groups, *Springer-Verlag, New York*, 1982.
- [17] P. SCHMID, Subgroups permutable with all Sylow subgroups, *J. Algebra* **207** (1998), 285–293.
- [18] A. SHAALAN, The influence of S-quasinormality of some subgroups on the structure of a finite group, *Acta Math. Hungar.* **56** (1990), 287–293.
- [19] ZHENCAI SHEN, SHIRONG LI and WUJIE SHI, Finite groups with normally embedded subgroups, *Journal of Group Theory* **13** (2010), 257–265.
- [20] ZHENCAI SHEN, SHIRONG LI and WUJIE SHI, S -quasinormality of finite groups, *Front. Math. China* **5** (2010), 329–339.
- [21] A. N. SKIBA, On weakly S -permutable subgroups of finite groups, *J. Algebra* **315** (2007), 192–209.

- [22] A. N. SKIBA and Q. V. TITOV, Finite groups with c -quasinormal subgroups, *Siberian Math. Journal* **48** (2007), 544–554.
- [23] S. SRINIVASAN, Two sufficient conditions for the supersolvability of finite groups, *Israel J. Math.* **35** (1980), 210–214.
- [24] Y. WANG, Finite groups with some subgroups of Sylow subgroups c -supplemented, *J. Algebra* **224** (2000), 464–478.
- [25] G. WALL, Groups with maximal subgroups of Sylow subgroups normal, *Israel J. Math.* **43** (1982), 166–168.

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