

A uniqueness theorem for meromorphic maps with moving hypersurfaces

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Abstract. In this paper, we establish a uniqueness theorem for algebraically non-degenerate meromorphic maps of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and slowly moving hypersurfaces $Q_j \subset \mathbb{C}P^n$, $j = 1, \dots, q$ in (weakly) general position, where q depends effectively on n and on the degrees d_j of the hypersurfaces Q_j .

1. Introduction

One of the most striking consequences of Nevanlinna's theory was his “five values” theorem, which says that if f and g are non-constant meromorphic functions on $\mathbb{C}$ such that $f^{-1}(a_i) = g^{-1}(a_i)$ for five distinct points a_i in the extended complex plane, then $f = g$. This theorem is an example of what is now known as “uniqueness theorem”. In 1975, Fujimoto generalized this result of Nevanlinna to the case of meromorphic maps of $\mathbb{C}^m$ into $\mathbb{C}P^n$. In the last years, many uniqueness theorems for meromorphic maps with hyperplanes (both for fixed and for moving ones) have been established. For the case of hypersurfaces, however, there are so far only the uniqueness theorem of THAI and TAN [10] for the case of Fermat moving hypersurfaces and the one of DULOCK and RU [5] for the case of (general) fixed hypersurfaces. More precisely, in [5], DULOCK and RU prove that one has a uniqueness theorem for algebraically non-degenerate holomorphic maps $f, g : \mathbb{C} \rightarrow \mathbb{C}P^n$ satisfying $f = g$ on $\cup_{i=1}^q (f^{-1}(Q_i) \cup g^{-1}(Q_i))$, with respect to

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$q > (n+1) + \frac{2Mn}{\tilde{d}} + \frac{1}{2}$ fixed hypersurfaces $Q_i \subset \mathbb{C}P^n$ in general position, where $\tilde{d}$ is the minimum of the degrees of these hypersurfaces and M is the truncation level in the Second Main Theorem for fixed hypersurface targets obtained by AN-PHUONG [1] with $\epsilon = \frac{1}{2}$. Their method of proof comes from their paper [4], where they prove a uniqueness theorem for holomorphic curves into abelian varieties.

In this paper, by a method different to the one used by DULOCK and RU, we prove a uniqueness theorem for the case of slowly moving hypersurfaces (Corollary 3.1 below). More precisely, we prove that one has a uniqueness theorem for algebraically non-degenerate meromorphic maps $f, g : \mathbb{C}^m \rightarrow \mathbb{C}P^n$ satisfying $f = g$ on $\cup_{i=1}^q (f^{-1}(Q_i) \cup g^{-1}(Q_i))$ with respect to $q > (n+1) + \frac{2nL}{\tilde{d}} + \frac{1}{2}$ moving hypersurfaces $Q_i \subset \mathbb{C}P^n$ in (weakly) general position, where $\tilde{d}$ is the minimum of the degrees of these hypersurfaces and L is the truncation level in the Second Main Theorem for moving hypersurface targets obtained by the authors in [2] with $\epsilon = \frac{1}{2}$. Moreover, under the additional assumption that the $f^{-1}(Q_i)$, $i = 1, \dots, q$ intersect properly, $q > (n+1) + \frac{2L}{\tilde{d}} + \frac{1}{2}$ moving hypersurfaces are sufficient. We remark that in the special case of fixed hypersurfaces, our result gives back the uniqueness theorem of Dulock and Ru (remark that $L \leq M$ in this case). Moreover, we give our uniqueness theorem in a slightly more general form (Theorem 3.1 below), requiring assumptions on the $(p-1)$ first derivatives of the maps, which gives in return a better bound on the number of moving hypersurfaces in $\mathbb{C}P^n$, namely $q > (n+1) + \frac{2nL}{pd} + \frac{1}{2}$ respectively $q > (n+1) + \frac{2L}{pd} + \frac{1}{2}$.

2. Preliminaries

For $z = (z_1, \dots, z_m) \in \mathbb{C}^m$, we set $\|z\| = (\sum_{j=1}^m |z_j|^2)^{1/2}$ and define

$$B(r) = \{z \in \mathbb{C}^m : \|z\| < r\}, \quad S(r) = \{z \in \mathbb{C}^m : \|z\| = r\},$$

$$d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial), \quad \mathcal{V} = (dd^c \|z\|^2)^{m-1}, \quad \sigma = d^c \log \|z\|^2 \wedge (dd^c \log \|z\|)^{m-1}.$$

Let L be a positive integer or $+\infty$ and ν be a divisor on $\mathbb{C}^m$. Set $|\nu| = \{z : \nu(z) \neq 0\}$. We define the counting function of ν by

$$N_\nu^{(L)}(r) := \int_1^r \frac{n^{(L)}(t)}{t^{2m-1}} dt \quad (1 < r < +\infty),$$

where

$$\begin{aligned} n^{(L)}(t) &= \int_{|\nu| \cap B(t)} \min\{\nu, L\} \cdot \mathcal{V} & \text{for } m \geq 2 \quad \text{and} \\ n^{(L)}(t) &= \sum_{|z| \leq t} \min\{\nu(z), L\} & \text{for } m = 1. \end{aligned}$$

Let F be a nonzero holomorphic function on $\mathbb{C}^m$. For a set $\alpha = (\alpha_1, \dots, \alpha_m)$ of nonnegative integers, we set $|\alpha| := \alpha_1 + \dots + \alpha_m$ and $\mathcal{D}^\alpha F := \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} z_1 \dots \partial^{\alpha_m} z_m} \cdot F$. We define the zero divisor ν_F of F by

$$\nu_F(z) = \max\{p : \mathcal{D}^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < p\}.$$

Let φ be a nonzero meromorphic function on $\mathbb{C}^m$. The zero divisor ν_φ of φ is defined as follows: For each $a \in \mathbb{C}^m$, we choose nonzero holomorphic functions F and G on a neighborhood U of a such that $\varphi = \frac{F}{G}$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq m - 2$, then we put $\nu_\varphi(a) := \nu_F(a)$.

Set $N_\varphi^{(L)}(r) := N_{\nu_\varphi}^{(L)}(r)$. For brevity we will omit the character $^{(L)}$ in the counting function if $L = +\infty$.

Let f be a meromorphic map of $\mathbb{C}^m$ into $\mathbb{C}P^n$. For arbitrary fixed homogeneous coordinates $(w_0 : \dots : w_n)$ of $\mathbb{C}P^n$, we take a reduced representation $f = (f_0 : \dots : f_n)$, which means that each f_i is a holomorphic function on $\mathbb{C}^m$ and $f(z) = (f_0(z) : \dots : f_n(z))$ outside the analytic set $\{z : f_0(z) = \dots = f_n(z) = 0\}$ of codimension ≥ 2 . Set $\|f\| = \max\{|f_0|, \dots, |f_n|\}$.

The characteristic function of f is defined by

$$T_f(r) := \int_{S(r)} \log \|f\| \sigma - \int_{S(1)} \log \|f\| \sigma, \quad 1 < r < +\infty.$$

For a meromorphic function φ on $\mathbb{C}^m$, the characteristic function $T_\varphi(r)$ of φ is defined by considering φ as a meromorphic map of $\mathbb{C}^m$ into $\mathbb{C}P^1$.

Let f be a nonconstant meromorphic map of $\mathbb{C}^m$ into $\mathbb{C}P^n$. We say that a meromorphic function φ on $\mathbb{C}^m$ is “small” with respect to f if $T_\varphi(r) = o(T_f(r))$ as $r \rightarrow \infty$ (outside a set of finite Lebesgue measure).

Denote by $\mathcal{M}$ the field of all meromorphic functions on $\mathbb{C}^m$ and by $\mathcal{K}_f$ the subfield of $\mathcal{M}$ which consists of all “small” (with respect to f) meromorphic functions on $\mathbb{C}^m$.

For a homogeneous polynomial $Q \in \mathcal{M}[x_0, \dots, x_n]$ of degree $d \geq 1$ we write $Q = \sum_{I \in \mathcal{T}_d} a_I x^I$, where $\mathcal{T}_d := \{(i_0, \dots, i_n) \in \mathbb{N}_0^{n+1} : i_0 + \dots + i_n = d\}$ and $x^I = x_0^{i_0} \dots x_n^{i_n}$ for $x = (x_0, \dots, x_n)$ and $I = (i_0, \dots, i_n) \in \mathcal{T}_d$. Denote by $Q(z) =$

$Q(z)(x_0, \dots, x_n) = \sum_{I \in \mathcal{T}_d} a_I(z)x^I$ the homogeneous polynomial over $\mathbb{C}$ obtained by evaluating the coefficients of Q at a specific point $z \in \mathbb{C}^m$ in which all coefficient functions of Q are holomorphic.

Let $Q \in \mathcal{M}[x_0, \dots, x_n]$ of degree $d \geq 1$ with $Q(f) := Q(f_0, \dots, f_n) \not\equiv 0$. We define

$$N_f^{(L)}(r, Q) := N_{Q(f)}^{(L)}(r) \quad \text{and} \quad f^{-1}(Q) := \{z : \nu_{Q(f)} > 0\}.$$

The *First Main Theorem* of Nevanlinna theory gives, for $Q = \sum_{I \in \mathcal{T}_d} a_I x^I$ with $Q(f) := Q(f_0, \dots, f_n) \not\equiv 0$:

$$N(r, Q) \leq d \cdot T_f(r) + O\left(\sum_{I \in \mathcal{T}_d} T_{a_I}(r)\right).$$

Let

$$Q_j = \sum_{I \in \mathcal{T}_{d_j}} a_{jI} x^I \quad (j = 1, \dots, q)$$

be homogeneous polynomials in $\mathcal{K}_f[x_0, \dots, x_n]$ with $\deg Q_j = d_j \geq 1$. Denote by $\mathcal{K}_{\{Q_j\}_{j=1}^q}$ the field over $\mathbb{C}$ of all meromorphic functions on $\mathbb{C}^m$ generated by all quotients $\left\{\frac{a_{jI_1}}{a_{jI_2}} : a_{jI_2} \neq 0, I_1, I_2 \in \mathcal{T}_{d_j}; j \in \{1, \dots, q\}\right\}$. We say that f is algebraically nondegenerate over $\mathcal{K}_{\{Q_j\}_{j=1}^q}$ if there is no nonzero homogeneous polynomial $Q \in \mathcal{K}_{\{Q_j\}_{j=1}^q}[x_0, \dots, x_n]$ such that $Q(f_0, \dots, f_n) \equiv 0$.

We say that a set $\{Q_j\}_{j=1}^q$ ($q \geq n+1$) of homogeneous polynomials in $\mathcal{K}_f[x_0, \dots, x_n]$ is admissible (or in (weakly) general position) if there exists $z \in \mathbb{C}^m$ in which all coefficient functions of all Q_j , $j = 1, \dots, q$ are holomorphic and such that for any $1 \leq j_0 < \dots < j_n \leq q$ the system of equations

$$\begin{cases} Q_{j_i}(z)(x_0, \dots, x_n) = 0 \\ 0 \leq i \leq n \end{cases} \quad (2.1)$$

has only the trivial solution $(x_0, \dots, x_n) = (0, \dots, 0)$ in $\mathbb{C}^{n+1}$. We remark that in this case this is true for the generic $z \in \mathbb{C}^m$.

In order to prove our result for (weakly) general position (under the stronger assumption of pointwise general position this can be avoided), we finally will need some classical results on resultants, see LANG [8], section IX.3, for the precise definition, the existence and for the principal properties of resultants, as well as EREMENKO-SODIN [6], page 127: Let $\{Q_j\}_{j=0}^n$ be a set of homogeneous polynomials of common degree $d \geq 1$ in $\mathcal{K}_f[x_0, \dots, x_n]$

$$Q_j = \sum_{I \in \mathcal{T}_d} a_{jI} x^I, \quad a_{jI} \in \mathcal{K}_f \quad (j = 0, \dots, n).$$

Let $T = (\dots, t_{kI}, \dots)$ ($k \in \{0, \dots, n\}$, $I \in \mathcal{T}_d$) be a family of variables. Set

$$\tilde{Q}_j = \sum_{I \in \mathcal{T}_d} t_{jI} x^I \in \mathbb{Z}[T, x], \quad j = 0, \dots, n.$$

Let $\tilde{R} \in \mathbb{Z}[T]$ be the resultant of $\tilde{Q}_0, \dots, \tilde{Q}_n$. This is a polynomial in the variables $T = (\dots, t_{kI}, \dots)$ ($k \in \{0, \dots, n\}$, $I \in \mathcal{T}_d$) with integer coefficients, such that the condition $\tilde{R}(T) = 0$ is necessary and sufficient for the existence of a nontrivial solution $(x_0, \dots, x_n) \neq (0, \dots, 0)$ in $\mathbb{C}^{n+1}$ of the system of equations

$$\begin{cases} \tilde{Q}_j(T)(x_0, \dots, x_n) = 0 \\ 0 \leq i \leq n. \end{cases} \quad (2.2)$$

From equations (2.2) and (2.1) it follows immediately that if

$$\{Q_j = \tilde{Q}_j(a_{jI})(x_0, \dots, x_n), \quad j = 0, \dots, n\}$$

is an admissible set,

$$R := \tilde{R}(\dots, a_{kI}, \dots) \neq 0. \quad (2.3)$$

Furthermore, since $a_{kI} \in \mathcal{K}_f$, we have $R \in \mathcal{K}_f$. We finally will use the following result on resultants, which is contained in Theorem 3.4 in [8] (see also EREMENKO-SODIN [6], page 127, for a similar result):

Proposition 1. *There exists a positive integer s and polynomials $\{\tilde{b}_{ij}\}_{0 \leq i, j \leq n}$ in $\mathbb{Z}[T, x]$, which are (without loss of generality) zero or homogenous in x of degree $s - d$, such that*

$$x_i^s \cdot \tilde{R} = \sum_{j=0}^n \tilde{b}_{ij} \tilde{Q}_j \quad \text{for all } i \in \{0, \dots, n\}.$$

If we still set

$$b_{ij} = \tilde{b}_{ij}((\dots, a_{kI}, \dots), (f_0, \dots, f_n)), \quad 0 \leq i, j \leq n,$$

we get

$$f_i^s \cdot R = \sum_{j=0}^n b_{ij} \cdot Q_j(f_0, \dots, f_n) \quad \text{for all } i \in \{0, \dots, n\}. \quad (2.4)$$

In particular, if $D \subset \mathbb{C}^m$ is a divisor contained in all divisors $f^{-1}(Q_j)$, $j = 0, \dots, n$, then R vanishes on D : This follows from (2.4) since $f = (f_0 : \dots : f_n)$ is a reduced representation (and it follows in principle already directly from the definition of the resultant).

3. Main result

Let f, g be nonconstant meromorphic maps of $\mathbb{C}^m$ into $\mathbb{C}P^n$. Let $\{Q_j\}_{j=1}^q$ be an admissible set of homogeneous polynomials in $\mathcal{K}_f[x_0, \dots, x_n]$ with $\deg Q_j = d_j \geq 1$. Denote by $d, d^*, \tilde{d}$ respectively the least common multiple, the maximum number and the minimum number of the d_j 's. Put $N = d \cdot (4(n+1)(2^n-1)(nd+1) + n+1)$. Set $t_{\{Q_j\}_{j=1}^q} = 1$ if the field $\mathcal{K}_{\{Q_j\}_{j=1}^q}$ coincides with the complex number field $\mathbb{C}$ (ie. all Q_j are fixed hypersurface targets) and

$$t_{\{Q_j\}_{j=1}^q} = \left(\binom{n+N}{n}^2 \cdot \binom{q}{n} + \left[\frac{\left(\binom{n+N}{n}^2 \cdot \binom{q}{n} - 1 \right) \cdot \log \left(\binom{n+N}{n}^2 \cdot \binom{q}{n} \right)}{\log \left(1 + \frac{1}{4 \binom{n+N}{n} N} \right)} + 1 \right]^2 \right)^{\binom{n+N}{n}^2 \cdot \binom{q}{n} - 1}$$

if $\mathcal{K}_{\{Q_j\}_{j=1}^q} \neq \mathbb{C}$, where we denote $[x] := \max\{k \in \mathbb{Z} : k \leq x\}$ for a real number x .

Let $L = \left\lceil \frac{d^* \cdot \binom{n+N}{n}^{t_{\{Q_j\}_{j=1}^q}} - d^*}{d} + 1 \right\rceil$.

With these notations, we state our main result:

Theorem 3.1. a) Assume that f, g are algebraically nondegenerate over $\mathcal{K}_{\{Q_j\}_{j=1}^q}$ and satisfy

- i) $\mathcal{D}^\alpha \left(\frac{f_k}{f_s} \right) = \mathcal{D}^\alpha \left(\frac{g_k}{g_s} \right)$ on $(\cup_{i=1}^q (f^{-1}(Q_i) \cup g^{-1}(Q_i))) \setminus (\text{Zero}(f_s \cdot g_s))$, for all $|\alpha| < p$, $0 \leq k \neq s \leq n$, where p is a positive integer and $(f_0 : \dots : f_n)$, $(g_0 : \dots : g_n)$ are reduced representations of f, g respectively.

Then for $q > n + \frac{2nL}{pd} + \frac{3}{2}$, we have $f \equiv g$.

- b) Assume that f, g as in a) satisfy i) and

- ii) $\dim(f^{-1}(Q_i) \cap f^{-1}(Q_j)) \leq m-2$ for all $1 \leq i < j \leq q$.

Then for $q > n + \frac{2L}{pd} + \frac{3}{2}$, we have $f \equiv g$.

We note that if $p = 1$ the condition i) becomes the following usual condition: $f = g$ on $\cup_{i=1}^q (f^{-1}(Q_i) \cup g^{-1}(Q_i))$, and we state this case again explicitly because of its importance:

Corollary 3.1. a) Assume that f, g are algebraically nondegenerate over $\mathcal{K}_{\{Q_j\}_{j=1}^q}$ and satisfy

- i) $f = g$ on $\cup_{i=1}^q (f^{-1}(Q_i) \cup g^{-1}(Q_i))$.

Then for $q > n + \frac{2nL}{d} + \frac{3}{2}$, we have $f \equiv g$.

- b) Assume that f, g as in a) satisfy i) and
- ii) $\dim(f^{-1}(Q_i) \cap f^{-1}(Q_j)) \leq m-2$ for all $1 \leq i < j \leq q$.
- Then for $q > n + \frac{2L}{d} + \frac{3}{2}$, we have $f \equiv g$.

In order to prove Theorem 3.1, we need the following two results. The first one is similar to Lemma 5.1 in Ji [7], the second one is a special case of our main result in [2].

Proposition 2. *Let $A_1, \dots, A_k$ be pure $(m-1)$ -dimensional analytic subsets of $\mathbb{C}^m$. Let f_1, f_2 be meromorphic maps of $\mathbb{C}^m$ into $\mathbb{C}P^n$. Then there exists a dense subset $\mathcal{C} \subset \mathbb{C}^{n+1} \setminus \{0\}$ such that for any $c = (c_0, \dots, c_n) \in \mathcal{C}$ the hyperplane H_c defined by $c_0 w_0 + \dots + c_n w_n = 0$ satisfies: $\dim(\cup_{j=1}^k A_j \cap f_i^{-1}(H_c)) \leq m-2$, $i \in \{1, 2\}$.*

PROOF OF PROPOSITION 2. For any irreducible pure $(m-1)$ -dimensional component σ of $\cup_{j=1}^k A_j$ we set

$$K_\sigma^i = \left\{ (t_0, \dots, t_n) \in \mathbb{C}^{n+1} : \sum_{s=0}^n t_s f_{is} = 0 \text{ on } \sigma \right\}, \quad i \in \{1, 2\},$$

where $(f_{i0} : \dots : f_{in})$ are reduced representations of f_i . Then K_σ^i is a complex vector subspace of $\mathbb{C}^{n+1}$. Since $\dim\{f_{i0} = \dots = f_{in} = 0\} \leq m-2$, we get that $\sigma \setminus \cup_{i \in \{1, 2\}} \{f_{i0} = \dots = f_{in} = 0\} \neq \emptyset$. This implies that $\dim K_\sigma^i \leq n$. Let $K = \cup_{i \in \{1, 2\}} \cup_\sigma K_\sigma^i$, then K is a union of at most a countable number of at most n -dimensional complex vector subspaces in $\mathbb{C}^{n+1}$. Let $\mathcal{C} = \mathbb{C}^{n+1} \setminus K$. Then $\mathcal{C}$ meets the requirement of the Proposition. $\square$

Theorem 3.2. *Under the same assumption as in Theorem 3.1, we have*

$$\left(q - n - \frac{3}{2}\right) T_f(r) \leq \sum_{j=1}^q \frac{1}{d_j} N_f^{(L)}(r, Q_j),$$

for all $r \in [1, +\infty)$ excluding a Borel subset E of $[1, +\infty)$ with $\int_E dr < +\infty$.

PROOF OF THEOREM 3.2. This is the special case of the Main Theorem and Proposition 1.2. in [2] for $\epsilon = \frac{1}{2}$ and where we estimate the different d_j 's in the numerators of the expressions entering into the truncation level L by d^* . $\square$

PROOF OF THEOREM 3.1. Assume that $f \neq g$. We first prove the following

Claim: There exist (fixed) hyperplanes $H_i : a_{i0} w_0 + \dots + a_{in} w_n = 0$ ($i = 1, 2$) in $\mathbb{C}P^n$ such that $S = S_{H_1, H_2}(f, g) := \frac{H_1(f)}{H_2(f)} - \frac{H_1(g)}{H_2(g)} \neq 0$ and

$$\dim(f^{-1}(Q_j) \cap f^{-1}(H_i)) \leq m-2, \quad \dim(g^{-1}(Q_j) \cap g^{-1}(H_i)) \leq m-2 \quad (3.1)$$

for all $j \in \{1, \dots, q\}$, $i \in \{1, 2\}$.

Proof of the Claim: By assumption i) of Theorem 3.1 we have pure $(m-1)$ -dimensional analytic sets

$$A_j := f^{-1}(Q_j) = g^{-1}(Q_j) \subset \mathbb{C}^m, \quad j = 1, \dots, q. \quad (3.2)$$

By Proposition 2 there exists a dense subset $\mathcal{C} \subset \mathbb{C}^{n+1} \setminus \{0\}$ such that for any $c = (c_0, \dots, c_n) \in \mathcal{C}$ the hyperplane H_c defined by $c_0 w_0 + \dots + c_n w_n = 0$ satisfies (3.1), that is

$$\dim(A_j \cap f^{-1}(H_c)) \leq m-2, \quad \dim(A_j \cap g^{-1}(H_c)) \leq m-2$$

for all $j \in \{1, \dots, q\}$. Since f, g are algebraically nondegenerate over $\mathcal{K}_{\{Q_j\}_{j=1}^q}$, so in particular algebraically nondegenerate over $\mathbb{C}$, we have that $L_c(f) \not\equiv 0$ and $L_c(g) \not\equiv 0$ are holomorphic functions for all $c = (c_0, \dots, c_n) \in \mathcal{C}$, where $L_c(f) := \sum_{i=0}^n c_i f_i$ with a reduced representation $f = (f_0 : \dots : f_n)$ and $L_c(g) := \sum_{i=0}^n c_i g_i$ with a reduced representation $g = (g_0 : \dots : g_n)$. Finally for $c^{(1)}, c^{(2)} \in \mathcal{C}$, we put $S_{c^{(1)}, c^{(2)}}(f, g) := \frac{L_{c^{(1)}}(f)}{L_{c^{(2)}}(f)} - \frac{L_{c^{(1)}}(g)}{L_{c^{(2)}}(g)}$. In order to prove the Claim it suffices to show that for some $c^{(1)}, c^{(2)} \in \mathcal{C}$, $S_{c^{(1)}, c^{(2)}}(f, g) \not\equiv 0$. Assume the contrary. Then for all $0 \leq i < j \leq n$ there exist sequences $(c^{(1)})_\nu, (c^{(2)})_\nu, \nu \in \mathbb{N}$, of elements in $\mathcal{C}$ such that $L_{(c^{(1)})_\nu}(f) \rightarrow f_i$ and $L_{(c^{(2)})_\nu}(f) \rightarrow f_j$. From this we get

$$0 \equiv S_{(c^{(1)})_\nu, (c^{(2)})_\nu}(f, g) \rightarrow \frac{f_i}{f_j} - \frac{g_i}{g_j},$$

what implies $0 \equiv \frac{f_i}{f_j} - \frac{g_i}{g_j}$ for all $0 \leq i < j \leq n$, contradicting the assumption $f \not\equiv g$. This proves the claim. $\square$

Since $f = g$ on $\cup_{j=1}^q f^{-1}(Q_j)$, for any generic point

$$z_0 \in \cup_{j=1}^q f^{-1}(Q_j) \setminus (f^{-1}(H_2) \cup g^{-1}(H_2))$$

(outside an analytic subset of codimension at least 2), there exists $s \in \{0, \dots, n\}$ such that both of $f_s(z_0)$ and $g_s(z_0)$ are different from zero. Then by assumption i) we have

$$\begin{aligned} \mathcal{D}^\alpha S(z_0) &= \mathcal{D}^\alpha \left(\frac{H_1(f)}{H_2(f)} - \frac{H_1(g)}{H_2(g)} \right) (z_0) \\ &= \mathcal{D}^\alpha \left(\frac{\sum_{v=0}^n \frac{f_v}{f_s} a_{1v}}{\sum_{v=0}^n \frac{f_v}{f_s} a_{2v}} - \frac{\sum_{v=0}^n \frac{g_v}{g_s} a_{1v}}{\sum_{v=0}^n \frac{g_v}{g_s} a_{2v}} \right) (z_0) = 0 \end{aligned}$$

for all $|\alpha| < p$.

This implies that

$$\nu_S \geq p \text{ on } \bigcup_{j=1}^q f^{-1}(Q_j) \setminus (A \cup f^{-1}(H_2) \cup g^{-1}(H_2)). \quad (3.3)$$

where A is an analytic subset of codimension at least 2.

Now we will estimate the divisors $\nu_{Q_j \circ f}$ by making use of the resultants: In fact, for any $J = \{j_0, \dots, j_n\} \subset \{1, 2, \dots, q\}$, let R_J be the resultant of $Q_{j_0}, \dots, Q_{j_n}$. Then if $D \subset \mathbb{C}^m$ is a divisor contained in all divisors $f^{-1}(Q_{j_k})$, $k = 0, \dots, n$, then R_J vanishes on D . Thus, we get

$$\sum_{j=1}^q \min\{1, \nu_{Q_j \circ f}\} \leq n \cdot \min\{1, \sum_{j=1}^q \nu_{Q_j \circ f}\} + (q - n) \cdot \min\{1, \sum_{|J|=n+1} \nu_{R_J}\} \quad (3.4)$$

By (3.1), (3.2), (3.3), (3.4), by the First Main Theorem and since $R_J \in \mathcal{K}_f$, we have

$$\sum_{j=1}^q N_g^{(1)}(r, Q_j) = \sum_{j=1}^q N_f^{(1)}(r, Q_j) \leq \frac{n}{p} N_S(r) + o(T_f(r)) \quad (3.5)$$

Furthermore, by the First Main Theorem

$$\begin{aligned} N_S(r) &\leq T_{\frac{H_1(f)}{H_2(f)} - \frac{H_1(g)}{H_2(g)}}(r) + O(1) \leq T_{\frac{H_1(f)}{H_2(f)}}(r) + T_{\frac{H_1(g)}{H_2(g)}}(r) + O(1) \\ &\leq T_f(r) + T_g(r) + O(1). \end{aligned} \quad (3.6)$$

Thus,

$$\sum_{j=1}^q (N_f^{(1)}(r, Q_j) + N_g^{(1)}(r, Q_j)) \leq \frac{2n}{p} (T_f(r) + T_g(r)) + o(T_f(r)). \quad (3.7)$$

By Theorem 3.2 and by the First Main Theorem, we have

$$\begin{aligned} \left(q - n - \frac{3}{2}\right) T_f(r) &\leq \sum_{j=1}^q \frac{1}{d_j} N_f^{(L)}(r, Q_j) \\ &\leq \sum_{j=1}^q \frac{L}{d_j} N_f^{(1)}(r, Q_j) = \sum_{j=1}^q \frac{L}{d_j} N_g^{(1)}(r, Q_j) \leq qLT_g(r) + o(T_f(r)) \end{aligned} \quad (3.8)$$

for all $r \in [1, +\infty)$ excluding a Borel subset E of $(1, +\infty)$ with $\int_E dr < +\infty$ (note that $Q_j \in \mathcal{K}_f[x_0, \dots, x_n]$).

This implies that $\mathcal{K}_f \subset \mathcal{K}_g$. Then $\{Q_j\}_{j=1}^q \subset \mathcal{K}_g[x_0, \dots, x_n]$. So we can apply Theorem 3.2 for both meromorphic maps f and g with moving hypersurfaces $\{Q_j\}_{j=1}^q$. By Theorem 3.2 and by the First Main Theorem, we have

$$\begin{aligned} (q - n - \frac{3}{2})(T_f(r) + T_g(r)) &\leq \sum_{j=1}^q \frac{1}{d_j} (N_f^{(L)}(r, Q_j) + N_g^{(L)}(r, Q_j)) \\ &\leq \frac{L}{\tilde{d}} \sum_{j=1}^q (N_f^{(1)}(r, Q_j) + N_g^{(1)}(r, Q_j)) \end{aligned} \quad (3.9)$$

for all $r \in [1, +\infty)$ excluding a Borel subset E of $(1, +\infty)$ with $\int_E dr < +\infty$.

Combining with (3.7), we get

$$\left(q - n - \frac{3}{2}\right) (T_f(r) + T_g(r)) \leq \frac{2nL}{p\tilde{d}} (T_f(r) + T_g(r)) + o(T_f(r)) \quad (3.10)$$

for all $r \in [1, +\infty)$ excluding a Borel subset E of $(1, +\infty)$ with $\int_E dr < +\infty$. This is a contradiction, since $q > n + \frac{2nL}{p\tilde{d}} + \frac{3}{2}$, thus finishing the proof of part a).

In order to prove b), we observe that under the additional assumption ii), we can improve (3.5), namely we get, by using (3.1), (3.2), (3.3) and assumption ii)

$$\sum_{j=1}^q N_g^{(1)}(r, Q_j) = \sum_{j=1}^q N_f^{(1)}(r, Q_j) \leq \frac{1}{p} N_S(r) \quad (3.11)$$

This improves (3.7), namely we get from (3.6) and (3.11):

$$\sum_{j=1}^q (N_f^{(1)}(r, Q_j) + N_g^{(1)}(r, Q_j)) \leq \frac{2}{p} (T_f(r) + T_g(r)) + O(1). \quad (3.12)$$

Using this (3.10) becomes, by using now (3.9) and (3.12):

$$\left(q - n - \frac{3}{2}\right) (T_f(r) + T_g(r)) \leq \frac{2L}{p\tilde{d}} (T_f(r) + T_g(r)) + O(1) \quad (3.13)$$

for all $r \in [1, +\infty)$ excluding a Borel subset E of $(1, +\infty)$ with $\int_E dr < +\infty$. This is a contradiction, since $q > n + \frac{2L}{p\tilde{d}} + \frac{3}{2}$, thus finishing the proof of part b). $\square$

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