

Approximation by q -parametric operators

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Abstract. We establish sufficient conditions to insure the convergence of a sequence of positive linear operators defined on $C[0, 1]$. As applications we obtain quantitative estimates for some q -parametric operators.

1. Introduction

The development of the q -calculus has led to the discovery of q -parametric operators. The first example in this direction was given by A. LUPAŞ in 1987. He introduced a q -analogue of the well-known Bernstein operators (see [10]), denoted by $R_{n,q}(f, x)$, where $n = 1, 2, \dots$, $q \in (0, 1)$, $f \in C[0, 1]$ and $x \in [0, 1]$. In [14], OSTROVSKA defined the limit Lupaş operator $\tilde{R}_{\infty,q}(f, x)$ and proved the convergence of the sequence $\{R_{n,q}(f, x)\}$ to $\tilde{R}_{\infty,q}(f, x)$ as $n \rightarrow \infty$, uniformly for $x \in [0, 1]$. The case $q \in (1, \infty)$ was also considered in [10] (resp. in [14]).

In 1992, L. LUPAŞ [11] and independently, in 2000, TRIF [17] considered the q -Meyer–König and Zeller operators $M_{n,q}(f, x)$, where $n = 1, 2, \dots$, $q \in (0, 1)$, $f \in C[0, 1]$ and $x \in [0, 1]$. WANG proved in [19], among others, that the sequence $\{M_{n,q}(f, x)\}$ converges to the limit q -Bernstein operator $B_{\infty,q}(f, x)$ as $n \rightarrow \infty$, uniformly for $x \in [0, 1]$.

Later, in 1997, PHILLIPS introduced a new generalization of the classical Bernstein operators, based on q -integers (see [15] and [16]). He defined the so-called q -Bernstein operators $B_{n,q}(f, x)$, where $n = 1, 2, \dots$, $q \in (0, 1)$, $f \in C[0, 1]$

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and $x \in [0, 1]$. From the properties of q -Bernstein operators, we mention the following one established by IL'INSKII and OSTROVSKA [8]: the sequence $\{B_{n,q}(f, x)\}$ converges to $B_{\infty,q}(f, x)$ as $n \rightarrow \infty$, uniformly for $x \in [0, 1]$.

Further important q -parametric operators were introduced and studied by S. LEWANOWICZ and P. WOŹNY [9], M.-M. DERRIENNIC [3], V. GUPTA [6], A. ARAL [1], V. GUPTA and H. WANG [7], A. ARAL and V. GUPTA [2], and G. NOWAK [12], respectively.

Let (L_n) be a sequence of positive linear operators such that $L_n : C[0, 1] \rightarrow C[0, 1]$, $n = 1, 2, \dots$. Motivated by the above convergence results, we propose to obtain sufficient conditions to insure the convergence of the sequence (L_n) to its limit operator denoted by L_∞ . Moreover, the rate of approximation $\|L_n f - L_\infty f\|$ will be estimated by the second order Ditzian–Totik modulus of smoothness of $f \in C[0, 1]$, defined by

$$\omega_\varphi^2(f, t) = \sup_{0 < h \leq t} \sup_{x \pm h\varphi(x) \in [0, 1]} |f(x + h\varphi(x)) - 2f(x) + f(x - h\varphi(x))|, \quad (1)$$

where $\|\cdot\|$ denotes the uniform norm on $C[0, 1]$ and φ is an admissible step-weight function on $[0, 1]$ (for details see [5]). The corresponding K -functional to (1) is defined for $f \in C[0, 1]$ and $t > 0$ as follows:

$$K_{2,\varphi}(f, t) = \inf \{ \|f - g\| + t \|\varphi^2 g''\| : g \in W^2(\varphi) \},$$

where $W^2(\varphi) = \{g \in C[0, 1] : g' \in AC_{loc}[0, 1], \varphi^2 g'' \in C[0, 1]\}$ and $g' \in AC_{loc}[0, 1]$ means that g is differentiable and g' is absolutely continuous on every interval $[a, b] \subset [0, 1]$. In view of [5, Theorem 2.1.1] there exists $C > 0$ such that

$$C^{-1} \omega_\varphi^2(f, \sqrt{t}) \leq K_{2,\varphi}(f, t) \leq C \omega_\varphi^2(f, \sqrt{t}). \quad (2)$$

Here we mention that C will denote throughout this paper a positive constant which can be different at each occurrence, and it is independent of n , f and x . Further, we use the notations $e_0(x) = 1$, $x \in [0, 1]$ and $e_2(x) = x^2$, $x \in [0, 1]$.

In Section 2 is established our main theorem, and in Section 3 is applied this theorem for some q -parametric operators.

2. Main result

The next result is our main theorem.

Theorem 2.1. *Let $(L_n)_{n \geq 1}$ be a sequence of positive linear operators on $C[0, 1]$ and let $(\alpha_n)_{n \geq 1}$ be a positive sequence such that $\alpha_n \rightarrow 0^+$ as $n \rightarrow \infty$. If the positive sequence $(\beta_n)_{n \geq 1}$ satisfies the conditions:*

- (i) $\beta_n + \beta_{n+1} + \cdots + \beta_{n+p-1} \leq C\alpha_n$ for every $n, p = 1, 2, \dots$,
- (ii) $\|L_n g - L_{n+1} g\| \leq C\beta_n \|\varphi^2 g''\|$ for every $g \in W^2(\varphi)$ and $n = 1, 2, \dots$,

then there exists a positive linear operator L_∞ on $C[0, 1]$ such that $\|L_n f - L_\infty f\| \rightarrow 0$ as $n \rightarrow \infty$, for every $f \in C[0, 1]$. Furthermore,

$$\|L_n f - L_\infty f\| \leq C \omega_\varphi^2(f, \sqrt{\alpha_n}), \quad (3)$$

where $f \in C[0, 1]$ and $n = 1, 2, \dots$ are arbitrary.

PROOF OF THEOREM 2.1. We mention that $\|L_n f - L_\infty f\| \rightarrow 0$ as $n \rightarrow \infty$, is a consequence of (3), because $\alpha_n \rightarrow 0^+$ as $n \rightarrow \infty$.

Furthermore, in view of (ii) and (i), we find for every $g \in W^2(\varphi)$ and $n, p = 1, 2, \dots$ that

$$\begin{aligned} \|L_n g - L_{n+p} g\| &\leq \|L_n g - L_{n+1} g\| + \|L_{n+1} g - L_{n+2} g\| + \cdots + \|L_{n+p-1} g - L_{n+p} g\| \\ &\leq C(\beta_n + \beta_{n+1} + \cdots + \beta_{n+p-1}) \|\varphi^2 g''\| \leq C\alpha_n \|\varphi^2 g''\|. \end{aligned} \quad (4)$$

Let $g = e_0$ in (4). Then $L_n e_0 = L_{n+p} e_0$ for all $n, p = 1, 2, \dots$. In this case the positivity of L_n implies that

$$\begin{aligned} |L_n(f, x)| &\leq L_n(|f|, x) \leq L_n(\|f\|e_0, x) = \|f\|L_n(e_0, x) = \|f\|L_1(e_0, x) \\ &\leq \|f\| \|L_1 e_0\| \end{aligned}$$

for $x \in [0, 1]$, $f \in C[0, 1]$ and $n = 1, 2, \dots$. Hence $\|L_n f\| \leq \|L_1 e_0\| \|f\|$ for $f \in C[0, 1]$ and $n = 1, 2, \dots$. This means that

$$\|L_n\| \leq \|L_1 e_0\| < \infty \quad (5)$$

for all $n = 1, 2, \dots$.

On the other hand, $W^2(\varphi)$ is dense in $C[0, 1]$. Then, by the well-known Banach-Steinhaus theorem (see [4]), it is sufficient to prove the convergence of the sequence $(L_n g)_{n \geq 1}$ in $C[0, 1]$, for each $g \in W^2(\varphi)$. Because $\alpha_n \rightarrow 0^+$ as $n \rightarrow \infty$, we get, in view of (4), that $(L_n g)_{n \geq 1}$ is a Cauchy-sequence and therefore

converges in $C[0, 1]$. In conclusion there exists an operator L_∞ on $C[0, 1]$ such that $\|L_n f - L_\infty f\| \rightarrow 0$ as $n \rightarrow \infty$, for all $f \in C[0, 1]$. This also implies that L_∞ is a positive linear operator on $C[0, 1]$, because L_n are positive linear operators for all $n = 1, 2, \dots$.

Further, by (5),

$$\|L_n f\| \leq \|L_n\| \|f\| \leq \|L_1 e_0\| \|f\| \quad (6)$$

for each $f \in C[0, 1]$. Because we have the convergence $L_n f \rightarrow L_\infty f$ in the uniform norm for all $f \in C[0, 1]$, then (6) implies

$$\|L_\infty f\| \leq \|L_1 e_0\| \|f\| \quad (7)$$

for each $f \in C[0, 1]$.

Now let $p \rightarrow \infty$ in (4). Then

$$\|L_n g - L_\infty g\| \leq C \alpha_n \|\varphi^2 g''\|, \quad (8)$$

where $g \in W^2(\varphi)$ and $n = 1, 2, \dots$. By combining (6), (7) and (8), we find for every $f \in C[0, 1]$, that

$$\begin{aligned} \|L_n f - L_\infty f\| &\leq \|L_n f - L_n g\| + \|L_n g - L_\infty g\| + \|L_\infty g - L_\infty f\| \\ &\leq 2\|L_1 e_0\| \|f - g\| + C \alpha_n \|\varphi^2 g''\| \leq C\{\|f - g\| + \alpha_n \|\varphi^2 g''\|\}. \end{aligned}$$

Taking the infimum on the right-hand side over all $g \in W^2(\varphi)$, we get

$$\|L_n f - L_\infty f\| \leq CK_{2,\varphi}(f, \alpha_n).$$

Hence, by (2), we obtain the estimate (3). This completes the proof of our theorem. $\square$

We mention that WANG established in [18] a Korovkin-type theorem, which insures for a sequence (L_n) of positive linear operators on $C[0, 1]$ that there exists an operator L_∞ on $C[0, 1]$ such that $\|L_n f - L_\infty f\| \rightarrow 0$ as $n \rightarrow \infty$, for each $f \in C[0, 1]$. Our theorem is different from Wang's result.

3. Applications

In this section we shall apply Theorem 2.1 for some q -parametric operators, namely for the q -Bernstein operator defined by PHILLIPS [15], for the q -Meyer-König and Zeller operator introduced by TRIF [17] (see also [11]) and for a q -analogue of the Bernstein operator considered by LUPAŞ in [10].

1° Let $0 < q \leq 1$. For each non-negative integer k , the q -integers $[k]$ and the q -factorials $[k]!$ are defined by

$$[k] = \begin{cases} 1 + q + \cdots + q^{k-1}, & \text{if } k \geq 1 \\ 0, & \text{if } k = 0 \end{cases}$$

and

$$[k]! = \begin{cases} [1][2] \cdots [k], & \text{if } k \geq 1 \\ 1, & \text{if } k = 0. \end{cases}$$

For integers $0 \leq k \leq n$, the q -binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

In [15], PHILLIPS defined the following generalization of the classical Bernstein operators, based on q -integers. For each $n = 1, 2, \dots$ and $f \in C[0, 1]$, we define the q -Bernstein operators as

$$(B_{n,q}f)(x) \equiv B_{n,q}(f, x) = \sum_{k=0}^n f\left(\frac{[k]}{[n]}\right) p_{n,k}(q, x),$$

where $x \in [0, 1]$ and

$$p_{n,k}(q, x) = \begin{cases} \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)(1-xq) \cdots (1-xq^{n-k-1}), & \text{if } 0 \leq k \leq n-1 \\ x^n, & \text{if } k = n. \end{cases}$$

For $q = 1$, we recover the well-known Bernstein operators.

By [15, (15)], we have $B_{n,q}(e_2, x) = x^2 + [n]^{-1}x(1-x)$, $x \in [0, 1]$. Hence

$$B_{n,q}(e_2, x) - x^2 = \frac{1}{[n]} \varphi^2(x), \quad (9)$$

where $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0, 1]$ and $n = 1, 2, \dots$

Using [13, (3.2)], we have for $g \in W^2(\varphi)$, $x \in [0, 1]$ and $n = 1, 2, \dots$,

$$B_{n,q}(g, x) - B_{n+1,q}(g, x) = \sum_{k=1}^n a_{n,k}(g) p_{n+1,k}(q, x), \quad (10)$$

where

$$a_{n,k}(g) = \frac{[n+1-k]}{[n+1]} g\left(\frac{[k]}{[n]}\right) + q^{n+1-k} \frac{[k]}{[n+1]} g\left(\frac{[k-1]}{[n]}\right) - g\left(\frac{[k]}{[n+1]}\right). \quad (11)$$

By Taylor's formula, we find

$$\begin{aligned} g\left(\frac{[k]}{[n]}\right) &= g\left(\frac{[k]}{[n+1]}\right) + \left(\frac{[k]}{[n]} - \frac{[k]}{[n+1]}\right) g'\left(\frac{[k]}{[n+1]}\right) \\ &\quad + \int_{[k]/[n+1]}^{[k]/[n]} \left(\frac{[k]}{[n]} - u\right) g''(u) \, du \end{aligned}$$

and

$$\begin{aligned} g\left(\frac{[k-1]}{[n]}\right) &= g\left(\frac{[k]}{[n+1]}\right) + \left(\frac{[k-1]}{[n]} - \frac{[k]}{[n+1]}\right) g'\left(\frac{[k]}{[n+1]}\right) \\ &\quad + \int_{[k]/[n+1]}^{[k-1]/[n]} \left(\frac{[k-1]}{[n]} - u\right) g''(u) \, du. \end{aligned}$$

Then, in view of (11), we obtain

$$\begin{aligned} a_{n,k}(g) &= \frac{[n+1-k]}{[n+1]} g\left(\frac{[k]}{[n]}\right) + q^{n+1-k} \frac{[k]}{[n+1]} g\left(\frac{[k-1]}{[n]}\right) \\ &\quad - \frac{[n+1-k] + q^{n+1-k}[k]}{[n+1]} g\left(\frac{[k]}{[n+1]}\right) \\ &= \frac{[n+1-k]}{[n+1]} \left(\frac{[k]}{[n]} - \frac{[k]}{[n+1]}\right) g'\left(\frac{[k]}{[n+1]}\right) \\ &\quad + \frac{[n+1-k]}{[n+1]} \int_{[k]/[n+1]}^{[k]/[n]} \left(\frac{[k]}{[n]} - u\right) g''(u) \, du \\ &\quad + \frac{q^{n+1-k}[k]}{[n+1]} \left(\frac{[k-1]}{[n]} - \frac{[k]}{[n+1]}\right) g'\left(\frac{[k]}{[n+1]}\right) \\ &\quad + \frac{q^{n+1-k}[k]}{[n+1]} \int_{[k]/[n+1]}^{[k-1]/[n]} \left(\frac{[k-1]}{[n]} - u\right) g''(u) \, du \\ &= \frac{[n+1-k]}{[n+1]} \int_{[k]/[n+1]}^{[k]/[n]} \left(\frac{[k]}{[n]} - u\right) g''(u) \, du \\ &\quad + \frac{q^{n+1-k}[k]}{[n+1]} \int_{[k]/[n+1]}^{[k-1]/[n]} \left(\frac{[k-1]}{[n]} - u\right) g''(u) \, du, \quad (12) \end{aligned}$$

because

$$\frac{[n+1-k]}{[n+1]} \left(\frac{[k]}{[n]} - \frac{[k]}{[n+1]}\right) + \frac{q^{n+1-k}[k]}{[n+1]} \left(\frac{[k-1]}{[n]} - \frac{[k]}{[n+1]}\right)$$

$$\begin{aligned}
&= \frac{[k]}{[n][n+1]^2} \{ [n+1-k]([n+1]-[n]) + q^{n+1-k}([k-1][n+1]-[k][n]) \} \\
&= \frac{[k]}{[n][n+1]^2} \{ [n+1-k]q^n + q^{n+1-k}(-q^{k-1}[n+1-k]) \} = 0.
\end{aligned}$$

Taking into account (12), (9) and the estimate

$$\left| \int_x^t (t-u)g''(u) du \right| \leq (t-x)^2 \varphi^{-2}(x) \|\varphi^2 g''\|, \quad (13)$$

(see [5, Lemma 9.6.1]), we have

$$\begin{aligned}
|a_{n,k}(g)| &\leq \frac{[n+1-k]}{[n+1]} \left| \int_{[k]/[n+1]}^{[k]/[n]} \left(\frac{[k]}{[n]} - u \right) g''(u) du \right| \\
&\quad + \frac{q^{n+1-k}[k]}{[n+1]} \left| \int_{[k]/[n+1]}^{[k-1]/[n]} \left(\frac{[k-1]}{[n]} - u \right) g''(u) du \right| \\
&\leq \frac{[n+1-k]}{[n+1]} \left(\frac{[k]}{[n]} - \frac{[k]}{[n+1]} \right)^2 \varphi^{-2} \left(\frac{[k]}{[n+1]} \right) \|\varphi^2 g''\| \\
&\quad + \frac{q^{n+1-k}[k]}{[n+1]} \left(\frac{[k-1]}{[n]} - \frac{[k]}{[n+1]} \right)^2 \varphi^{-2} \left(\frac{[k]}{[n+1]} \right) \|\varphi^2 g''\| \\
&= \left\{ \frac{[n+1-k][k]([n+1]-[n])^2}{[n]^2[n+1]([n+1]-[k])} \right. \\
&\quad \left. + \frac{q^{n+1-k}([k-1][n+1]-[k][n])^2}{[n]^2[n+1]([n+1]-[k])} \right\} \|\varphi^2 g''\| \\
&= \left\{ \frac{[n+1-k][k]q^{2n}}{[n]^2[n+1]q^k[n+1-k]} \right. \\
&\quad \left. + \frac{q^{n+1-k}(-q^{k-1}[n+1-k])^2}{[n]^2[n+1]q^k[n+1-k]} \right\} \|\varphi^2 g''\| \\
&= \frac{q^{n-1}}{[n]^2[n+1]} \{ q^{n+1-k}[k] + [n+1-k] \} \|\varphi^2 g''\| = \frac{q^{n-1}}{[n]^2} \|\varphi^2 g''\|. \quad (14)
\end{aligned}$$

Then, by (10), (14) and $B_{n+1,q}(e_0, x) = 1$ (see [15, (13)]), we have

$$\begin{aligned}
|B_{n,q}(g, x) - B_{n+1,q}(g, x)| &\leq \sum_{k=1}^n |a_{n,k}(g)| p_{n+1,k}(q, x) \\
&\leq \sum_{k=1}^n \frac{q^{n-1}}{[n]^2} \|\varphi^2 g''\| p_{n+1,k}(q, x) \\
&\leq \frac{q^{n-1}}{[n]^2} B_{n+1,q}(e_0, x) \|\varphi^2 g''\| = \frac{q^{n-1}}{[n]^2} \|\varphi^2 g''\|. \quad (15)
\end{aligned}$$

Let $\beta_n = q^{n-1}/[n]^2$, $n = 1, 2, \dots$, where $0 < q < 1$. Then

$$\begin{aligned}\beta_n + \beta_{n+1} + \dots + \beta_{n+p-1} &\leq \frac{q^{n-1}}{[n]^2} (1 + q + \dots + q^{p-1}) \\ &\leq \frac{1}{[n]^2} \frac{q^{n-1}}{1-q} \leq \frac{q^{n-1}}{(1-q^n)^2}\end{aligned}$$

for all $n, p = 1, 2, \dots$, i.e. we obtain the condition (i) of Theorem 2.1 with $\alpha_n = q^{n-1}/(1-q^n)^2$, $n = 1, 2, \dots$. Obviously $\alpha_n \rightarrow 0^+$ as $n \rightarrow \infty$.

Due to (15), we get $\|B_{n,q}g - B_{n+1,q}g\| \leq \frac{q^{n-1}}{[n]^2} \|\varphi^2 g''\|$, which is the condition (ii) of Theorem 2.1 for the q -Bernstein operator. Applying Theorem 2.1, we have the following statement.

Let $q \in (0, 1)$ be given. Then there exists a positive linear operator $L_{\infty,q}$ on $C[0, 1]$ such that

$$\|B_{n,q}f - L_{\infty,q}f\| \leq C\omega_\varphi^2(f, \sqrt{q^{n-1}}/(1-q^n))$$

for all $f \in C[0, 1]$ and $n = 1, 2, \dots$. The operator $L_{\infty,q}$ coincides with the limit q -Bernstein operator $B_{\infty,q}$, because in [8] it is proved that $\{B_{n,q}f\}$ converges to $B_{\infty,q}f$ as $n \rightarrow \infty$, uniformly on $[0, 1]$.

2° Let $0 < q \leq 1$. For each $n = 1, 2, \dots$ and $f \in C[0, 1]$, we define the q -Meyer-König and Zeller operators [17] as follows.

$$\begin{aligned}(M_{n,q}f)(x) &\equiv M_{n,q}(f, x) \\ &= \begin{cases} \prod_{s=0}^n (1-xq^s) \sum_{k=0}^{\infty} f\left(\frac{[k]}{[n+k]}\right) \begin{bmatrix} n+k \\ k \end{bmatrix} x^k, & \text{if } 0 \leq x < 1 \\ f(1), & \text{if } x = 1. \end{cases}\end{aligned}$$

For $q = 1$, we recover the well-known Meyer-König and Zeller operators.

By [17, (2.3)–(2.4)], we have

$$\begin{aligned}|M_{n,q}(e_2, x) - x^2| &\leq \frac{1}{[n-1]} x(1-x)(1-xq^n) \\ &\quad + \frac{[2]q^{n-1}}{[n-1][n-2]} x(1-x)(1-xq)(1-xq^n) \\ &\leq \frac{1}{[n-1]} \varphi^2(x) + \frac{1}{[n-1]} \varphi^2(x) \leq \frac{4}{[n]} \varphi^2(x),\end{aligned}\tag{16}$$

where $n \geq 4$ and $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0, 1]$.

On the other hand, by [17, Theorem 3.3], we have for $x \in [0, 1]$, $n = 1, 2, \dots$ and $g \in W^2(\varphi)$,

$$\begin{aligned} & M_{n,q}(g, x) - M_{n+1,q}(g, x) \\ &= x \prod_{s=0}^n (1 - xq^s) \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix} x^k \left\{ q^{n+1} \frac{[n+k+1]}{[n+1]} g\left(\frac{[k]}{[n+k+1]}\right) \right. \\ &\quad - q^{n+1} \frac{[n+k+1]}{[n+1]} g\left(\frac{[k+1]}{[n+k+2]}\right) - \frac{[n+k+1]}{[k+1]} g\left(\frac{[k+1]}{[n+k+2]}\right) \\ &\quad \left. + \frac{[n+k+1]}{[k+1]} g\left(\frac{[k+1]}{[n+k+1]}\right) \right\}. \end{aligned} \quad (17)$$

By Taylor's formula, we obtain

$$\begin{aligned} & g\left(\frac{[k]}{[n+k+1]}\right) \\ &= g\left(\frac{[k+1]}{[n+k+2]}\right) + \left(\frac{[k]}{[n+k+1]} - \frac{[k+1]}{[n+k+2]}\right) g'\left(\frac{[k+1]}{[n+k+2]}\right) \\ &\quad + \int_{[k+1]/[n+k+2]}^{[k]/[n+k+1]} \left(\frac{[k]}{[n+k+1]} - u\right) g''(u) du \\ &= g\left(\frac{[k+1]}{[n+k+2]}\right) - \frac{q^k [n+1]}{[n+k+1][n+k+2]} g'\left(\frac{[k+1]}{[n+k+2]}\right) \\ &\quad + \int_{[k+1]/[n+k+2]}^{[k]/[n+k+1]} \left(\frac{[k]}{[n+k+1]} - u\right) g''(u) du \end{aligned} \quad (18)$$

and

$$\begin{aligned} & g\left(\frac{[k+1]}{[n+k+1]}\right) \\ &= g\left(\frac{[k+1]}{[n+k+2]}\right) + \left(\frac{[k+1]}{[n+k+1]} - \frac{[k+1]}{[n+k+2]}\right) g'\left(\frac{[k+1]}{[n+k+2]}\right) \\ &\quad + \int_{[k+1]/[n+k+2]}^{[k+1]/[n+k+1]} \left(\frac{[k+1]}{[n+k+1]} - u\right) g''(u) du \\ &= g\left(\frac{[k+1]}{[n+k+2]}\right) + \frac{q^k [n+1]}{[n+k+1][n+k+2]} g'\left(\frac{[k+1]}{[n+k+2]}\right) \\ &\quad + \int_{[k+1]/[n+k+2]}^{[k+1]/[n+k+1]} \left(\frac{[k+1]}{[n+k+1]} - u\right) g''(u) du, \end{aligned} \quad (19)$$

respectively. Then, in view of (17), (18), (19), (16), (13) and $M_{n,q}(e_0, x) = 1$, $x \in [0, 1]$ (see [17, (2.1)]), we have

$$\begin{aligned}
& |M_{n,q}(g, x) - M_{n+1,q}(g, x)| \leq x \prod_{s=0}^n (1 - xq^s) \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix} x^k \\
& \quad \times \left| q^{n+1} \frac{[n+k+1]}{[n+1]} \left(\frac{[k]}{[n+k+1]} - \frac{[k+1]}{[n+k+2]} \right) + \frac{[n+k+1]}{[k+1]} \right. \\
& \quad \times \left. \left(\frac{[k+1]}{[n+k+1]} - \frac{[k+1]}{[n+k+2]} \right) \right| \left| g' \left(\frac{[k+1]}{[n+k+2]} \right) \right| \\
& \quad + x \prod_{s=0}^n (1 - xq^s) \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix} x^k \\
& \quad \times \left\{ q^{n+1} \frac{[n+k+1]}{[n+1]} \left| \int_{[k+1]/[n+k+2]}^{[k]/[n+k+1]} \left(\frac{[k]}{[n+k+1]} - u \right) g''(u) du \right| \right. \\
& \quad \left. + \frac{[n+k+1]}{[k+1]} \left| \int_{[k+1]/[n+k+2]}^{[k+1]/[n+k+1]} \left(\frac{[k+1]}{[n+k+1]} - u \right) g''(u) du \right| \right\} \\
& = x \prod_{s=0}^n (1 - xq^s) \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix} x^k \\
& \quad \times \left\{ q^{n+1} \frac{[n+k+1]}{[n+1]} \left| \int_{[k+1]/[n+k+2]}^{[k]/[n+k+1]} \left(\frac{[k]}{[n+k+1]} - u \right) g''(u) du \right| \right. \\
& \quad \left. + \frac{[n+k+1]}{[k+1]} \left| \int_{[k+1]/[n+k+2]}^{[k+1]/[n+k+1]} \left(\frac{[k+1]}{[n+k+1]} - u \right) g''(u) du \right| \right\} \\
& \leq x \prod_{s=0}^n (1 - xq^s) \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix} x^k \\
& \quad \times \left\{ q^{n+1} \frac{[n+k+1]}{[n+1]} \left(\frac{[k]}{[n+k+1]} - \frac{[k+1]}{[n+k+2]} \right)^2 + \frac{[n+k+1]}{[k+1]} \right. \\
& \quad \times \left. \left(\frac{[k+1]}{[n+k+1]} - \frac{[k+1]}{[n+k+2]} \right)^2 \right\} \varphi^{-2} \left(\frac{[k+1]}{[n+k+2]} \right) \|\varphi^2 g''\| \\
& = x \prod_{s=0}^n (1 - xq^s) \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix} x^k \\
& \quad \times \left\{ q^{n+1} \frac{[n+k+1]}{[n+1]} \frac{q^{2k} [n+1]^2}{[n+k+1]^2 [n+k+2]^2} \frac{[n+k+2]}{[k+1]} \right.
\end{aligned}$$

$$\begin{aligned}
& \times \frac{[n+k+2]}{[n+k+2]-[k+1]} + \frac{[n+k+1]}{[k+1]} \frac{q^{2n+2k+2}[k+1]^2}{[n+k+1]^2[n+k+2]^2} \\
& \times \frac{[n+k+2]}{[k+1]} \frac{[n+k+2]}{[n+k+2]-[k+1]} \Big\} \|\varphi^2 g''\| \\
& = x \prod_{s=0}^n (1-xq^s) \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix} x^k \frac{q^n}{[n+1]} \frac{q^k}{[k+1]} \frac{[n+k+2]}{[n+k+1]} \|\varphi^2 g''\| \\
& \leq \frac{2q^n}{[n+1]} x \prod_{s=0}^n (1-xq^s) \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix} x^k \|\varphi^2 g''\| \\
& \leq \frac{2q^n}{[n+1]} M_{n,q}(e_0, x) \|\varphi^2 g''\| = \frac{2q^n}{[n+1]} \|\varphi^2 g''\|.
\end{aligned}$$

Hence

$$\|M_{n,q}g - M_{n+1,q}g\| \leq \frac{2q^n}{[n+1]} \|\varphi^2 g''\| \quad (20)$$

for all $n = 1, 2, \dots$ and $g \in W^2(\varphi)$.

In this case we consider $\beta_n = q^n/[n+1]$, $n = 1, 2, \dots$ and $0 < q < 1$. Then

$$\beta_n + \beta_{n+1} + \dots + \beta_{n+p-1} \leq \frac{q^n}{[n+1]} (1 + q + \dots + q^{p-1}) \leq \frac{q^n}{1 - q^{n+1}}$$

for all $n, p = 1, 2, \dots$. We set $\alpha_n = q^n/(1 - q^{n+1})$, $n = 1, 2, \dots$. Then $\alpha_n \rightarrow 0^+$ as $n \rightarrow \infty$. In conclusion, in view of (20), we can apply Theorem 2.1.

Let $q \in (0, 1)$ be given. Then there exists a positive linear operator $L_{\infty,q}$ on $C[0, 1]$ such that

$$\|M_{n,q}f - L_{\infty,q}f\| \leq C\omega_{\varphi}^2(f, \sqrt{q^n/(1 - q^{n+1})})$$

for all $f \in C[0, 1]$ and $n = 4, 5, \dots$

In this case $L_{\infty,q}$ is identical with $B_{\infty,q}$, because in [19] it is proved that $\{M_{n,q}f\}$ converges to $B_{\infty,q}f$ as $n \rightarrow \infty$, uniformly on $[0, 1]$.

3° Let $0 < q \leq 1$. Following [10], the positive linear operators $R_{n,q} : C[0, 1] \rightarrow C[0, 1]$, defined by

$$(R_{n,q}f)(x) \equiv R_{n,q}(f, x) = \sum_{k=0}^n f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{k(k-1)/2} x^k (1-x)^{n-k}}{(1-x+xq) \dots (1-x+xq^{n-1})}$$

are called the q -analogue of the Bernstein operators. For $q = 1$, we recover the well-known Bernstein operators. Due to [14, Lemma 1], we have

$$R_{n,q}(e_0, x) = 1 \quad (21)$$

and

$$|R_{n,q}(e_2, x) - x^2| = \frac{1}{[n]}x(1-x) \cdot \frac{1-x+xq^n}{1-x+xq} \leq \frac{1}{[n]}x(1-x). \quad (22)$$

Thus we set $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0, 1]$. Taking into account [10, (2)], we have for $g \in W^2(\varphi)$,

$$\begin{aligned} R_{n+1,q}(g, x) - R_{n,q}(g, x) &= \frac{x(1-x)}{(1-x+xq) \dots (1-x+xq^n)} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{k(k-1)/2} x^k \\ &\times (1-x)^{n-1-k} \left\{ \frac{q^k[n]}{[k+1]} g\left(\frac{[k+1]}{[n]}\right) - \frac{q^k[n][n+1]}{[k+1][n-k]} g\left(\frac{[k+1]}{[n+1]}\right) \right. \\ &\left. + \frac{q^n[n]}{[n-k]} g\left(\frac{[k]}{[n]}\right) \right\}. \end{aligned} \quad (23)$$

By Taylor's formula, we find

$$\begin{aligned} g\left(\frac{[k+1]}{[n]}\right) &= g\left(\frac{[k+1]}{[n+1]}\right) + \left(\frac{[k+1]}{[n]} - \frac{[k+1]}{[n+1]}\right) g'\left(\frac{[k+1]}{[n+1]}\right) \\ &+ \int_{[k+1]/[n+1]}^{[k+1]/[n]} \left(\frac{[k+1]}{[n]} - u\right) g''(u) du \end{aligned} \quad (24)$$

and

$$\begin{aligned} g\left(\frac{[k]}{[n]}\right) &= g\left(\frac{[k+1]}{[n+1]}\right) + \left(\frac{[k]}{[n]} - \frac{[k+1]}{[n+1]}\right) g'\left(\frac{[k+1]}{[n+1]}\right) \\ &+ \int_{[k+1]/[n+1]}^{[k]/[n]} \left(\frac{[k]}{[n]} - u\right) g''(u) du, \end{aligned} \quad (25)$$

respectively. Because

$$\frac{q^k[n]}{[k+1]} + \frac{q^n[n]}{[n-k]} = \frac{q^k[n][n+1]}{[k+1][n-k]}$$

and

$$\frac{q^k[n]}{[k+1]} \left(\frac{[k+1]}{[n]} - \frac{[k+1]}{[n+1]}\right) + \frac{q^n[n]}{[n-k]} \left(\frac{[k]}{[n]} - \frac{[k+1]}{[n+1]}\right) = 0,$$

by combining (23), (24), (25), (22), (13) and (21), we obtain

$$\begin{aligned} |R_{n+1,q}(g, x) - R_{n,q}(g, x)| &\leq \frac{x(1-x)}{(1-x+xq) \dots (1-x+xq^n)} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{k(k-1)/2} \\ &\times x^k (1-x)^{n-1-k} \left\{ \frac{q^k[n]}{[k+1]} \left| \int_{[k+1]/[n+1]}^{[k+1]/[n]} \left(\frac{[k+1]}{[n]} - u\right) g''(u) du \right| \right. \\ &\left. + \frac{q^n[n]}{[n-k]} \left| \int_{[k+1]/[n+1]}^{[k]/[n]} \left(\frac{[k]}{[n]} - u\right) g''(u) du \right| \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{x(1-x)}{(1-x+xq)\dots(1-x+xq^n)} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{k(k-1)/2} x^k (1-x)^{n-1-k} \\
&\quad \times \left\{ \frac{q^k[n]}{[k+1]} \left(\frac{[k+1]}{[n]} - \frac{[k+1]}{[n+1]} \right)^2 \varphi^{-2} \left(\frac{[k+1]}{[n+1]} \right) \|\varphi^2 g''\| \right. \\
&\quad \left. + \frac{q^n[n]}{[n-k]} \left(\frac{[k]}{[n]} - \frac{[k+1]}{[n+1]} \right)^2 \varphi^{-2} \left(\frac{[k+1]}{[n+1]} \right) \|\varphi^2 g''\| \right\} \\
&= \frac{x(1-x)}{(1-x+xq)\dots(1-x+xq^n)} \|\varphi^2 g''\| \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{k(k-1)/2} \\
&\quad \times x^k (1-x)^{n-1-k} \left\{ \frac{q^k[n]}{[k+1]} \frac{[k+1]^2 q^{2n}}{[n]^2 [n+1]^2} \frac{[n+1]^2}{[k+1]([n+1]-[k+1])} \right. \\
&\quad \left. + \frac{q^n[n]}{[n-k]} \frac{q^{2k} [n-k]^2}{[n]^2 [n+1]^2} \frac{[n+1]^2}{[k+1]([n+1]-[k+1])} \right\} \\
&= \frac{x(1-x)}{(1-x+xq)\dots(1-x+xq^n)} \|\varphi^2 g''\| \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{k(k-1)/2} \\
&\quad \times x^k (1-x)^{n-1-k} \frac{q^{n-1+k} [n+1]}{[n][n-k][k+1]} \\
&= \frac{\|\varphi^2 g''\|}{(1-x+xq)\dots(1-x+xq^n)} \sum_{k=0}^{n-1} \frac{q^{n-1}}{[n]^2} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} q^{(k+1)k/2} x^{k+1} (1-x)^{n-k} \\
&= \frac{q^{n-1}}{[n]^2} \|\varphi^2 g''\| \sum_{k=1}^n \begin{bmatrix} n+1 \\ k \end{bmatrix} \frac{q^{k(k-1)/2} x^k (1-x)^{n+1-k}}{(1-x+xq)\dots(1-x+xq^n)} \\
&\leq \frac{q^{n-1}}{[n]^2} \|\varphi^2 g''\| R_{n+1,q}(e_0, x) = \frac{q^{n-1}}{[n]^2} \|\varphi^2 g''\|.
\end{aligned}$$

Similarly to the case 1° , we can choose the sequences $(\alpha_n)_{n \geq 1}$ and $(\beta_n)_{n \geq 1}$ as $\alpha_n = q^{n-1}/(1-q^n)^2$ and $\beta_n = q^{n-1}/[n]^2$, where $n = 1, 2, \dots$ and $0 < q < 1$. By Theorem 2.1, we have the following statement.

Let $q \in (0, 1)$ be given. Then there exists a positive linear operator $L_{\infty,q}$ on $C[0, 1]$ such that

$$\|R_{n,q}f - L_{\infty,q}f\| \leq C\omega_\varphi^2\left(f, \sqrt{q^{n-1}}/(1-q^n)\right)$$

for all $f \in C[0, 1]$ and $n = 1, 2, \dots$

The operator $L_{\infty,q}$ is identical with the limit q -analogue of the Bernstein operator, denoted by $\tilde{R}_{\infty,q}$, because in [14] it is proved that $\{R_{n,q}f\}$ converges to $\tilde{R}_{\infty,q}f$ as $n \rightarrow \infty$, uniformly on $[0, 1]$.

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