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Semi-parallel vector fields and conformally flat Randers metrics

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This paper is dedicated to Professor Masao Hashiguchi on the occasion of his 80th birthday

Abstract. In this article, we investigate the conformal flatness of Randers metrics as an application of the geometry of Riemannian spaces admitting a semi-parallel vector field.

1. Introduction

A Riemannian manifold (M, g) is said to be conformally flat if there exists a local function $\sigma(x)$ defined on a neighborhood U of an arbitrary point in M such that the local metric $e^{\sigma(x)}g$ is a flat metric on U. As is well-known, the conformal-flatness is characterized by the vanishing of the Weyl's conformal curvature. On the other hand, YANO [Ya2] characterized the conformal-flatness by the existence of some special linear connection, that is, a Riemannian metric g is conformally flat if and only if there exists a semi-symmetric metrical connection ∇ whose curvature vanishes identically.

In Finsler geometry, the conformal curvature which characterize the conformal-flatness of Finsler metric has not been obtained yet. The conformal-flatness of a Finsler metric is characterized by [Ha-Ic2] as a generalization of

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Yano's theorem as follows: A Finsler manifold (M, L) is conformally flat if and only if there exists a semi-symmetric Finsler connection ∇ whose curvature vanishes identically.

A typical example of Finsler metrics are given by the *Funk metric* or *Hilbert metric* on a strictly convex domain. These metrics have some special properties, that is, these metrics are projectively flat, and they are of negative constant flag curvature. In particular, the Funk metric is a special class of the so-called *Randers metric* if the boundary of the domain is given by a quadratic equation. A positive function L defined on the total space TM of tangent bundle of a smooth manifold M is said to be a *Randers metric* if L is given by the form $L(X) = \sqrt{g(X,X)} + \beta(X)$ $(X \in TM)$ for a Riemannian metric g on M and a one-form β on M. A characterization of Randers metric of negative constant flag curvature is given by [Ha-Sa-Sh] under some assumption.

Theorem 1.1 ([Ha-Sa-Sh]). *Let $L(X) = \sqrt{g(X,X)} + \beta(X)$ be a Randers metric on M with a one-form β satisfying $d\beta(E, X) = 0$ for all vector filed X on M, where E is the dual of β with respect to g. Then (M, L) has negative constant flag curvature $K = -\rho^2/4$ if and only if*

(1) *the base Riemannian manifold (M, g) has negative constant sectional curvature $-\rho^2$,*

(2) *the one-form β is semi-parallel, that is, β satisfies*

$$\nabla^g \beta = \rho(g - \beta \otimes \beta), \tag{1.1}$$

where ∇^g is the Riemannian connection of (M, g).

The form β satisfying (1.1) is a special case of the so-called *torse-forming one-form*, that is,

$$\nabla^g \beta = \rho\left(g + \varepsilon \beta \otimes \beta\right), \tag{1.2}$$

where ρ is a constant and $\varepsilon = \pm 1$ ([Ya1]). It is easily shown that one-form β satisfying (1.2) is closed. In the second section, we shall show that a connected complete Riemannian manifold which admits such a one-form β is given by the warped product space $M = N \times_{\psi(t)} \mathbb{R}$, where N is a totally umbilic hypersurface. In the third and fourth sections, we shall investigate the conformal flatness of Randers metric as an application of geometry of Riemannian manifolds admitting a semi-parallel vector field.

2. Semi-parallel vector fields

Let (M, g) be a connected complete Riemannian manifold of $\dim M = n$. In this section, we shall investigate the structure of a Riemannian manifold (M, g) admitting a semi-parallel vector filed E.

Definition 2.1 ([Ha1]). A vector field E is said to be *semi-parallel* if it satisfies

$$\nabla^g_X E = \rho\,(X + \varepsilon\beta(X)E)\,, \tag{2.1}$$

where β is the dual of E, that is, $\beta(X) = g(X, E)$ for any vector field X on M, and ρ is a constant and $\varepsilon = \pm 1$. If $\rho \equiv 0$, then E is a *parallel* vector field.

The condition (2.1) is equivalent to (1.2) which implies $d\beta = 0$, and thus $\beta = df$ for some local function f. Then E is the gradient vector filed $\nabla^g f$ of f, that is, $g(X, \nabla^g f) = X(f)$ every vector field X, and E is given by

$$E = \nabla^g f = \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} \tag{2.2}$$

in local coordinate on M.

Example 2.1. Let $\mathbb{B}$ be the unit ball in the n-dimensional Euclidean space $\mathbb{R}^n$:

$$\mathbb{B} = \left\{ (x^1, \ldots, x^n) \in \mathbb{R}^n \mid 1 - \sum_{i=1}^{n} (x^i)^2 > 0 \right\}.$$

For every point $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$, we set $\|x\|^2 = \sum_{i=1}^n (x^i)^2$ and $f(x) = \log(1 - \|x\|^2)$. The *Hilbert metric* g_H on $\mathbb{B}$ is defined by

$$g_H = \frac{(1 - \|x\|^2)\big(\sum_{i=1}^n dx^i\big)^2 + \big(\sum_{i=1}^n x^i dx^i\big)^2}{(1 - \|x\|^2)^2}, \tag{2.3}$$

which is a hyperbolic Riemannian metric induced on $\mathbb{B}$ from the Minkowski metric $g_M = \sum (dx^i)^2 - (dt)^2$ on the hyperboloid $\sum (x^i)^2 - t^2 = -1$ (cf. [KN]). As is well-known, the space $(\mathbb{B}, g_H)$ has negative constant curvature $K = -1$. The vector field $E = \nabla^g f$ satisfies (2.1), and the one-form β defined by

$$\beta = -\frac{1}{2} df = \frac{1}{1 - \|x\|^2} \sum_{i=1}^{n} x^i dx^i \tag{2.4}$$

satisfies (1.2) for the case of $\rho = 1$ and $\varepsilon = -1$ (cf. [Ok]). The norm $\|\beta\|$ of this one-form β is given by $\|\beta\| = \|x\| < 1$. $\square$

Since any torse-forming one-form β is closed, the integrability condition $\nabla^g_X \nabla^g_Y E - \nabla^g_Y \nabla^g_X E - \nabla^g_{[X,Y]} E = R^g(X,Y)E$ for (2.1) is given by

$$R^g(X,Y)E = -\varepsilon\rho^2\big[g(X,E)Y - g(Y,E)X\big]. \tag{2.5}$$

Then, the sectional curvature $K(X \wedge E)$ of the 2-plane $X \wedge E$ is given by

$$K(X \wedge E) = \frac{g(R^g(X,E)E,X)}{\|X \wedge E\|^2} = \frac{g(R^g(X,E)E,X)}{\|X\|^2\|E\|^2 - g(X,E)^2} = \varepsilon\rho^2 \tag{2.6}$$

for all $X \in TM$. In particular, if (M,g) is a Riemannian manifold of constant curvature $\varepsilon\rho^2$, that is, if its curvature R^g satisfies

$$R^g(X,Y)Z = \varepsilon\rho^2\big[g(Y,Z)X - g(X,Z)Y\big]$$

for all $X, Y, Z \in TM$, then the integrability condition (2.5) is satisfied, and thus, if (M,g) is of constant curvature, there exists a semi-parallel vector field around every point in M.

Every Riemannian manifold of constant curvature c can be locally expressed as a warped product whose warping function satisfies $\Delta f = cf$ (cf. [Ch]). In the sequel, we shall show that a Riemannian manifold (M,g) admitting a semi-parallel vector field E can be locally expressed as a warped product space $\mathbb{R} \times_\rho N$, where N is a totally umbilic submanifold.

We suppose that there exists a smooth function $f : M \to \mathbb{R}$ such that $E = \nabla^g f$ satisfies (2.1) and f has no critical points. Then the set $N = f^{-1}(t)$ is a hypersurface in M with normal vector field E for every $t \in \mathbb{R}$. Let $p \in N = f^{-1}(t)$ be an arbitrary point. The second fundamental form S of N at $p \in N$ with respect to the normal vector field $E = \nabla^g f$ is defined by

$$S(X,Y) = (\nabla^g_X Y)^\perp \tag{2.7}$$

for $X, Y \in T_pN$, where $\perp$ is the projection to the orthogonal line bundle spanned by E. Since ∇^g is metrical and $\beta(X) = g(X,E) = g(X, \nabla^g f) = 0$ for every $X \in T_pN$, we have

$$\begin{aligned} g\left(\nabla^g_X Y, E\right) &= Xg(Y,E) - g(Y, \nabla^g_X E) = X\beta(Y) - g\left(Y, \rho\left\{X + \varepsilon\beta(X)E\right\}\right) \\ &= -\rho g(X,Y), \end{aligned}$$

and thus we have

$$S(X,Y) = -\frac{\rho g(X,Y)}{\|E\|^2}E. \tag{2.8}$$

Hence we have

Lemma 2.1. *Let (M, g) be a connected complete Riemannian space. Suppose that there exists a smooth function $f : M \to \mathbb{R}$ such that its gradient $E = \nabla^g f$ is a semi-parallel vector field. Then the complete hypersurface $N = f^{-1}(t)$ is totally umbilical for every $t \in \mathbb{R}$. If E is parallel, that is, $\rho \equiv 0$, then N is a totally geodesic hypersurface.*

If a semi-parallel vector field E has constant length, then its length must be unit and $\varepsilon = -1$ (cf. [Ha1]). Thus, in the sequel, we shall restrict our discussions to the case of $\varepsilon = -1$. Then, since the gradient of f has constant norm $\|\nabla^g f\| = 1$, we have a splitting theorem by the same method as in [In], [Sa] and [Ya1] as follows.

Since the function f admits no critical points, $f^{-1}(t) = N$ is a complete hypersurface for every $t \in \mathbb{R}$, and M is diffeomorphic to the product space $N \times \mathbb{R}$. In the case of $\rho \equiv 0$, the gradient $\nabla^g f$ is parallel, and the Hessian $\nabla^g(df)$ vanishes everywhere. Such a function f is called an *affine function* (cf. [Sa]).

Let γ_E be the integral curve of E through $p \in N = f^{-1}(0)$. By the assumption of $\|E\|^2 = \beta(E) = 1$, we get $\nabla^g_E E = 0$, and thus the integral curve γ_E of E is a geodesic orthogonal to the hypersurface N. Since (M, g) is complete, $\gamma_E(t)$ is defined for all $t \in \mathbb{R}$. Then, since $E(f) = g(E, \nabla^g f) = \|E\|^2 = 1$ and

$$E(f) = \frac{d}{dt} f\left(\gamma_E(t)\right),$$

we have $f(\gamma_E(t)) = f(p) + t = t$ for all $t \in \mathbb{R}$.

Moreover, the integral curve γ_E is a minimal geodesic between the hypersurface $f^{-1}(0)$ and $f^{-1}(t)$ (cf. [Sa]). Indeed, let $c : [0, l] \longrightarrow M$ be any smooth curve parametrized by its arc-length s joining a point $p = c(0) \in f^{-1}(0)$ and a point $q = c(l) \in f^{-1}(t)$. Then the length $L(c)$ of c satisfies

$$\begin{aligned} L(c) &= \int_0^l \|\dot{c}(s)\|\, ds \geq \left| \int_0^l g\left(\dot{c}(s), \nabla^g f(c(s))\right) \right| \\ &= \left| \int_0^l \frac{d}{ds} f(c(s)) ds \right| = |f(q) - f(p)| = t \\ &= \int_0^t \|E\|\, dt = L(\gamma_E), \end{aligned}$$

namely $dist\left(N, f^{-1}(t)\right) = L(\gamma_E)$.

Lemma 2.2 ([Sa])**.** *Let (M, g) be a connected complete Riemannian manifold. Suppose that there exists a smooth function $f : M \to \mathbb{R}$ such that its gradient $\nabla^g f$ is a semi-parallel vector field of unit length. Then f is the signed distance function to the connected complete hypersurface $N = f^{-1}(0)$.*

With respect to a suitable local coordinate system $(x^1, \dots, x^n)$ on $M \cong N \times \mathbb{R}$, the given metric g has the form

$$g = \sum_{i,j=1}^{n-1} g_{ij}(x^1, \dots, x^n) dx^i \otimes dx^j + g_{nn}(x^1, \dots, x^n) dx^n \otimes dx^n.$$

Since the orthogonal trajectories to N defined by $x^i = constant$ $(i = 1, \dots, n-1)$ are geodesics, the coefficients $\Gamma^{\lambda}_{\beta\gamma}$ of the Riemannian connection ∇^g of (M, g) satisfy

$$\Gamma^i_{nn} = \frac{1}{2} \sum_{\lambda=1}^{n} g^{\lambda i} \left(\frac{\partial g_{\lambda n}}{\partial t} + \frac{\partial g_{n\lambda}}{\partial t} - \frac{\partial g_{nn}}{\partial x^{\lambda}} \right) = 0,$$

and this implies $g_{nn} = g_{nn}(x^n)$. Replacing $\sqrt{g_{nn}(x^n)}dx^n$ by dt, the metric has the following form

$$g = \sum_{i,j=1}^{n-1} g_{ij}(x, t) dx^i \otimes dx^j + dt \otimes dt. \tag{2.9}$$

Denoting by $g_0 = dt \otimes dt$ the canonical metric on $\mathbb{R}$, the map $f : (M, g) \longrightarrow (\mathbb{R}, g_0)$ is a Riemannian submersion.

With respect to such a coordinate system $(x, t) = (x^1, \dots, x^{n-1}, t)$, we have $E = \partial/\partial t$ and

$$\nabla^g_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} = \sum_{h=1}^{n-1} \Gamma^h_{ij}(x, t) \frac{\partial}{\partial x^h} + \Gamma^n_{ij}(x, t) E.$$

Consequently the second fundamental form S is given by

$$S \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \Gamma^n_{ij}(x, t) E,$$

and (2.9) implies that $\Gamma^n_{ij} = \rho(x, t) g_{ij}$, that is,

$$\frac{1}{2} \sum_{\lambda=1}^{n} g^{\lambda n} \left(\frac{\partial g_{i\lambda}}{\partial x^j} + \frac{\partial g_{\lambda j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^{\lambda}} \right) = \rho g_{ij}.$$

Since $g^{in} = g_{jn} = 0$ for $i, j = 1, \dots, n-1$ and $g^{nn} = 1$, we have

$$\frac{\partial g_{ij}}{\partial t} = -2\rho g_{ij}(x, t).$$

This implies that $g_{ij}(x,t) = \psi(t)^2 g^*_{ij}(x)$ for a Riemannian metric $g_N = \sum g^*_{ij}(x) dx^i \otimes dx^j$ on N, where ψ is given by $\psi(t) = e^{-\rho t}$. Thus the metric g is of the form

$$g = \left(e^{-\rho t}\right)^2 \sum_{i,j=1}^{n-1} g^*_{ij}(x) dx^i \otimes dx^j + dt \otimes dt = \left(e^{-\rho t}\right)^2 g_N + dt \otimes dt. \tag{2.10}$$

In particular, if $\rho = 0$, then g is the product metric $g = g_N + dt \otimes dt$ on the product space $M = N \times \mathbb{R}$. Consequently Lemma 2.1 and 2.2 imply the following.

Theorem 2.1. *Let (M, g) be a connected complete Riemannian manifold. Then, there exists a smooth function $f : M \to \mathbb{R}$ such that its gradient $\nabla^g f$ is a semi-parallel vector field of unit length if and only if there exists a connected complete hypersurface N which is totally umbilic with constant mean curvature ρ and M is isometric to the warped product space $N \times_{\psi(t)} \mathbb{R}$ with the warping function $\psi(t) = e^{-\rho t}$. In this case, the function f is the signed distance function to N.*

By the exponential function, the manifold $N \times \mathbb{R}$ is diffeomorphic to $N \times \mathbb{R}_+$ by sending $N \times \mathbb{R} \ni (x,t) \longrightarrow (x, e^t) \in N \times \mathbb{R}_+$.

Example 2.2. Let $\mathbb{H} = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n > 0\}$ be the upper half plane with the Riemannian connection ∇^g of the Poincaré metric

$$g_P = \left(\frac{1}{x^n}\right)^2 \sum_{\alpha=1}^{n} dx^\alpha \otimes dx^\alpha. \tag{2.11}$$

Setting $t = \log x^n$, the metric g_P is written as

$$g_P = (e^{-t})^2 \sum_{i=1}^{n-1} dx^i \otimes dx^i + dt \otimes dt. \tag{2.12}$$

Comparing this metric with (2.11), we define a function $f : \mathbb{H} \longrightarrow \mathbb{R}$ by $f(x) = \log x^n = t$. Since ∇^g is given by the Christoffel symbols

$$\Gamma^i_{jk} = -\frac{1}{x^n}\left(\delta_{jn}\delta^i_k + \delta_{kn}\delta^i_j - \delta_{jk}\delta^i_n\right),$$

we can easily show that the gradient

$$\nabla^g f = \frac{\partial}{\partial t} = x^n \frac{\partial}{\partial x^n}$$

is a semi-parallel vector field on $\mathbb{H}$ of constant norm $\|\nabla^g f\| = 1$, namely $E = \nabla^g f$ satisfies $\nabla^g_X E = \beta(X)E - X$. The hypersurface $f^{-1}(t)$ is a totally umbilical hypersurface with constant mean curvature $H = 1$ for every $t \in \mathbb{R}$. □

3. Finsler metrics and connections

3.1. Finsler metrics. First we shall review the variational problems from [He]. Let $\pi : TM \to M$ be the tangent bundle of a connected smooth manifold M. We denote by $v = (x, y)$ the points in TM if $y \in \pi^{-1}(x) = T_xM$. We denote by $z(M)$ the zero section of TM, and by $TM^\times$ the slit tangent bundle $TM \backslash z(M)$. We introduce a coordinate system on TM as follows. Let $U \subset M$ be an open set with local coordinate $(x^1, \dots, x^n)$. By setting $v = \sum_{i=1}^n y^i \left(\partial/\partial x^i\right)_x$ for every $v \in \pi^{-1}(U)$, we introduce a local coordinate $(x, y) = (x^1, \dots, x^n, y^1, \dots, y^n)$ on $\pi^{-1}(U)$.

Let $L : TM \times \mathbb{R} \to \mathbb{R}$ be a Lagrangian. The *Cartan form* θ_L of L is defined by

$$\theta_L = \sum_{i=1}^n \frac{\partial L}{\partial y^i}(dx^i - y^i dt) + L dt. \tag{3.1}$$

For every smooth curve $c : [0, 1] \to M$, we define a natural lift $\phi : [0, 1] \to TM \times \mathbb{R}$ by $\phi(t) = (c'(t), t)$. Since $\phi^*(dx^i - y^i dt) = 0$, we obtain

$$\int_0^1 \phi^*(\theta_L) = \int_0^1 L\left(c'(t), t\right) dt.$$

For an arbitrary proper variational field X, we denote by $\phi_s : (-\varepsilon, \varepsilon) \times I \to TM \times \mathbb{R}$ the variation of ϕ. The critical point of this variation is calculated as follows:

$$\begin{aligned} \frac{d}{ds}\Big|_{s=0} \int_0^1 \phi_s^* \theta_L = \int_0^1 \frac{d}{ds}\Big|_{s=0} \phi_s^* \theta_L &= \int_0^1 \left[\phi^*(\iota_X d\theta_L) + d\phi^*(\iota_X \theta_L)\right] \\ &= \int_0^1 \phi^*\left(\iota_X d\theta_L\right), \end{aligned}$$

since X is proper. Then we define a 2-form Θ_L on $TM \times \mathbb{R}$ by $\Theta_L = d\theta_L$:

$$\Theta_L = \sum \left[d\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} dt \right] \wedge (dx^i - y^i dt). \tag{3.2}$$

Then $\phi = \phi(t)$ is called a *characteristic curve* of L if it satisfies $\phi^*\Theta_L = 0$. A smooth curve $c = c(t)$ on M is called a *extremal curve* of L if c is the projection of a characteristic curve $\phi = \phi(t)$ into M. A smooth curve $c = c(t)$ is a extremal if and only if it satisfies the *Euler–Lagrange equation*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) = \frac{\partial L}{\partial x^i}.$$

Secondly we shall investigate the case where the Lagrangian L is independent of the parameter t and, moreover, L satisfies the homogeneity condition $L(\lambda X) = \lambda L(X)$ for all $\lambda > 0$. In this case, the extremal of L is independent of the parametrization, and, since the homogeneity of L implies $\sum y^i \partial L/\partial y^i - L = 0$, the Cartan form θ_L is given by

$$\theta_L = \sum_{i=1}^{n} \frac{\partial L}{\partial y^i} dx^i. \tag{3.3}$$

Definition 3.1. A function $L : TM \longrightarrow \mathbb{R}$ is called a *Finsler metric* on M if

1. $L(x, y) \geq 0$, and $L(x, y) = 0$ if and only if $y = 0$,
2. $L(x, \lambda y) = \lambda L(x, y)$ for ${}^\forall \lambda \in \mathbb{R}^+ = \{\lambda \in \mathbb{R} : \lambda > 0\}$,
3. $L(x, y)$ is smooth on $TM^\times$

are satisfied. The pair (M, L) is called a *Finsler space*.

For each $X \in T_x M$, its norm $\|X\|$ is defined by $\|X\| = L(x, X)$. The length $l(c)$ of a smooth curve $c = c(t)$ is defined by $l(c) = \int_0^1 L(c'(t))dt$. An extremal curve in a Finsler manifold (M, L) is called a *geodesic* in (M, L).

Example 3.1. Let g be a Riemannian metric on M. Since there exists a 1-from β satisfying $\beta(X) \leq \sqrt{g(X, X)}$, the function $L = \sqrt{g(X, X)} + \beta(X)$ defines a Finsler metric on M so-called *Randers metric*. A Randers metric L is strictly convex if and only the norm $\|\beta\|$ of β with respect to the metric g satisfies $\|\beta\| < 1$. $\square$

Let L and $\widetilde{L}$ be two Finsler metrics on M. Then (M, L) is *projectively equivalent* to $(M, \widetilde{L})$ if (M, L) and $(M, \widetilde{L})$ have the same geodesics as point sets. An appropriate "sufficient condition" for (M, L) to be projectively equivalent to $(M, \widetilde{L})$ is

$$\Theta_L = \Theta_{\widetilde{L}}. \tag{3.4}$$

Definition 3.2. A Finsler space (M, L) is said to be *strictly projective-equivalent* to a Finsler space $(M, \widetilde{L})$ if (3.4) is satisfied.

The condition (3.4) is written in the form:

$$\sum_{i=1}^{n} d\left(\frac{\partial L}{\partial y^i}\right) \wedge dx^i = \sum_{i=1}^{n} d\left(\frac{\partial \widetilde{L}}{\partial y^i}\right) \wedge dx^i.$$

Therefore we have

$$\frac{\partial^2(\widetilde{L} - L)}{\partial y^i \partial y^j} = 0, \quad \frac{\partial^2(\widetilde{L} - L)}{\partial y^i \partial x^j} = \frac{\partial^2(\widetilde{L} - L)}{\partial x^i \partial y^j}.$$

These equations imply $\theta_{\widetilde{L}} = \theta_L + \pi^*\beta$ for a closed one-form β on M and

$$\widetilde{L}(X) = L(X) + \beta(X) \tag{3.5}$$

for all $X \in TM$. Consequently we obtain the following theorem:

Theorem 3.1. *Let L and $\widetilde{L}$ be two Finsler metrics on a smooth manifold M. Then (M, L) is strictly projective-equivalent to (M, L) if and only if $\widetilde{L}$ is given by (3.5) for a closed one-form β on M. In particular, a Finsler manifold (M, L) is strictly projective-equivalent to a Riemannian manifold (M, g) if and only if there exists a closed one-form β on M satisfying*

$$L(X) = \sqrt{g(X, X)} + \beta(X) \tag{3.6}$$

for all $X \in TM$.

3.2. Chern-Finsler connection. Let $V = \ker \pi_*$ be the *vertical sub-bundle* of the tangent bundle over TM, where π_* is the differential of the projection $\pi : TM \to M$. Since we have the natural identification $V \cong \pi^*TM = \{(y, v) \in TM \times TM \mid v \in T_{\pi(y)}M\}$ and V is locally spanned by $\{e_j = \partial/\partial y^j\}$ $(j = 1, \ldots, n)$ on each $\pi^{-1}(U)$, we may consider the differential π_* as a V-valued one-form $\pi_* = \sum e_i \otimes dx^i$ on TM. We denote by $A^k(V)$ the space of smooth V-valued k-forms.

A Finsler metric L is said to be *convex* if $G = L^2/2$ is *strictly convex* on each tangent space T_xM, that is, the Hessian (G_{ij}) defined by

$$G_{ij}(x, y) = \frac{\partial^2 G}{\partial y^i \partial y^j} \tag{3.7}$$

is positive-definite.

In the sequel, we assume the convexity of L. Then each fiber T_xM is a Riemannian space with a metric $G_x = \sum G_{ij}(x, y) dy^i \otimes dy^j$, and the family $\{G_x\}_{x \in M}$ defines a metric G on V by $G(Y, Z) = \sum G_{ij} Y^i Z^j$ for every section $Y = \sum Y^i e_i$ and $Z = \sum Z^j e_j$. We define a symmetric tensor $C : \otimes^3 V \to \mathbb{R}$ by

$$C(e_i, e_j, e_k) = \frac{1}{2}\frac{\partial G_{ij}}{\partial y^k} := C_{ijk}. \tag{3.8}$$

It is trivial C vanishes identically if and only if G is a Riemannian metric on M.

The multiplier group $\mathbb{R}^+ \cong \{cI \in GL(TM);\ c \in \mathbb{R}^+\} \subset GL(TM)$ acts on the total space by multiplication $m_\lambda : TM \ni v = (x, y) \to \lambda v = (x, \lambda y) \in TM$ for $^\forall \lambda \in \mathbb{R}^+$. This action induces a canonical vector filed $\mathcal{E}$ defined by

$$\mathcal{E}(x, y) = \sum_{i=1}^{n} y^i \frac{\partial}{\partial y^i},$$

which is called the *tautological section* of V. By the homogeneity of L, we have

$$\mathcal{E}(L) = \frac{d}{dt}\Big|_{t=0} L(x, y + t\mathcal{E}) = L.$$

Moreover it is easily shown that $L = \sqrt{G(\mathcal{E}, \mathcal{E})}$ and $C(\mathcal{E}, \bullet, \bullet) \equiv 0$.

Definition 3.3 ([Ba-Ch-Sh]). The *Chern connection* on (M, L) is a connection $D : \Gamma(V) \to A^1(V)$ uniquely determined by the following conditions.

(1) D is *almost G-compatible*:

$$DG = 2C. \tag{3.9}$$

(2) D is *symmetric*:

$$D\pi_* = 0, \tag{3.10}$$

where we consider π_* as a V-valued one-form on TM.

We define $\theta \in A^1(V)$ by $\theta = D\mathcal{E}$. Then, $H = \ker\theta$ defines a *horizontal subbundle* H which is complementary to V. Denoting by $\omega^i_j = \sum_{k=1}^n \Gamma^i_{jk}(x, y)dx^k$ the connection form of the Chern connection D with respect to the frame $\{e_1, \ldots, e_n\}$, the differentiation d_D in the horizontal direction is given by

$$d_D F := \sum_{k=1}^n \left(\frac{\partial F}{\partial x^k} - \sum_{l=1}^n y^l \Gamma^i_{lk}(x, y) \frac{\partial F}{\partial y^i} \right) dx^k$$

for any smooth function F on $TM^\times$. Then the Chern–Finsler connection D of (M, L) satisfies

$$d_D L \equiv 0. \tag{3.11}$$

Remark 3.1. In the case of $C = 0$, the metric L is the norm function of a Riemannian metric g, and the Chern connection D is given by $D = \pi^*\nabla^g$ for the Riemannian connection ∇^g of (M, g). Then we have

$$d_{\nabla^g} L_g \equiv 0, \tag{3.12}$$

where L_g is defined by $L_g(X) = \sqrt{g(X, X)}$ for every $X \in TM$. □

Let D be the Chern–Finsler connection of a Finsler space (M, L). The 2-plane $\mathcal{F}(X)$ spanned by $X \in V$ and $\mathcal{E}$ is called the *flag* with the *flagpole* $\mathcal{E}$. For the curvature tensor R of D, the sectional curvature

$$K(X \wedge \mathcal{E}) = \frac{\langle R(X, \mathcal{E})\mathcal{E}, X\rangle}{\|X\|^2\|\mathcal{E}\|^2 - \langle X, \mathcal{E}\rangle^2}$$

is called the *flag curvature* of the flag $\mathcal{F}(X)$. A Finsler manifold (M, L) is said to be of *constant flag curvature* if $K(X \wedge \mathcal{E})$ is constant for every $X \in V$.

Example 3.2 ([Ok]). The Hilbert metric g_H defined by (2.3) has negative constant curvature. The *Funk metric* $L_{\mathbb{B}}$ on $\mathbb{B}$ is defined by

$$L_{\mathbb{B}}(X) = \sqrt{g_H(X,X)} + \beta(X) \tag{3.13}$$

for the one-form β defined by (2.4). The norm $\|\beta\|_H$ with respect to g_H is given by $\|\beta\|_H = \|x\| < 1$. Since β is closed, the Funk metric $L_{\mathbb{B}}$ is strictly projectively-equivalent to Hilbert metric g_H. Furthermore, β satisfies the condition (1.1). Therefore, by Theorem 1.1, $(\mathbb{B}, L_{\mathbb{B}})$ has negative constant flag curvature $K = -1/4$. □

3.3. Berwald spaces and Wagner spaces. A Finsler metric L is said to be *flat* or *locally Minkowski* if its Chern-Finsler connection D is flat, that is, its curvature R vanishes identically. The flatness of L is equivalent to the fact that there exists an open covering of M such that the metric L is independent of the base point $x \in M$ (cf. [Ma]).

Definition 3.4. A Finsler metric L is said to be *Berwald* if the Chern–Finsler connection D is given by $D = \pi^*\nabla$ for a symmetric linear connection ∇ in TM.

The following theorem plays an important role in the study of Berwald spaces.

Theorem 3.2 ([Sz])**.** *Let (M, L) be a Berwald space. Then there exists a Riemannian metric g satisfying $D = \pi^*\nabla^g$, where ∇^g is the Riemannian connection of (M, g).*

It is obvious that if (M, L) is a Berwald space, then (M, L) is projective equivalent to the associated Riemannian space (M, g). In Theorem 3.2, without loss of generality, we may assume $L(X) > L_g(X)$ for every $X \in TM^{\times}$. Then we have

Theorem 3.3. *Suppose that a Berwald space (M, L) is strictly projective-equivalent to the associated Riemannian space (M, g). Then L has the form of*

$$L(X) = \sqrt{g(X,X)} + \beta(X) \tag{3.14}$$

for a parallel one-form β on (M, g).

Proof. We assume that (M, L) is strictly projective-equivalent to (M, L_g), that is, $\Theta_L = \Theta_{L_g}$. Then, by Theorem 3.1, there exists a closed one-form $\beta = \sum \beta_k(x)dx^k$ on M satisfying (3.5). Furthermore, by Theorem 3.2, the Chern–Finsler connection D is given by $D = \pi^*\nabla^g$ for the Riemannian connection ∇^g

of the associated Riemannian space (M, g). Therefore we have $d_{\nabla^g} L = 0$. Consequently, because of (3.12), the one-form β satisfies

$$\begin{aligned} 0 = d_{\nabla^g}[\beta(\mathcal{E})] &= \sum_{k=1}^{n} \left(\frac{\partial \beta(\mathcal{E})}{\partial x^k} - \sum_{l=1}^{n} y^l \Gamma^i_{lk}(x,y) \frac{\partial \beta(\mathcal{E})}{\partial y^i} \right) dx^k \\ &= \sum_{k,l=1}^{n} y^l \left(\frac{\partial \beta_l}{\partial x^k} - \sum_{h=1}^{n} \Gamma^h_{kl}(x) \beta_h \right) dx^k = (\nabla^g \beta)(\mathcal{E}). \end{aligned}$$

Consequently β must be parallel with respect to ∇^g. □

A Finsler metric (M, L) is said to be *generalized Berwald* if there exists a linear connection ∇ with torsion T such that $D = \pi^* \nabla$ (cf. [Ha2]). By the same argument as that in [Sz], we can prove the following:

Theorem 3.4 ([Ai]). *Let (M, L) be a generalized Berwald space whose Chern connection D is given by $D = \pi^* \nabla$ for a linear connection ∇ with torsion T. Then there exists a Riemannian metric g such that ∇ is compatible with g.*

In particular, a generalized Berwald space is said to be a *Wagner space* if the linear connection ∇ is *semi-symmetric*, that is, there exists a one-form γ such that its torsion T is given by

$$T(X, Y) = \gamma(X)Y - \gamma(Y)X. \tag{3.15}$$

Since a semi-symmetric linear connection that is compatible with a Riemanian metric g is uniquely determined, we obtain the following as an application of Theorem 3.4.

Theorem 3.5. *Let (M, L) be a Wagner space, and let ∇ be the semi-symmetric linear connection whose torsion form T is given by* (3.15)*. Then there exists a Riemannian metric g such that ∇ is given by*

$$\nabla_X Y = \nabla^g_X Y - \gamma(Y)X + g(X, Y)\gamma^{\#}, \tag{3.16}$$

where $\gamma^{\#}$ is the dual of γ with respect to g.

The notion of Wagner spaces has a deep relation with the conformal flatness of Finsler spaces.

4. Conformally flat Randers metrics

A *conformal change* of Finsler metric L is defined by the change $L \mapsto \widetilde{L} = e^{\sigma(x)} L$ for a smooth function $\sigma(x)$ on M.

Definition 4.1. A Finsler space (M, L) is said to be *locally conformal to a Berwald space* if there exists a local function $\sigma_U(x)$ on an open subset $U \subset M$ such that $\widetilde{L}_U = e^{\sigma_U(x)} L$ is Berwald on U. A Finsler space (M, L) is said to be *conformally flat* if (M, L) is locally conformal to a flat Finsler space.

The following theorem is fundamental in the rest of this paper.

Theorem 4.1 ([Ha-Ic2]). *A Finsler space (M, L) is locally conformal to a Berwald space if and only if (M, L) is a Wagner space with respect to a closed one-form β. In particular, (M, L) is conformally flat if and only if (M, L) is a Wagner space whose semi-symmetric connection is flat.*

Let E be a semi-parallel vector field on a Rimeannian space (M, g) with unit length, that is,

$$\nabla^g_X E = \rho\big[X - \beta(X)E\big] \tag{4.1}$$

for a constant ρ. The dual β of E is a closed one-form satisfying (1.1). For this one-form β, we define a linear connection ∇ by

$$\nabla_X Y = \nabla^g_X Y + \rho\big[g(X, Y)E - \beta(Y)X\big] \tag{4.2}$$

for the Riemannian connection ∇^g of (M, g).

Then we have

Proposition 4.1. *The curvature tensor R of the connection ∇ defined by (4.2) is given by*

$$R(X, Y)Z = R^g(X, Y)Z + \rho^2\big[g(Y, Z)X - g(X, Z)Y\big] \tag{4.3}$$

in term of the curvature R^g of ∇^g.

Proof. Because of $\nabla_X E = 0$, we have

$$\begin{aligned}
\nabla_X \nabla_Y Z = \nabla^g_X \nabla^g_Y Z &+ \rho\big[g(X, \nabla^g_Y Z)E - \beta(\nabla^g_Y Z)X - X(\beta(Z))Y \\
&+ X(g(Y, Z))E - \beta(Z)\nabla^g_X Y\big] - \rho^2\beta(Z)\big[g(X, Y)E - \beta(Y)X\big], \\
\nabla_Y \nabla_X Z = \nabla^g_Y \nabla^g_X Z &+ \rho\big[g(Y, \nabla^g_X Z)E - \beta(\nabla^g_X Z)Y - Y(\beta(Z))X \\
&+ Y(g(X, Z))E - \beta(Z)\nabla^g_Y X\big] - \rho^2\beta(Z)\big[g(Y, X)E - \beta(X)Y\big]
\end{aligned}$$

and

$$\nabla_{[X,Y]} Z = \nabla^g_{[X,Y]} Z + \rho\big[g([X, Y], Z)E - \beta(Z)[X, Y]\big].$$

Furthermore, because of

$$\begin{aligned}
g(X, \nabla^g_Y Z) + X(g(Y, Z)) - Y(g(X, Z)) - g(Y, \nabla^g_X Z) - g([X, Y], Z) \\
= g(\nabla^g_X Y, Z) - g(\nabla^g_Y X, Z) - g([X, Y], Z) = 0,
\end{aligned}$$

$$-\beta(\nabla^g_Y Z) + Y(\beta(Z)) = g(Z, \nabla^g_Y E)$$

and

$$-X(\beta(Z)) + \beta(\nabla^g_X Z) = -g(Z, \nabla^g_X E),$$

we obtain

$$\begin{aligned} R(X,Y)Z &= R^g(X,Y)Z + \rho\big[g(Z,\nabla^g_Y E)X - g(Z,\nabla^g_X E)Y\big] \\ &\quad + \rho^2\beta(Z)\big[\beta(Y)X - \beta(X)Y\big] \\ &= R^g(X,Y)Z + \rho^2\big[g(Z,Y)X - g(Z,X)Y\big]. \end{aligned}$$

By this proposition, we see that the connection ∇ defined by (4.2) is flat if and only if (M,g) is of negative constant curvature $K = -\rho^2$. Then we have

Theorem 4.2. *Suppose that a Riemannian space (M,g) admits a semi-parallel vector field E of unit length. Then the Finsler metric L defined by*

$$L(X) = \sqrt{g(X,X)} + c\cdot\beta(X) \quad (0 < c < 1) \tag{4.4}$$

is locally conformal to a Berwald metric. Furthermore, if (M,g) is a space of negative constant curvature, then (M,L) is conformally flat.

Proof. Suppose that (M,g) admits a vector field E satisfying (4.1). Then it is easily shown that ∇ defined by (4.2) is compatible with g, and that ∇ has the torsion $T(X,Y) = \rho[\beta(X)Y - \beta(Y)X]$. Furthermore, by direct computation, we can show that the dual β of E is parallel with respect to ∇. Therefore, similarly to the proof of Theorem 3.3, we have

$$d_\nabla L = 0.$$

Consequently, by Theorem 4.1, L is locally conformal to a Berwald metric.

In particular, if (M,g) is a space of negative constant curvature, then (2.6) implies the constant curvature K must be $K = -\rho^2$. Therefore (4.3) implies that the connection ∇ defined by (4.2) is flat. Consequently (M,L) is conformally flat. □

Example 4.1. Let $\mathbb{H}$ the upper half plane with the Poincaré metric g_P. For the function $f(x^1,\dots,x^n) = \log x^n$, the one-form β defined by $\beta = c\cdot df$ for a constant c such that $0 < c < 1$ has constant norm $\|\beta\|_P = c < 1$ with respect to g_P. We shall define a Randers metric $L_{\mathbb{H}}$ on $\mathbb{H}$ by

$$L_{\mathbb{H}}(X) = \sqrt{g_P(X,X)} + c\cdot df(X) \quad (0 < c < 1). \tag{4.5}$$

Then, as shown in Example 2.2, $\nabla^g f$ is semi-parallel vector field on $\mathbb{H}$ with unit length. Since $(\mathbb{H}, g_P)$ is negative constant curvature -1 and the form β satisfies the condition (4.1), $(\mathbb{H}, L_{\mathbb{H}})$ is conformally flat. In deed, $L_{\mathbb{H}}$ is given by

$$L_{\mathbb{H}}(X) = \frac{1}{x^n}\sqrt{\sum_{i=1}^{n}(X^i)^2} + c\,\frac{X^n}{x^n} \quad (0 < c < 1)$$

for every $X \in T\mathbb{H}$. □

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