

A note on n -clean group rings

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Abstract. Let R be an associative ring with identity. An element $x \in R$ is clean if x can be written as the sum of a unit and an idempotent in R . R is said to be clean if all of its elements are clean. Let n be a positive integer. An element $x \in R$ is n -clean if it can be written as the sum of an idempotent and n units in R . R is said to be n -clean if all of its elements are n -clean. In this paper we obtain conditions which are necessary or sufficient for a group ring to be n -clean.

1. Introduction

Throughout this paper all rings are associative with identity. The notion of clean rings was first introduced by NICHOLSON in [4]. An element x in a ring R is said to be clean if x can be written as the sum of a unit and an idempotent in R . The ring R is clean if every element in R is clean. In [6], XIAO and TONG generalised clean rings to n -clean rings. For a positive integer n , an element x in a ring R is n -clean if x can be written as the sum of an idempotent and n units in R . A ring R is n -clean if all of its elements are n -clean. Clearly, a 1-clean ring is a clean ring and vice versa.

It is known by work of CHEN and ZHOU [1], as well as MCGOVERN [3], that for a commutative ring R and an abelian group G , if RG is clean, then G is locally finite. We extend this result to n -clean rings in this paper. We also show that a partial converse of this result is true. That is, we show that if R is a commutative clean ring and G is a locally finite p -group where p is some prime with $p \in J(R)$,

Mathematics Subject Classification: 16E50, 16S34, 16U99.

Key words and phrases: clean, n -clean, group ring.

then RG is n -clean. The notation $J(R)$ as usual denotes the Jacobson radical of the ring R .

2. Some preliminary results

In this section we obtain some results which will be used in the proofs of the main results.

Proposition 2.1. *Let n be a positive integer. Then every homomorphic image of an n -clean ring is n -clean.*

PROOF. Let R be an n -clean ring and let $\phi : R \rightarrow S$ be a ring epimorphism. Let $x \in S$. Then $x = \phi(y)$ for some $y \in R$. Since R is n -clean, then $y = e + u_1 + \cdots + u_n$ for some idempotent e and units $u_1, \dots, u_n$ in R . Since ϕ is an epimorphism, we then have that $\phi(e)$ is an idempotent, $\phi(u_i)$ is a unit in S ($i = 1, \dots, n$) and $x = \phi(y) = \phi(e) + \phi(u_1) + \cdots + \phi(u_n)$, that is, x is n -clean in S . It follows that $\phi(R) = S$ is n -clean. $\square$

Proposition 2.2. *Let R be a ring and let n be a positive integer. If R is local, then R is n -clean.*

PROOF. Since R is local, the only idempotents in R are 0 and 1. Let $x \in R$.

Case 1: x is a unit

Note that

$$x = \begin{cases} 0 + (x + (-x)) + \cdots + (x + (-x)) + x, & \text{if } n \text{ is odd} \\ 1 + (x + (-1)) + (x + (-x)) + \cdots + (x + (-x)), & \text{if } n \text{ is even.} \end{cases}$$

Case 2: x is not a unit

In this case $1 - x$ is a unit. Hence $x - 1$ is also a unit and we have

$$x = \begin{cases} 1 + ((1 - x) + (x - 1)) + \cdots + ((1 - x) + (x - 1)) + (x - 1), & \text{if } n \text{ is odd} \\ 0 + ((x - 1) + 1) + ((1 - x) + (x - 1)) + \cdots + ((1 - x) + (x - 1)), & \text{if } n \text{ is even.} \end{cases}$$

In both cases, we have that x can be written as the sum of an idempotent and n units; hence x is n -clean. $\square$

Proposition 2.3. *A clean ring without any nontrivial idempotents is local.*

PROOF. Let R be a clean ring with no nontrivial idempotents. Then for any $x \in R$, either x or $1 - x$ is a unit. Suppose that R has two distinct maximal right ideals M_1 and M_2 . Then there exists an element $a \in M_1$, $a \notin M_2$. Thus for every $r \in R$, $ar \in M_1$ and since M_1 is a maximal right ideal (hence, a proper ideal), it follows that ar is not a unit. Therefore $1 - ar$ is a unit for every $r \in R$. Now from $M_2 + aR = R$, we have that $1 - as \in M_2$ for some $s \in R$. But since $1 - as$ is a unit, we have that $M_2 = R$ which contradicts the fact that M_2 is proper. Hence, R only has one maximal right ideal, that is, R must be local. $\square$

Proposition 2.4. *Let R be a ring, let G be a group and let n be a positive integer. If $x \in RG$ is not n -clean, then there exists a proper ideal J of R such that R/J does not have any nontrivial central idempotent and $x + JG \in RG/JG$ is not n -clean.*

PROOF. Suppose that $x \in RG$ is not n -clean. Let $\mathcal{C} = \{I \triangleleft R \mid I \neq R, x + IG \in RG/IG \text{ is not } n\text{-clean}\}$. Then $\mathcal{C} \neq \emptyset$ because $\{0\} \in \mathcal{C}$. Let $I_1 \subseteq I_2 \subseteq \dots \subseteq I_j \subseteq I_{j+1} \subseteq \dots$ be a chain of ideals in $\mathcal{C}$ and let $I = \cup I_i$. Then I is a proper ideal of R and $\bar{x} = x + IG \in RG/IG$ is not n -clean. Indeed, if $\bar{x}$ is n -clean, then $x + I_t G \in RG/I_t G$ is n -clean for some $t \in \mathbb{N}$; a contradiction. Therefore $I \in \mathcal{C}$. By Zorn's Lemma, $\mathcal{C}$ has a maximal element, say J .

If R/J has a nontrivial central idempotent $e + J$, then

$$R/J = (e + J)R/J \oplus ((1 - e) + J)R/J \cong I_1/J \times I_2/J$$

for some ideals I_1, I_2 of R properly containing J . Then

$$RG/JG \cong (R/J)G \cong (I_1/J)G \times (I_2/J)G \cong I_1G/JG \times I_2G/JG. \quad (1)$$

Note that

$$RG/I_1G \cong (RG/JG) / (I_1G/JG) \cong I_2G/JG \quad (2)$$

and

$$RG/I_2G \cong (RG/JG) / (I_2G/JG) \cong I_1G/JG. \quad (3)$$

Let $(x_1 + JG, x_2 + JG) \in I_1G/JG \times I_2G/JG$ be the image of $x + JG$ under the isomorphism in (1). By the maximality of J in $\mathcal{C}$, $x + I_t G$ is n -clean in $RG/I_t G$ ($t = 1, 2$). Hence, by (2) and (3), $x_t + JG$ is n -clean in $I_t G/JG$ ($t = 1, 2$). It follows by (1) that $x + JG \in RG/JG$ is n -clean. This is a contradiction since $J \in \mathcal{C}$. Hence, R/J does not have any nontrivial central idempotent. $\square$

Corollary 2.5. *Let R be a commutative clean ring, let G be a group and let n be a positive integer. If $x \in RG$ is not n -clean, then there exists a proper ideal J of R such that R/J is local and $x + JG \in RG/JG$ is not n -clean.*

PROOF. By Proposition 2.4, there exists a proper ideal J of R such that R/J does not have any nontrivial idempotent and $x + JG \in RG/JG$ is not n -clean. Since R/J is clean (by Proposition 2.1), it follows by Proposition 2.3 that R/J is local. $\square$

In [5], NICHOLSON obtained sufficient conditions for a group ring to be local as follows:

Proposition 2.6. *Let R be a local ring and let G be a locally finite p -group where p is some prime with $p \in J(R)$. Then RG is local.*

3. Main results

For a ring R , let $Id(R)$ and $U(R)$ denote the set of all idempotents and the set of all units of R , respectively. We first extend an idea in [1] to obtain necessary conditions for a commutative group ring to be n -clean.

Theorem 3.1. *Let R be a commutative ring, let G be an abelian group and let n be a positive integer. If RG is n -clean, then R is n -clean and G is locally finite.*

PROOF. Since $RG/\Delta \cong R$ where Δ is the augmentation ideal of RG , it follows readily by Proposition 2.1 that R is n -clean. Suppose that G is not locally finite. Then G is not torsion; hence $G/t(G)$ is nontrivial and torsion-free, where $t(G)$ is the torsion subgroup of G . Since $R(G/t(G)) \cong RG/R(t(G))$ is a homomorphic image of RG and RG is n -clean, it follows by Proposition 2.1 that $R(G/t(G))$ is n -clean. We may therefore assume that G is torsion-free. If G has rank greater than 1, then G has a torsion-free quotient G' of rank 1. But since RG' is also n -clean, we can assume that G is of rank 1. Thus, G is isomorphic to a subgroup of $(\mathbb{Q}, +)$. Since R is commutative, then R/M is a field where M is a maximal ideal of R . Furthermore, $(R/M)G$ is n -clean because $(R/M)G \cong RG/MG$ is a homomorphic image of RG (by Proposition 2.1). Hence, we can assume that R is a field. Since G is torsion-free, there exists a $g \in G$ such that $g^{-1} \neq g$. Now since $g + \cdots + g^n + g^{-1} + \cdots + g^{-n}$ is n -clean in RG , there exists a finitely generated subgroup G_1 of G such that $g \in G_1$ and $g + \cdots + g^n + g^{-1} + \cdots + g^{-n}$ is n -clean in RG_1 . From above, G_1 is isomorphic to a finitely generated subgroup of $(\mathbb{Q}, +)$. Since every finitely generated subgroup of $(\mathbb{Q}, +)$ is cyclic, so is G_1 , and we can write $G_1 = \langle h \rangle$. Thus, $g = h^k$, $g^{-1} = h^{-k}$ for some $k \in \mathbb{N}$. Note that there is a natural isomorphism $R\langle h \rangle \cong R[x, x^{-1}]$ with

$h^k + \dots + h^{nk} + h^{-k} + \dots + h^{-nk} \leftrightarrow x^k + \dots + x^{nk} + x^{-k} + \dots + x^{-nk}$. This implies that $x^k + \dots + x^{nk} + x^{-k} + \dots + x^{-nk}$ is n -clean in $R[x, x^{-1}]$ which is impossible because $Id(R[x, x^{-1}]) \subseteq R$ and $U(R[x, x^{-1}]) \subseteq \{ax^i \mid 0 \neq a \in R, i \in \mathbb{Z}\}$. Hence, G must be locally finite. $\square$

In [2], it was shown that if R is a clean ring and G is a finite group such that $|G|$ is a unit in R , then RG is not necessarily clean. Here we prove the following:

Theorem 3.2. *Let R be a commutative clean ring and let G be a locally finite p -group with $p \in J(R)$. Then RG is an n -clean ring for any positive integer n .*

PROOF. Let n be a positive integer and suppose that $x \in RG$ is not n -clean. By Corollary 2.5, there exists a proper ideal I of R such that R/I is local and $x + IG \in RG/IG$ is not n -clean. If $p + I$ is a unit in R/I , then $pr - 1 \in I$ for some $r \in R$. But $p \in J(R)$ implies that $pr - 1$ is a unit in R . Hence $I = R$; a contradiction. Thus $p + I$ is not a unit in R/I and therefore, $p + I \in J(R/I)$. By Proposition 2.6, $RG/IG \cong (R/I)G$ is local. It follows by Proposition 2.2 that RG/IG is n -clean; a contradiction. Hence, $x \in RG$ must be n -clean. $\square$

Remark. Let $\mathbb{Z}_{(7)} = \{\frac{m}{n} \in \mathbb{Q} \mid 7 \text{ does not divide } n\}$ and let C_3 be the cyclic group of order 3. The example in [2] that the group ring $\mathbb{Z}_{(7)}C_3$ is not clean shows that the condition $p \in J(R)$ in Theorem 3.2 is not superfluous.

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(Received December 15, 2009; revised July 31, 2010)