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Nonaliquot numbers

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Abstract. For any positive integer n, let $\sigma(n)$ be the sum of the positive divisors of n. It is known that almost all odd numbers can be represented in the form $\sigma(m) - m$ for some natural number m. In this paper, we prove that the number of even numbers which are less than x and not of the form $\sigma(m) - m$ is at least $0.06x + o(x)$. This improves the lower bound $\frac{1}{48}x + o(x)$ obtained by Banks and Luca.

1. Introduction

For any positive integer n, let $\sigma(n)$ be the sum of divisors function, and let $\phi(n)$ be the Euler totient function. A positive integer n is called an aliquot number if $n = \sigma(m) - m$ for some positive integer m, otherwise it is called a nonaliquot number. Nonaliquot numbers are also known as untouchable numbers (see [3, B10]). In this paper we study the set of nonaliquot numbers defined by

$$N_a(x) = \{1 \le n \le x : n \text{ is a nonaliquot number}\}.$$

It is easy to see that almost all odd numbers are aliquot numbers, and thus $|N_a(x)| \le \frac{1}{2}x + o(x)$. Indeed, it is well known that almost all even numbers can be represented as the sum of two distinct primes (for example, see Vaughan [5]). If $2n = p + q$ for distinct primes p and q, then $2n + 1 = \sigma(pq) - pq$. Hence $2n + 1$ is an aliquot number.

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Concerning lower bounds, Erdős [2] showed that $|N_a(x)| \geq cx$ for some positive constant c and all sufficiently large x. Banks and Luca [1] proved that

$$|N_a(x)| \geq \frac{x}{48}(1+o(1)) = 0.020833\cdots x, \quad x \to \infty.$$

P. G. Walsh commented in this review [MR2148946] on the paper [1] that it would be interesting to know if this is indeed the correct constant.

The main result of this paper is the following.

Theorem 1. *For any positive integer M, we have*

$$|N_a(x)| \geq g_M x + o_M(x),$$

where

$$g_M = \sum_{d|M} \frac{\phi(M/d)}{M/d} \max\left\{0, \frac{1}{2d} - \frac{1}{\sigma(2d)-2d}\right\}.$$

Taking $M = 2^6 \times 3^5 \times 5^4 \times 7^3 \times 11^2 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41$, we have $g_M > 0.0602757$. Let $g = \sup g_M$. One can prove that $g_M < g$ for any positive integer M. We conjecture that $g < 0.07$.

Question 1. *Is it true that $|N_a(x)| = gx + o(x)$?*

Question 2. *Are there a positive proportion even numbers which are aliquot numbers?*

Question 3. *What is an approximate numerical value for the constant g?*

Question 4. *Is the constant g irrational?*

2. Proof of Theorem 1

For a set U of positive integers and $x > 0$, let

$$U(x) = \{a \leq x : a \in U\}.$$

First we state the following lemma.

Lemma 1. *Let k be a positive integer. Then $|\{n \leq x : k \mid \sigma(n)\}| = x + o_k(x)$.*

Lemma 1 is a weak form of [4, Lemma 4]. Erdős [2] proved that for any fixed prime p, $|\{n \le x : p \mid \sigma(n)\}| = x + o_p(x)$.

Now we return to the proof of Theorem 1.

Let M be a given integer. Let $2n$ be an even number such that $2n \le x$ and $2n = \sigma(m) - m$ for some positive integer m. If m is odd, then $\sigma(m)$ is odd, and in this case Banks and Luca [1] proved that the number of such $2n \le x$ is $o(x)$. Now we assume that m is even. Then $\sigma(m) - m \ge m/2$. So $m \le 2x$ since $2n \le x$. By Lemma 1, the number of $m \le 2x$ with $2M \nmid \sigma(m)$ is $o(x)$. Next, we assume that $2M \mid \sigma(m)$. Let

$$H_M(x) = \{2n \le x : 2n = \sigma(m) - m \text{ for some integer } m \text{ with } 2M \mid \sigma(m)\}.$$

For $d \mid M$ let

$$A_d(x) = \{2n \le x : (n, M) = d\}$$

and $B_d(x) = A_d(x) \cap H_M(x)$. For $2n \in A_d(x)$, let $n = dn_1$. Then $n_1 \le x/(2d)$ and $(n_1, M/d) = 1$. So

$$\frac{\phi(M/d)}{M/d}\frac{x}{2d} - \phi(M/d) \le |A_d(x)| \le \frac{\phi(M/d)}{M/d}\frac{x}{2d} + \phi(M/d). \tag{1}$$

For $2n \in B_d(x)$, we have $2n = \sigma(m) - m$ with $(m, 2M) = 2d$ since $2M \mid \sigma(m)$. Let $m = 2dm_1$. Then $(m_1, M/d) = 1$ and

$$2n = \sigma(m) - m = \sigma(2dm_1) - 2dm_1 \ge \sigma(2d)m_1 - 2dm_1.$$

As $2n \le x$ we have

$$m_1 \le \frac{x}{\sigma(2d) - 2d}.$$

Since $(m_1, M/d) = 1$, the number of m with $\sigma(m) - m = 2n \in B_d(x)$ is less than

$$\frac{\phi(M/d)}{M/d}\frac{x}{\sigma(2d) - 2d} + \phi(M/d).$$

Then

$$|B_d(x)| \le \frac{\phi(M/d)}{M/d}\frac{x}{\sigma(2d) - 2d} + \phi(M/d).$$

It is also clear that

$$|B_d(x)| \le |A_d(x)| \le \frac{\phi(M/d)}{M/d}\frac{x}{2d} + \phi(M/d).$$

Hence

$$|B_d(x)| \le \frac{\phi(M/d)}{M/d} \min\left\{\frac{1}{2d}, \frac{1}{\sigma(2d)-2d}\right\} x + \phi(M/d). \tag{2}$$

By (1) and (2) we have

$$\begin{aligned}|A_d(x) \setminus B_d(x)| &\ge \frac{\phi(M/d)}{M/d}\left(\frac{1}{2d} - \min\left\{\frac{1}{2d}, \frac{1}{\sigma(2d)-2d}\right\}\right) x - 2\phi(M/d)\\ &= x\frac{\phi(M/d)}{M/d} \max\left\{0, \frac{1}{2d} - \frac{1}{\sigma(2d)-2d}\right\} - 2\phi(M/d).\end{aligned}$$

Thus

$$\begin{aligned}|N_a(x)| &= \sum_{d|M} |A_d(x) \setminus B_d(x)| + o(x)\\ &\ge x\sum_{d|M} \frac{\phi(M/d)}{M/d} \max\left\{0, \frac{1}{2d} - \frac{1}{\sigma(2d)-2d}\right\} - 2\sum_{d|M}\phi(M/d) + o(x).\end{aligned}$$

Since $\sum_{d|M} \phi(M/d) = M = o(x)$, this completes the proof.

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