

Diffeomorphic theorems for open Riemannian manifolds with curvature decay

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Abstract. In this paper, we study the topology of complete non-compact Riemannian manifolds with curvature decay to a non-positive constant. We show that such a complete open manifold M is diffeomorphic to a Euclidean n -space $\mathbb{R}^n$ if it contains enough rays starting from the base point. As applications, we also show that this kind of manifolds with Ricci curvature bounded from below by a non-positive constant are diffeomorphic to $\mathbb{R}^n$ if the volumes of geodesic balls in M grow properly. Our results generalize the main theorems of Wang–Xia for manifolds with quadratic curvature decay to zero.

1. Introduction

Let (M, g) be a complete non-compact $n(\geq 2)$ -dimensional Riemannian manifold. For a fixed point $p \in M$ and any $r > 0$, denote by $B(p, r)$ the open geodesic ball around p with radius r in M , and $S(p, r)$ the corresponding geodesic sphere.

Denote by K_M the sectional curvature of M , and let

$$k_p(r) = \inf_{M \setminus B(p, r)} K_M,$$

where the infimum is taken over all the sections at all points on $M \setminus B(p, r)$. It is easy to see that we can choose $k_p(r)$ to be a non-positive monotone increasing function of r .

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In this paper, we consider complete open Riemannian manifolds with curvature decay to a non-positive constant, that is to say, there exists a positive monotone decreasing function $K(r)$ with $\lim_{r \rightarrow \infty} K(r) \geq 0$ satisfying

$$k_p(r) \geq -(K(r))^2, \quad \text{for all } r > 0.$$

Let R_p denote the (point set) union of rays issuing from p . One can show that R_p is a closed subset of M . Define a function h_p on M by

$$h_p(x) = d(x, R_p),$$

where d is the distance function on M . We set for $r > 0$ (cf. [8], [17], [21])

$$\mathcal{H}(p, r) = \max_{x \in S(p, r)} d(x, R_p). \quad (1.1)$$

By definition, we always have

$$\mathcal{H}(p, r) \leq \max_{x \in S(p, r)} d(x, p) = r, \quad \text{for all } r > 0.$$

If M is a complete simply connected Riemannian manifold with non-positive sectional curvature, then $\mathcal{H}(p, x) \equiv 0$. This follows from the fact that any point in M lies in some ray starting from p .

For a constant $c \geq 0$, we denote by $M^n(-c)$ an n -dimensional complete simply connected Riemannian manifold of constant curvature $-c$. If the Ricci curvature of M satisfies $\text{Ric}_M \geq -(n-1)c$, the relative volume comparison theorem [4] tells us that the function $r \rightarrow \frac{\text{vol}[B(p, r)]}{\alpha_n(r, -c)}$ is monotone decreasing, where $\text{vol}[B(p, r)]$ is the volume of $B(p, r)$ and $\alpha_n(r, -c)$ the volume of a geodesic ball of radius r in $M^n(-c)$. It is well known that

$$\alpha_n(r, -c) = \omega_{n-1} \int_0^r f_{-c}^{n-1}(t) dt,$$

where

$$f_{-c}(t) = \begin{cases} t, & c = 0, \\ \frac{\sinh(\sqrt{c}t)}{\sqrt{c}}, & c > 0, \end{cases} \quad (1.2)$$

and ω_m is the volume of $S^m(1)$.

For any $p \in M$, we set

$$v_{-c}(p) = \lim_{r \rightarrow \infty} \frac{\text{vol}[B(p, r)]}{\alpha_n(r, -c)},$$

and define

$$v_{-c} = \inf_{p \in M} v_{-c}(p). \quad (1.3)$$

One always has

$$\frac{\text{vol}[B(p, r)]}{\alpha_n(r, -c)} \geq v_{-c}(p) \geq v_{-c}, \quad \forall r > 0, \quad \forall p \in M.$$

ABRESCH [1] proved that if $\int_0^\infty r k_p(r) dr > -\infty$, then M is of finite topological type. XIA [21] proved that if M is an n -dimensional complete open Riemannian manifold with nonnegative sectional curvature in which there exists a $p \in M$ such that $\mathcal{H}(p, r) < r$, for all $r > 0$, then M is diffeomorphic to $\mathbb{R}^n$. WANG–XIA [19] proved that there exists a constant $\epsilon = \epsilon(n) > 0$ such that an n -dimensional open manifold with quadratic curvature decay to zero and $\mathcal{H}(p, r) < \epsilon r$ for all $r > 0$ is diffeomorphic to $\mathbb{R}^n$. For recent progress on manifolds with quadratic curvature decay, we refer to the paper of YEGANEFAR [24] for more details.

In this paper, we first obtain the following pinching theorem, which generalizes the result of WANG–XIA in [19]

Theorem 1.1. *Given a positive monotone decreasing function $K(r)$, there are positive constants $\epsilon, \delta \in (0, 1)$ with $\epsilon + \delta < 1$, such that any n -dimensional complete open manifold M satisfying*

$$k_p(r) \geq -(K(r))^2,$$

and

$$\mathcal{H}(p, r) < \epsilon r - \frac{1}{2K(\delta r)} \ln \left\{ \cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right\}, \quad (1.4)$$

for some $p \in M$ and all $r > 0$ is diffeomorphic to $\mathbb{R}^n$.

Our second theorem is on a Riemannian manifold with large volume growth, i.e. $v_{-c} > 0$. There have many articles studying complete noncompact Riemannian manifold with large volume growth (cf. [3], [7], [8], [11], [12], [14]–[23]). If M has nonnegative Ricci curvature, it has been proven by LI [12] and ANDERSON [3] that $\pi_1(M)$ is finite. PERELMAN [15] has shown that there is a small constant $\epsilon(n) > 0$ depending only on n such that if $v_0 > 1 - \epsilon(n)$, then M is contractible, and CHEEGER–COLDING [7] showed that the manifold in Perelman’s theorem is actually diffeomorphic to $\mathbb{R}^n$. SHEN [17] has shown that M has finite topological type, provided that $\frac{\text{vol}[B(p, r)]}{\omega_n r^n} = v_0 + o(\frac{1}{r^{n-1}})$ and, either the conjugate radius

$\text{conj}_M \geq c > 0$ or the sectional curvature $K_M \geq K_0 > -\infty$. SHA-SHEN [16] proved that these manifolds have finite topological type if in addition the manifolds have quadratic curvature decay to zero. One can find some other topological uniqueness theorems about M , e.g. in [8], [14], [22].

Given a point $p \in M$, we denote by crit_p the criticality radius of M at p , i.e., crit_p is the smallest critical value for the distance function $d(p, \cdot) : M \rightarrow \mathbb{R}$. Recall that q is not a critical point of this distance function iff there exists a vector $\mathbf{v} \in S_q M$ such that for all minimizing geodesics σ from $\sigma(0) = q$ to p , we have $\angle(\mathbf{v}, \sigma'(0)) > \frac{\pi}{2}$ (cf. [6, 9]). In view of Theorem 1.1, we have the following topological rigidity theorem for manifolds with Ricci curvature bounded from below by a non-positive constant and large volume growth.

Theorem 1.2. *Given a positive monotone decreasing function $K(r)$, there are positive constants $\epsilon, \delta \in (0, 1)$ with $\epsilon + \delta < 1$, such that any n -dimensional complete open manifold M with $\text{Ric}_M \geq -(n-1)c$ ($c \geq 0$), $v_{-c} > 0$,*

$$k_p(r) \geq -(K(r))^2,$$

and

$$\frac{\text{vol}[B(p, r)]}{\alpha_n(r, -c)} < \left\{ 1 + \frac{\int_0^A f_{-c}^{n-1}(t) dt}{\int_0^{2r} f_{-c}^{n-1}(t) dt} \right\} v_{-c}, \quad (1.5)$$

for some $p \in M$ and all $r > 0$ is diffeomorphic to $\mathbb{R}^n$, where

$$A = \epsilon r - \frac{1}{2K(\delta r)} \ln \left\{ \cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right\}.$$

Especially, if M is of nonnegative Ricci curvature, we have

Theorem 1.3. *Given a positive monotone decreasing function $K(r)$, there are positive constants $\epsilon, \delta \in (0, 1)$ with $\epsilon + \delta < 1$, such that any n -dimensional complete open manifold M with $\text{Ric}_M \geq 0$, $v_0 > 0$,*

$$k_p(r) \geq -(K(r))^2,$$

and

$$\begin{aligned} \frac{\text{vol}[B(p, r)]}{\omega_n r^n} < \left\{ 1 + 2^{2-3n} r^{1-n} \left[\epsilon r - \frac{1}{2K(\delta r)} \ln \left(\cosh^2(K(\delta r)\epsilon r) \right. \right. \right. \\ \left. \left. \left. + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right) \right]^{n-1} \right\} v_0, \quad (1.6) \end{aligned}$$

for some $p \in M$ and all $r > 0$ is diffeomorphic to $\mathbb{R}^n$.

Remark 1.4. (i) If the curvature of M is quadratic decay to zero, i.e., there exists a constant $C > 0$ such that $K(r)r \leq C$ for all $r > 0$, then we can choose ϵ in Theorem 1.1 small enough so that (1.4) becomes

$$\mathcal{H}(p, r) < \tilde{\epsilon}r,$$

for some $\tilde{\epsilon} \in (0, 1)$ depending on ϵ , δ and C . Then by Theorem 1.1, M is diffeomorphic to $\mathbb{R}^n$, and this recovers WANG–XIA’s result in [19].

(ii) If the function $K(r) = Cr^{-\beta}$ for $C > 0, \beta \in [0, 1]$ and the Ricci curvature of M is nonnegative, (1.6) can be written in an explicit form, therefore we again obtain a WANG–XIA’s type theorem in [19].

2. Preliminaries

Let (M, g) be a complete non-compact n -dimensional Riemannian manifold. For a fixed point $p \in M$. We say that $K_p^{\min} \geq c$ if for any minimal geodesic γ issuing from p all sectional curvatures of planes which are tangent to γ are greater than or equal to c . For $p, q \in M$, the excess function $e_{pq}(x)$ is defined by

$$e_{pq}(x) = d(p, x) + d(q, x) - d(p, q).$$

We denote by $M^2(c)$ the complete simply connected surface of constant curvature c . Throughout this paper, all geodesics are assumed to have unit speed. In [13], MACHIGASHIRA proved the following Toponogov-type comparison theorem for complete manifolds with $K_p^{\min} \geq c$.

Lemma 2.1. *Let M be a complete n -manifold and p be a point of M with $K_p^{\min} \geq c$. Let $\gamma_i : [0, l_i] \rightarrow M$, $i = 0, 1, 2$ be minimal geodesics with $\gamma_1(0) = \gamma_2(l_2) = p$, $\gamma_0(0) = \gamma_1(l_1)$ and $\gamma_0(l_0) = \gamma_2(0)$. Then there exist minimal geodesics $\tilde{\gamma}_i : [0, l_i] \rightarrow M^2(c)$, $i = 0, 1, 2$ with $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(l_2)$, $\tilde{\gamma}_0(0) = \tilde{\gamma}_1(l_1)$ and $\tilde{\gamma}_0(l_0) = \tilde{\gamma}_2(0)$ which are such that*

$$L(\gamma_i) = L(\tilde{\gamma}_i) \quad \text{for } i = 0, 1, 2,$$

and

$$\begin{aligned} \angle(-\gamma'_1(l_1), \gamma'_0(0)) &\geq \angle(-\tilde{\gamma}'_1(l_1), \tilde{\gamma}'_0(0)), \\ \angle(-\gamma'_0(l_0), \gamma'_2(0)) &\geq \angle(-\tilde{\gamma}'_0(l_0), \tilde{\gamma}'_2(0)). \end{aligned}$$

Lemma 2.2 ([22]). *Let M be a complete n -manifold with $\text{Ric}_M \geq 0$ and $v_0 > 0$. Then for any $r > 0$ and any $x \in S(p, r)$, we have*

$$d(x, R_p) \leq 2v_0^{-\frac{1}{n}} \left\{ \frac{\text{vol}[B(p, r)]}{\omega_n r^n} - v_0 \right\}^{\frac{1}{n}} r.$$

Lemma 2.3 ([2]). *Let (M, g) be a complete n -manifold with $\text{Ric}_M \geq 0$. Let $\gamma : [0, a] \rightarrow M$ be a minimal geodesic from p to q . Then for any $x \in M$*

$$e_{pq}(x) \leq 8 \left(\frac{s^n}{r} \right)^{\frac{1}{n-1}},$$

where $s = d(x, \gamma)$ and $r = \min(d(p, x), d(q, x))$.

Let Σ be a closed subset of the unit tangent sphere $S_p M$ of M at p . Denote by $B_\Sigma(p, r)$ the set of points $x \in B(p, r)$ such that there is a minimizing geodesic γ from p to x with $\gamma'(0) \in \Sigma$. For $0 < r \leq \infty$, let $\Sigma_p(r)$ denote the set of unit vectors $\mathbf{v} \in \Sigma$ such that the geodesic $\gamma(t) = \exp_p(t\mathbf{v})$ is minimizing on $[0, r)$. Notice that

$$\Sigma_p(r_2) \subset \Sigma_p(r_1), \text{ for } 0 < r_1 < r_2; \quad \Sigma_p(\infty) = \bigcap_{r>0} \Sigma_p(r).$$

The standard argument [4, 5] gives the following generalized Bishop's comparison theorem.

Lemma 2.4 ([22]). *Let (M, g) be a complete n -manifold with $\text{Ric}_M \geq 0$ and $v_0 > 0$. Then*

$$\frac{\text{vol}[B_{\Sigma_p(\infty)}(p, r)]}{\alpha_n(r, 0)} \geq v_0.$$

Lemma 2.5 ([23]). *Let (M, g) be a complete n -manifold with $\text{Ric}_M \geq -(n-1)$ and $v_{-1} > 0$. Then*

$$\frac{\text{vol}[B_{\Sigma_p(\infty)}(p, r)]}{\alpha_n(r, -1)} \geq v_{-1}.$$

It is not difficult to check that Lemma 2.5 also holds for $\text{Ric}_M \geq -(n-1)c$ ($c > 0$). We then have the following corollary.

Corollary 2.6. *Let (M, g) be a complete n -manifold with $\text{Ric}_M \geq -(n-1)c$, $c \geq 0$ and $v_{-c} > 0$. Then for any $p \in M$ and any $r > 0$, we have*

$$\int_0^{\mathcal{H}(p, r)} f_{-c}^{n-1}(t) dt \leq v_{-c}^{-1} \int_0^{2r} f_{-c}^{n-1}(t) dt \left\{ \frac{\text{vol}[B(p, r)]}{\alpha_n(r, -c)} - v_{-c} \right\},$$

where $f_{-c}(t)$ is defined in (1.2), and v_{-c} is defined in (1.3).

PROOF. Fix an point $x \in S(p, r)$, and set $s = d(x, R_p)$, then $s \leq r$ and

$$B(p, r) \bigcup B_{\Sigma_p(\infty)}(p, 2r) \subset B(p, 2r).$$

The left hand side is a disjoint union. We have

$$\text{vol}[B(p, r)] \geq v_{-c}\{\alpha_n(s, -c)\}.$$

From Lemma 2.4 and 2.5, one obtains

$$\begin{aligned} \text{vol}[B(p, 2r)] &\geq \text{vol}[B(x, s)] + \text{vol}[B_{\Sigma_p(\infty)}(p, 2r)] \\ &\geq v_{-c}\alpha_n(s, -c) + v_{-c}\alpha_n(2r, -c). \end{aligned}$$

By Bishop's comparison theorem, we have

$$\frac{\text{vol}[B(p, 2r)]}{\alpha_n(2r, -c)} \leq \frac{\text{vol}[B(p, r)]}{\alpha_n(r, -c)}.$$

We then obtain

$$\int_0^{\mathcal{H}(p,r)} f_{-c}^{n-1}(t)dt \leq v_{-c}^{-1} \int_0^{2r} f_{-c}^{n-1}(t)dt \left\{ \frac{\text{vol}[B(p, r)]}{\alpha_n(r, -c)} - v_{-c} \right\}. \quad \square$$

3. Proofs of Theorems

PROOF OF THEOREM 1.1. We shall prove that M contains no critical points of the distance function $d(p, \cdot)$ other than p , and therefore it is diffeomorphic to $\mathbb{R}^n$ (cf. [9], Disk Theorem). We refer to [6], [9], [10] for the notion of critical points of the distance functions and their applications.

For any $\epsilon, \delta \in (0, 1)$ with $\epsilon + \delta < 1$ and any $r > 0$, we see

$$\cosh(2K(\delta r)\epsilon r) - \cosh^2(K(\delta r)\epsilon r) = \cosh^2(K(\delta r)\epsilon r) - 1 > 0. \quad (3.1)$$

By definition, $\cosh(2K(\delta r)\epsilon r) = \frac{1}{2}(e^{2K(\delta r)\epsilon r} + e^{-2K(\delta r)\epsilon r})$, we have from (3.1)

$$(e^{2K(\delta r)\epsilon r})^2 - 2\cosh^2(K(\delta r)\epsilon r)e^{2K(\delta r)\epsilon r} + 1 > 0,$$

which implies

$$\epsilon r - \frac{1}{2K(\delta r)} \ln \left\{ \cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right\} > 0. \quad (3.2)$$

Take an arbitrary point $x(\neq p) \in M$ and set $r = d(p, x)$. By (1.1), $\mathcal{H}(p, r)$ must be nonnegative. By (3.2), we see that our condition (1.4) is reasonable, and this enables us to find a ray $\gamma : [0, +\infty) \rightarrow M$ such that

$$s = d(x, \gamma) < \epsilon r - \frac{1}{2K(\delta r)} \ln \left\{ \cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right\}. \quad (3.3)$$

Fix a minimizing geodesic σ from x to $q = \gamma(2r)$. For any minimal geodesic σ_1 from x to p , let $\tilde{p} = \sigma_1(\epsilon r)$ and $\tilde{q} = \sigma(\epsilon r)$. The choice of ϵ and δ indicates that $\sigma|_{[0, \epsilon r]}$ and $\sigma_1|_{[0, \epsilon r]}$ are disjoint with $B(p, \delta r)$. Moreover, the sectional curvature of M satisfies $K_M \geq -(K(\delta r))^2$ on $M \setminus B(p, \delta r)$. Applying the Toponogov comparison theorem Lemma 2.1 to the hinge $(\sigma|_{[0, \epsilon r]}, \sigma_1|_{[0, \epsilon r]})$ in $M \setminus B(p, \delta r)$, we have

$$\cos \theta \sinh^2(K(\delta r)\epsilon r) \leq \cosh^2(K(\delta r)\epsilon r) - \cosh(K(\delta r)d(\tilde{p}, \tilde{q})), \quad (3.4)$$

where $\theta = \angle(\sigma'(0), \sigma_1'(0))$ is the angle of σ and σ_1 at x .

Let $m \in \gamma$ be such that $d(x, m) = d(x, \gamma)$, then $m \in \gamma(0, 2r)$. It follows from the triangle inequality that

$$\begin{aligned} d(\tilde{p}, \tilde{q}) &\geq d(p, q) - d(p, \tilde{p}) - d(q, \tilde{q}) \\ &= d(p, m) + d(q, m) - [d(p, x) - d(\tilde{p}, x)] - [d(x, q) - d(x, \tilde{q})] \\ &= 2\epsilon r + [d(p, m) - d(p, x)] + [d(q, m) - d(q, x)] \geq 2\epsilon r - 2d(x, m). \end{aligned}$$

Introducing (3.3) into the above inequality we see that

$$d(\tilde{p}, \tilde{q}) > \frac{1}{K(\delta r)} \ln \left\{ \cosh^2(K\epsilon r) + \sqrt{\cosh^4(K\epsilon r) - 1} \right\}. \quad (3.5)$$

This implies that

$$\cosh^2(K(\delta r)\epsilon r) - \cosh(K(\delta r)d(\tilde{p}, \tilde{q})) < 0. \quad (3.6)$$

From (3.4) and (3.6), we obtain

$$\cos \theta \sinh^2(K(\delta r)\epsilon r) \leq \cosh^2(K(\delta r)\epsilon r) - \cosh(K(\delta r)d(\tilde{p}, \tilde{q})) < 0.$$

Thus

$$\theta > \frac{\pi}{2}.$$

Therefore any minimizing geodesic σ_1 , from x to p has $\angle(\sigma_1'(0), \sigma'(0)) > \frac{\pi}{2}$, which implies that x is not a critical point of $d(p, \cdot)$. Theorem 1.1 follows. $\square$

PROOF OF THEOREM 1.2. We take the constants ϵ and δ in Theorem 1.2 to be the same as in Theorem 1.1. Therefore in order to prove Theorem 1.2, it suffices to show that

$$\mathcal{H}(p, r) < \epsilon r - \frac{1}{2K(\delta r)} \ln \left\{ \cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right\}.$$

Since $\text{Ric}_M \geq -(n-1)c$, where $c \geq 0$, we have by Corollary 2.6

$$\int_0^{\mathcal{H}(p, r)} f_{-c}^{n-1}(t) dt \leq v_{-c}^{-1} \int_0^{2r} f_{-c}^{n-1}(t) dt \left\{ \frac{\text{vol}[B(p, r)]}{\alpha_n(r, -c)} - v_{-c} \right\}.$$

Substituting the assumption (1.5) into the above inequality, we have

$$\int_0^{\mathcal{H}(p, r)} f_{-c}^{n-1}(t) dt < \int_0^A f_{-c}^{n-1}(t) dt.$$

Thus

$$\mathcal{H}(p, r) < \epsilon r - \frac{1}{2K(\delta r)} \ln \left\{ \cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right\}.$$

This completes the proof of Theorem 1.2. $\square$

Before proving Theorem 1.3, we need the following lemma.

Lemma 3.1. *Given a positive monotone decreasing function $K(r)$, there are positive constants $\epsilon, \delta \in (0, 1)$ with $\epsilon + \delta < 1$, such that any n -dimensional complete open manifold M with $\text{Ric}_M \geq 0$,*

$$k_p(r) \geq -(K(r))^2,$$

and

$$\mathcal{H}(p, r) < r^{\frac{1}{n}} \left[\frac{1}{4} \epsilon r - \frac{1}{8K(\delta r)} \ln \left(\cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right) \right]^{n-1},$$

for some $p \in M$ and all $r > 0$ is diffeomorphic to $\mathbb{R}^n$.

PROOF. We take the constants ϵ and δ as same in Theorem 1.1. Fix any point $q \in M$ and set $r = d(p, q)$. We only need to show that q is not a critical point of $d(p, \cdot)$.

Take a point $m \in R_p$ so that $d(q, m) = d(q, R_p)$. Set $s = d(q, m)$. It then follows from the assumption of Lemma 3.1 that

$$s < r^{\frac{1}{n}} \left[\frac{1}{4} \epsilon r - \frac{1}{8K(\delta r)} \ln \left(\cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right) \right]^{n-1}. \quad (3.7)$$

Take a ray $\gamma : [0, +\infty) \rightarrow M$ starting from p and passing through m . It follows from the triangle inequality that $\min(d(p, q), d(\gamma(t), q)) = r$ for all $t \geq 2r$. Thus $m \in \gamma(0, 2r)$ and so $d(q, \gamma|_{[0, 2r]}) = s$. By Lemma 2.3 and (3.7) we have

$$e_{p, \gamma(2r)}(q) < 2\epsilon r - \frac{1}{K(\delta r)} \ln \left(\cosh^2(K(\delta r)\epsilon r) + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right). \quad (3.8)$$

Set $z = \gamma(2r)$ and take a minimal geodesic $\tilde{\sigma}$ from q to z . For any minimal geodesic $\tilde{\sigma}_1$ from q to p , let $p' = \tilde{\sigma}_1(\epsilon r)$ and $z' = \tilde{\sigma}(\epsilon r)$, and set $\tilde{\theta} = \angle(\tilde{\sigma}'(0), \tilde{\sigma}_1'(0))$. Since $K_M \geq -(K(\delta r))^2$ on $M \setminus B(p, \delta r)$ we can apply the Toponogov comparison theorem to the hinge $(\tilde{\sigma}|_{[0, \epsilon r]}, \tilde{\sigma}_1|_{[0, \epsilon r]})$ in $M \setminus B(p, \delta r)$ to get

$$\cos \tilde{\theta} \sinh^2(K(\delta r)\epsilon r) \leq \cosh^2(K(\delta r)\epsilon r) - \cosh(K(\delta r)d(p', z')). \quad (3.9)$$

It follows from the triangle inequality that

$$\begin{aligned} d(p', z') &\geq -d(p, p') - d(z, z') + d(p, z) \\ &= -d(p, q) + d(p', q) - d(q, z) + d(z', q) + d(p, z) = 2\epsilon r - e_{p, z}(q). \end{aligned}$$

Inserting (3.8) into the above inequality and noticing (3.9), one obtains

$$\cos \tilde{\theta} \sinh^2(K(\delta r)\epsilon r) \leq \cosh^2(K(\delta r)\epsilon r) - \cosh(K(\delta r)(2\epsilon r - e_{p, z}(q))) < 0.$$

Thus $\tilde{\theta} > \frac{\pi}{2}$. Consequently, q is not a critical point of $d(p, \cdot)$. Lemma 3.1 follows. $\square$

PROOF OF THEOREM 1.3. We take the constants ϵ and δ to be the same as in Lemma 3.1. In order to prove Theorem 1.3, it suffices to show that

$$\begin{aligned} \mathcal{H}(p, r) &< r^{\frac{1}{n}} \left[\frac{1}{4}\epsilon r - \frac{1}{8K(\delta r)} \ln \left(\cosh^2(K(\delta r)\epsilon r) \right. \right. \\ &\quad \left. \left. + \sqrt{\cosh^4(K(\delta r)\epsilon r) - 1} \right) \right]^{n-1}. \end{aligned} \quad (3.10)$$

By Lemma 2.2

$$d(x, R_p) \leq 2v_0^{-\frac{1}{n}} \left\{ \frac{\text{vol}[B(p, r)]}{\omega_n r^n} - v_0 \right\}^{\frac{1}{n}} r.$$

By the definition (1.1), it then follows from the assumption (1.6) of Theorem 1.3 that (3.10) holds, and therefore we complete the proof of the theorem. $\square$

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