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Nilpotency class of symmetric units of group algebras

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Dedicated to Professor Adalbert Bovdi on his 75th birthday

Abstract. Let F be a field of odd prime characteristic p, G a group, U the group of units in the group algebra FG, and U^+ the subgroup of U generated by the elements of U fixed by the anti-automorphism of FG which inverts all elements of G. It is known that U is nilpotent if G is nilpotent and the commutator subgroup G' has p-power order, and then the nilpotency class of U is at most the order of G'; this bound is attained if and only if G' is cyclic and not a Sylow subgroup of G. Adalbert Bovdi and János Kurdics proved the 'if' part of this last statement by exhibiting a nontrivial commutator of the relevant weight. For the special case when G is a nonabelian torsion group (so G' cannot possibly be a Sylow subgroup), the present paper identifies such a commutator in U^+, showing (Theorem 1) that the same bound is attained even by the nilpotency class of this subgroup. We do not know what happens when G' is not a Sylow subgroup but G is not torsion.

It can happen that U^+ is nilpotent even though U is not. The torsion groups G which allow this are known (from the work of Gregory T. Lee) to be precisely the direct products of a finite p-group P, a quaternion group Q of order 8, and an elementary abelian 2-group. Theorem 2: in this case, the nilpotency class of U^+ is strictly smaller than the nilpotency index of the augmentation ideal of the group algebra FP, and this bound is attained whenever P is a powerful p-group. The nonabelian group P of order 27 and exponent 3 is not powerful, yet the $G = P \times Q$ formed with this P also leads to a U^+ attaining the general bound, so here a necessary and sufficient condition remains elusive.

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1. Introduction

Let G be a group and let $g_1, \dots, g_n \in G$. By the symbol $(g_1, \dots, g_n)$ we denote the commutator of the elements $g_1, \dots, g_n$ which is defined inductively as $(g_1, \dots, g_n) = ((g_1, \dots, g_{n-1}), g_n)$ with $(g_1, g_2) = g_1^{-1} g_2^{-1} g_1 g_2$. As usual, for the subsets X, Y of G by the commutator (X, Y) we mean the subgroup generated by all commutators (x, y) with $x \in X, y \in Y$. This allows us to define the lower central series of a nonempty subset H of G: let $\gamma_{n+1}(H) = (\gamma_n(H), H)$ with $\gamma_1(H) = H$. We say that H is nilpotent if $\gamma_n(H) = 1$ for some n. It is not so hard to show the equivalence of the following statements: (i) H is a nilpotent subset; (ii) H satisfies the group identity $(g_1, g_2, \dots, g_n) = 1$ for some $n \geq 2$; (iii) $\langle H \rangle$ is a nilpotent group (see [14]). For a nilpotent subset $H \subseteq G$ the number $\mathrm{cl}(H) = \min\{n \in \mathbb{N}_0 : \gamma_{n+1}(H) = 1\}$ is called the nilpotency class of H.

Let R be an associative ring with unity. Then R can be considered as a Lie ring with the Lie commutator defined by $[x, y] = xy - yx$ for all $x, y \in R$. For $X, Y \subseteq R$, by $[X, Y]$ we denote the additive subgroup generated by all Lie commutators $[x, y]$ with $x \in X, y \in Y$. The upper Lie powers of a nonempty subset S of R are defined inductively: set $[S]_1 = S$ and for $n \geq 2$ let $[S]_n$ be the associative ideal of R generated by all Lie commutators $[x, y]$ with $x \in [S]_{n-1}$, $y \in S$. S is said to be upper Lie nilpotent if some upper Lie power of S vanishes; the minimal n for which $[S]_n = 0$ is called the upper Lie nilpotency index of S (in notation $t^L(S)$). Denote by $U(S)$ the set of units in the subset S and suppose that it is nonempty. By the equality $(x, y) = 1 + x^{-1}y^{-1}[x, y]$, where $x, y \in U(S)$, it is easy to see that $\gamma_n(U(S)) \subseteq 1 + [S]_n$ for all $n \geq 2$, which implies that the set of units of an upper Lie nilpotent subset S is nilpotent, and $\mathrm{cl}(U(S)) \leq t^L(S) - 1$.

Let F be a field and let G be a group. For the noncommutative group algebra FG the equivalence of the following statements follows from [12], [16]: (i) FG is upper Lie nilpotent; (ii) $\mathrm{char}\, F = p > 0$, G is nilpotent and its commutator subgroup G' has p-power order; (iii) FG is modular and $U(FG)$ is nilpotent. As the reader can see in [2], [3], [9], [17], [18], [19], significant developments have been achieved concerning the study of the nilpotency class of $U(FG)$, however a complete description is not yet known.

Let $*$ be the canonical involution on FG; that is, the F-linear extension of the anti-automorphism of G sending each element to its inverse. We will denote by S^+ the set of symmetric elements of $S \subseteq FG$; that is, $S^+ = \{x \in S : x^* = x\}$. A number of interesting results on the symmetric units of group rings can be found, for example, in the articles [4], [6], [7], [14], [15] and in the book [13]. This paper is devoted to the study of the nilpotency class of $U^+(FG)$. Assume first that FG is a modular group algebra with a nilpotent unit group. Then $U^+(FG)$ is nilpotent as well, but we do not know if $\mathrm{cl}(U^+(FG))$ reaches $\mathrm{cl}(U(FG))$ all the time or not. Furthermore, FG is upper Lie nilpotent, and by [20], $t^L(FG) \leq |G'| + 1$. This gives that $|G'|$ is an upper bound on $\mathrm{cl}(U(FG))$ and so on $\mathrm{cl}(U^+(FG))$. We prove the following theorem.

Theorem 1. *Let FG be the group algebra of a torsion group G over a field F of characteristic $p > 2$ such that $U(FG)$ is nilpotent. Then $\mathrm{cl}(U^+(FG)) = |G'|$ if and only if G' is cyclic.*

We cannot expect that this theorem remains true for non-torsion groups. Indeed, by Theorem 4.3 of [3], if G' is a cyclic group of order $p^n > 2$ and $\mathrm{Syl}_p(G) = G'$, then $\mathrm{cl}(U^+(FG)) \leq \mathrm{cl}(U(FG)) = |G'| - 1$. It is obvious that G' cannot possibly be a Sylow subgroup whenever G is torsion.

Corollary 1. *Let FG be the group algebra of a torsion group G over a field F of characteristic $p > 2$ such that $U(FG)$ is nilpotent. If G' is cyclic, then $\mathrm{cl}(U^+(FG)) = \mathrm{cl}(U(FG))$.*

Now assume that $U^+(FG)$ is nilpotent, but $U(FG)$ is not. According to [14], if $\mathrm{char}\, F = p \neq 2$ and G is a torsion group, then $G \cong Q_8 \times E \times P$, where Q_8 is the quaternion group of order 8, E is an elementary abelian 2-group and P is a finite p-group as long as $p > 0$, otherwise $P = 1$. For the non-torsion case the characterization is only known when F is infinite by [15]. It is easy to verify that if P is trivial, then the elements of $U^+(FG)$ commute for any field F, so $\mathrm{cl}(U^+(FG)) = 1$. Our next result is about the case when P is nontrivial. In order to state it, we require a couple of definitions. By the augmentation ideal of a group algebra FG we mean the ideal in FG, generated by the set $\{g - 1 \mid g \in G\}$, and it will be denoted by $\omega(FG)$. In [10] it was proved that $\omega(FG)$ is nilpotent if and only if G is a finite p-group and $\mathrm{char}\, F = p$. In this case, the nilpotency index of $\omega(FG)$ will be denoted by $t_N(G)$. We also recall that a finite p-group G is called powerful if either p is odd and $G' \subseteq G^p$, or $p = 2$ and $G' \subseteq G^4$.

Theorem 2. *Let F be a field of characteristic $p > 2$, and let G be a torsion group with a nontrivial Sylow p-subgroup P such that $U^+(FG)$ is nilpotent but $U(FG)$ is not. Then $\mathrm{cl}(U^+(FG)) \leq t_N(P) - 1$. In addition, if P is powerful, then the equality holds.*

We should remark that the assumption P to be powerful is not necessary for the equality. Using the LAGUNA [5] software package in the GAP [21] computer algebra system, it is easy to verify that if P is the noncommutative group of order 27 with exponent 3 and char $F = 3$, then the equality holds, although this group is not powerful.

It is well known that if P has order p^n, then $1+n(p-1) \le t_N(P) \le p^n$, with equality on the left (right) hand side if and only if P is elementary abelian (resp. cyclic). Furthermore, if P is the direct product of cyclic groups of order p^{m_i} $(1 \le i \le n)$, then $t_N(P) = 1 + \sum_{i=1}^{n}(p^{m_i} - 1)$. In general, there is a formula for $t_N(P)$ which gives its exact value in terms of the orders of the so-called dimension subgroups of P. In the case when P is powerful, its dimension subgroups are its powers.

The identities

$$ab - 1 = (a-1)(b-1) + (a-1) + (b-1);$$

$$[ab, c] = a[b, c] + [a, c]b \quad \text{and} \quad [a, bc] = b[a, c] + [a, b]c;$$

$$[a, b] = ba\big((a, b) - 1\big) \quad \text{and} \quad (a, b) = 1 + a^{-1}b^{-1}[a, b] \quad (\text{here } a,\ b \text{ are units}),$$

hold for all elements a, b, c of an arbitrary associative ring R, and they will be used freely. We denote by $\zeta(G)$ and $\zeta(FG)$ the centers of the group G and the group algebra FG, respectively. Throughout this paper by p we always mean an *odd prime* and by F a field of characteristic p.

2. Proof of Theorem 1

First of all, we collect and examine those Lie commutators of associative powers of the augmentation ideal that we need in the proof. By definition, $\omega(FG)^0 = FG$.

Lemma 1. *Let G be a finite p-group such that $\gamma_3(G) \subseteq (G')^p$. Then for all $k, l, m, n \ge 1$*

$$[\omega(FG')^m, \omega(FG)^l] \subseteq \omega(FG)^{l-1}\omega(FG')^{m+1};$$

$$[\omega(FG)^k, \omega(FG)^l] \subseteq \omega(FG)^{k+l-2}\omega(FG');$$

$$[\omega(FG)^k\omega(FG')^m, \omega(FG)^l] \subseteq \omega(FG)^{k+l-2}\omega(FG')^{m+1};$$

$$[FG\omega(FG')^m, \omega(FG)^l] \subseteq FG\omega(FG')^{m+1}.$$

Proof. The first two inclusions were proved in [1], and they are followed by the last two, because

$$\begin{gathered}[\omega(FG)^k\omega(FG')^m, \omega(FG)^l] \\ \subseteq \omega(FG)^k[\omega(FG')^m, \omega(FG)^l] + [\omega(FG)^k, \omega(FG)^l]\omega(FG')^m,\end{gathered}$$

and

$$FG\omega(FG')^m = \omega(FG)\omega(FG')^m + \omega(FG')^m. \qquad \square$$

We can also easily observe that

$$g^m - 1 \equiv m(g-1) \pmod{\omega(FG)^2} \tag{1}$$

for every $g \in G$ and integer m.

Let now G be a finite p-group with derived subgroup $G' = \langle x \rangle$, and let $a, b \in G$ such that $(a,b) = x$. It is easy to check (see e.g. [11] p. 252) that

$$[a^m, b^s] \equiv ms \cdot b^s a^m (x-1) \pmod{FG\omega(FG')^2}. \tag{2}$$

For $n \geq 2$ denote by I_n the ideal $\omega(FG)^3\omega(FG')^{n-1} + FG\omega(FG')^n$ of FG. In the next lemma we need the congruences

$$\begin{aligned}
&[(a-1)(b-1), (a-1)(a^{-1}-1)] \equiv 2(a-1)^2(x-1) \pmod{I_2},\\
&[(a-1)(a^{-1}-1), (b-1)(b^{-1}-1)] \equiv 4(a-1)(b-1)(x-1) \pmod{I_2},\\
&[(a-1)^2, (b-1)(b^{-1}-1)] \equiv -4(a-1)(b-1)(x-1) \pmod{I_2}.
\end{aligned} \tag{3}$$

Now we prove the first one, the last two can be obtained analogously. Applying (2) we can calculate that

$$\begin{aligned}
[(a-1)(b-1),&(a-1)(a^{-1}-1)]\\
&= (a-1)^2[b, a^{-1}] + (a-1)[b,a](a^{-1}-1)\\
&\equiv (a-1)^2ba^{-1}(x-1) - (a-1)ba(x-1)(a^{-1}-1) \pmod{I_2}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&(a-1)^2ba^{-1}(x-1) - (a-1)ba(x-1)(a^{-1}-1)\\
&\quad = (a-1)^2(ba^{-1}-1)(x-1) + (a-1)^2(x-1)\\
&\qquad - (a-1)(ba-1)(x-1)(a^{-1}-1) - (a-1)(x-1)(a^{-1}-1).
\end{aligned}$$

Clearly, $(a-1)^2(ba^{-1}-1)(x-1) \in \omega(FG)^3\omega(FG') \subseteq I_2$, and using the fact that the value of the product $(g-1)(h-1)(x-1)$ is independent of the order of its factors modulo I_2, we have that $(a-1)(ba-1)(x-1)(a^{-1}-1)$ is also belongs to I_2. Hence, applying (1) we have

$$\begin{aligned}
[(a-1)(b-1), (a-1)(a^{-1}-1)] &\equiv (a-1)^2(x-1) - (a-1)(x-1)(a^{-1}-1)\\
&\equiv 2(a-1)^2(x-1) \pmod{I_2}.
\end{aligned}$$

Lemma 2. *Let G be a finite p-group with cyclic derived subgroup. Then*

$$\mathrm{cl}(U^+(FG)) \geq |G'|.$$

Proof. Let us choose the elements x, a and b in G such that $x = (a, b)$ and $\langle x\rangle = G'$. We are going to prove that for $n \geq 2$ there exist $z_n \in \gamma_n(U^+(FG))$ such that

$$z_n \equiv \begin{cases} 1 + \alpha_n(a-1)^2(x-1)^{n-1} \pmod{I_n} & \text{if } n \text{ is odd;} \\ 1 + \alpha_n(a-1)(b-1)(x-1)^{n-1} \pmod{I_n} & \text{if } n \text{ is even,} \end{cases} \tag{4}$$

where $\alpha_n \in F \setminus \{0\}$.

For $n \geq 1$ let

$$u_n = \begin{cases} (a-1)(a^{-1}-1) & \text{if } n \text{ is odd;} \\ (b-1)(b^{-1}-1) & \text{if } n \text{ is even.} \end{cases}$$

Evidently, u_n is a nilpotent symmetric element and so $1 + u_n$ is a symmetric unit for all n. Applying (3) we have

$$\begin{aligned} (1+u_1, 1+u_2) &= 1 + (1+u_1)^{-1}(1+u_2)^{-1}[u_1, u_2] \\ &= 1 + ((1+u_1)^{-1}(1+u_2)^{-1} - 1)[u_1, u_2] + [u_1, u_2] \\ &\equiv 1 + 4(a-1)(b-1)(x-1) \pmod{I_2}, \end{aligned}$$

which confirms (4) for $n = 2$. Assume by induction the truth of (4) for some i ($i \geq 2$); i.e., there exist $\mu \in I_i$ and $\alpha_i \in F \setminus \{0\}$ such that

$$z_i = 1 + \alpha_i v_i (x-1)^{i-1} + \mu \in \gamma_i(U^+(FG)),$$

where either $v_i = (a-1)^2$ or $v_i = (a-1)(b-1)$ when i is odd or even, respectively. Applying Lemma 1 and (3) we have

$$\begin{aligned} (z_i, 1+u_{i+1}) &= 1 + z_i^{-1}(1+u_{i+1})^{-1}[z_i, u_{i+1}] \\ &= 1 + (z_i^{-1}(1+u_{i+1})^{-1} - 1)([\alpha_i v_i(x-1)^{i-1}, u_{i+1}] + [\mu, u_{i+1}]) \\ &\quad + [\alpha_i v_i(x-1)^{i-1}, u_{i+1}] + [\mu, u_{i+1}] \\ &\equiv 1 + \alpha_i [v_i, u_{i+1}](x-1)^{i-1} \equiv 1 + \alpha_{i+1} v_{i+1}(x-1)^i \pmod{I_{i+1}}, \end{aligned}$$

where $\alpha_{i+1} = -4\alpha_i$ if i is odd, else $\alpha_{i+1} = 2\alpha_i$. Thus, (4) is true for all $n \geq 2$.

We finish the proof by showing that z_m is not zero for $m = |G'|$. To this we show that the element $y = v_m(x-1)^{m-1}$ does not belong to I_m. Since now

$m = |G'|$, so $FG\omega(FG')^m = 0$ and $I_m = \omega(FG)^3\omega(FG')^{m-1}$. According to [10], the element $x-1$ is of weight $t \geq 2$, so y has weight $2+t(m-1)$, which means that $y \in \omega(FG)^{2+t(m-1)} \setminus \omega(FG)^{3+t(m-1)}$. Since $\omega(FG)^i$ has an F-basis consisting of regular elements of weight not less than i, the inclusion $\omega(FG)^3\omega(FG')^{m-1} \subseteq \omega(FG)^{3+t(m-1)}$ holds. Therefore y cannot be in I_m. $\square$

Proof of Theorem 1. According to [8], if G' is not cyclic, then $t^L(FG) < |G'|+1$, which forces the inequality $\mathrm{cl}(U^+(FG)) < |G'|$. Conversely, if G' is cyclic, we can choose the elements x, a and b in G such that $x = (a,b)$ and $\langle x\rangle = G'$. As a finitely generated torsion nilpotent group, $N = \langle a,b\rangle$ is finite, and it is the direct product of its Sylow subgroups. Let us denote by P the Sylow p-subgroup of N. Since G' is a p-group we have $P' = N' = G'$. Now if G' is cyclic, then by Lemma 2 we are done, because

$$|G'| = |P'| \leq \mathrm{cl}(U(FP)^+) \leq \mathrm{cl}(U^+(FG)). \qquad \square$$

3. Proof of Theorem 2

Assume that G is a torsion group such that $U^+(FG)$ is nilpotent but $U(FG)$ is not. Then $G \cong Q_8 \times E \times P$, where $E^2 = 1$ and P is a finite p-group. In what follows we suppose that P is nontrivial. Set $N = Q_8 \times E$ and $\mathfrak{I}(P) = FG\omega(FP)$. Obviously, $\mathfrak{I}(P)$ is a nilpotent ideal, so the set $\{1+x : x \in \mathfrak{I}(P)\}$ is a normal subgroup of the unit group $U(FG)$.

The upper bound $t_N(P) - 1$ on $\mathrm{cl}(U^+(FG))$ is a consequence of the next lemma.

Lemma 3. $t^L(FG^+) \leq t_N(P)$.

Proof. As it is well known, FG^+ is generated as an F-space by the set

$$S = \{g + g^{-1} : g \in G\}.$$

Now, in our case

$$S = \{a(h + a^2h^{-1}) : a \in N, h \in P\}.$$

Since

$$a(h + a^2h^{-1}) = a(h-1) + a^3(h^{-1}-1) + a + a^3,$$

and the element $a + a^3$ is central in FG, so we obtain that

$$FG^+ \subseteq \mathfrak{I}(P) + \zeta(FG).$$

Hence by induction one can easily get that $[FG^+]_n \subseteq \mathfrak{I}(P)^n$ for all $n \geq 2$, which forces the desired inequality. $\square$

Proof of Theorem 2. It remains to show that if P is powerful, then $\mathrm{cl}(U^+(FG)) \geq t_N(P) - 1$. Denote by c the generator element of N^2. We are going to prove by induction that for any $a \in N \setminus \zeta(N)$ and $h_1, \dots, h_n \in P$ there exists $u \in \gamma_n(U^+(FG))$ such that

$$u \equiv 1 - a(1-c)(h_1 - 1)\cdots(h_n - 1) \pmod{\mathfrak{I}(P)^{n+1}}.$$

Indeed, for any $a \in N \setminus \zeta(N)$ and $h \in P$ we have

$$\begin{aligned} 1 - a(h-1) - a^3(h^{-1} - 1) &\equiv 1 - a(h-1) + a^3(h-1) \\ &= 1 - a(1-c)(h-1) \pmod{\mathfrak{I}(P)^2} \end{aligned}$$

and we are done for $n = 1$.

Assume the statement for some $n \geq 1$. Let $a \in N \setminus \zeta(N), h_1, \dots, h_n, h_{n+1} \in P$ and choose $a_1, a_2 \in N \setminus \zeta(N)$ such that $(a_1, a_2) \neq 1$ and $a_1 a_2 = a$. Then, by the induction, there exist $u \in \gamma_n(U^+(FG))$ and $v \in U^+(FG)$ such that

$$\begin{aligned} u &\equiv 1 - a_1(1-c)(h_1 - 1)\cdots(h_n - 1) \pmod{\mathfrak{I}(P)^{n+1}}, \\ v &\equiv 1 - a_2(1-c)(h_{n+1} - 1) \pmod{\mathfrak{I}(P)^2}. \end{aligned}$$

Since $u^{-1}v^{-1} - 1 \in \mathfrak{I}(P)$, it is clear that

$$(u, v) = 1 + (u^{-1}v^{-1} - 1)[u, v] + [u, v] \equiv 1 + [u, v] \pmod{\mathfrak{I}(P)^{n+2}}. \tag{5}$$

Further,

$$\begin{aligned} &[a_1(1-c)(h_1 - 1)\cdots(h_n - 1), a_2(1-c)(h_{n+1} - 1)] \\ &\quad = a_1(1-c)[(h_1 - 1)\cdots(h_n - 1), a_2(1-c)(h_{n+1} - 1)] \\ &\qquad + [a_1(1-c), a_2(1-c)(h_{n+1} - 1)](h_1 - 1)\cdots(h_n - 1) \\ &\quad = a_1 a_2 (1-c)^2[(h_1 - 1)\cdots(h_n - 1), (h_{n+1} - 1)] \\ &\qquad + [a_1, a_2](1-c)^2(h_{n+1} - 1)(h_1 - 1)\cdots(h_n - 1), \end{aligned}$$

and using the equality $(1-c)^2 = 2(1-c)$ we get

$$\begin{aligned} [u, v] &\equiv 2a_1 a_2(1-c)(h_1 - 1)\cdots(h_n - 1)(h_{n+1} - 1) \\ &\quad + 2a_1 a_2(1-c)(h_{n+1} - 1)(h_1 - 1)\cdots(h_n - 1) \pmod{\mathfrak{I}(P)^{n+2}}. \end{aligned}$$

Recall that P is assumed to be powerful and $\mathrm{char}\, F = p \geq 3$, thus

$$(h_i, h_j) - 1 \in \omega(P') \subseteq \omega(P^p) \subseteq \omega(P)^p \subseteq \mathfrak{I}(P)^3$$

and

$$(h_i-1)(h_j-1) = (h_j-1)(h_i-1) + h_jh_i((h_i,h_j)-1)$$
$$\equiv (h_j-1)(h_i-1) \pmod{\mathfrak{I}(P)^3}$$

for all i, j, therefore

$$[u,v] \equiv 4a_1a_2(1-c)(h_1-1)\cdots(h_n-1)(h_{n+1}-1) \pmod{\mathfrak{I}(P)^{n+2}},$$

and by (5)

$$(u,v) \equiv 1 + 4a(1-c)(h_1-1)\cdots(h_{n+1}-1) \pmod{\mathfrak{I}(P)^{n+2}}.$$

Keeping in mind that p is an odd prime we can choose an integer s such that $4s \equiv -1 \pmod{p}$ and we can apply the binomial theorem to have

$$(u,v)^s \equiv 1 - a(1-c)(h_1-1)\cdots(h_{n+1}-1) \pmod{\mathfrak{I}(P)^{n+2}}.$$

Since $(u,v)^s \in \gamma_n(U^+(FG))$ the induction is done.

For $n < t_N(P)$ there exist $h_1,\dots,h_n \in P$ such that $(h_1-1)\cdots(h_n-1) \neq 0$ and we get that $\mathrm{cl}(U^+(FG)) \geq t_N(P) - 1$. □

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