Publ. Math. Debrecen
78/3-4 (2011), 743–753
DOI: 10.5486/PMD.2011.5017

Minkowski-type inequalities for means generated by two functions and a measure

By LÁSZLÓ LOSONCZI (Debrecen) and ZSOLT PÁLES (Debrecen)

Abstract. Given two continuous functions $f, g : I \to \mathbb{R}$ such that g is positive and f/g is strictly monotone, and a probability measure μ on the Borel subsets of $[0,1]$, the two variable mean $M_{f,g;\mu} : I^2 \to I$ is defined by

$$M_{f,g;\mu}(x,y) := \left(\frac{f}{g}\right)^{-1} \left(\frac{\displaystyle\int_0^1 f\big(tx+(1-t)y\big)\,d\mu(t)}{\displaystyle\int_0^1 g\big(tx+(1-t)y\big)\,d\mu(t)} \right) \qquad (x,y \in I).$$

The aim of this paper is to study Minkowski-type inequalities for these means, i.e., to find conditions for the generating functions $f_0, g_0 : I_0 \to \mathbb{R}$, $f_1, g_1 : I_1 \to \mathbb{R}$, $\dots$, $f_n, g_n : I_n \to \mathbb{R}$, and for the measure μ such that

$$M_{f_0,g_0;\mu}(x_1+\cdots+x_n, y_1+\cdots+y_n) \underset{[\geq]}{\leq} M_{f_1,g_1;\mu}(x_1,y_1)+\cdots+M_{f_n,g_n;\mu}(x_n,y_n)$$

holds for all $x_1, y_1 \in I_1, \dots, x_n, y_n \in I_n$ with $x_1+\cdots+x_n, y_1+\cdots+y_n \in I_0$. The particular case when the generating functions are power functions, i.e., when the means are generalized Gini means is also investigated.

Mathematics Subject Classification: Primary: 39B12, 39B22.
Key words and phrases: generalized Cauchy means, equality and homogeneity problem.
This research has been supported by the Hungarian Scientific Research Fund (OTKA) Grant NK81402 and by the TÁMOP 4.2.1./B-09/1/KONV-2010-0007 project implemented through the New Hungary Development Plan co-financed by the European Social Fund, and the European Regional Development Fund.

1. Introduction

Throughout this paper the classes of continuous strictly monotone and continuous positive real-valued functions defined on a nonempty open real interval I will be denoted by $\mathcal{CM}(I)$ and $\mathcal{CP}(I)$, respectively. Given two continuous functions $f, g : I \to \mathbb{R}$ with $g \in \mathcal{CP}(I)$, $f/g \in \mathcal{CM}(I)$ and a probability measure μ on the Borel subsets of $[0, 1]$, the two variable mean $M_{f,g;\mu} : I^2 \to I$ is defined by

$$M_{f,g;\mu}(x, y) := \left(\frac{f}{g}\right)^{-1} \left(\frac{\displaystyle\int_0^1 f\bigl(tx + (1-t)y\bigr) d\mu(t)}{\displaystyle\int_0^1 g\bigl(tx + (1-t)y\bigr) d\mu(t)} \right) \qquad (x, y \in I).$$

If $\mu = \frac{\delta_0+\delta_1}{2}$ (where δ_s denotes the Dirac measure concentrated at $s \in [0, 1]$), $\varphi \in \mathcal{CM}(I)$, and $p \in \mathcal{CP}(I)$, then

$$M_{p\varphi,p;\mu}(x, y) = \varphi^{-1}\left(\frac{p(x)\varphi(x) + p(y)\varphi(y)}{p(x) + p(y)}\right) \qquad (x, y \in I),$$

which was introduced and studied by Bajraktarević [Baj58], [Baj69]. In the particular case $p = 1$, we get the well-known quasi-arithmetic means (cf. [HLP34]).

If μ is the Lebesgue measure on $[0, 1]$ and $\varphi, \psi : I \to \mathbb{R}$ are continuously differentiable functions with $\psi' \in \mathcal{CP}(I)$ and $\varphi'/\psi' \in \mathcal{CM}(I)$, then, by the Fundamental Theorem of Calculus, one can easily see that

$$M_{\varphi',\psi';\mu}(x, y) = \begin{cases} \left(\dfrac{\varphi'}{\psi'}\right)^{-1} \left(\dfrac{\varphi(y) - \varphi(x)}{\psi(y) - \psi(x)}\right) & \text{if } x \neq y \\ x & \text{if } x = y \end{cases} \qquad (x, y \in I),$$

which is called a Cauchy or difference mean in the literature (cf. [BM00], [Los00]). When $\psi(x) = x$, then this mean goes over into a Lagrangian mean (cf. [BM98], [Ber98]).

Consider now the setting when $I = \mathbb{R}_+$ and the functions f, g are power functions, more precisely, for $p, q \in \mathbb{R}$, define

$$\begin{aligned} f(x) &= x^p, & g(x) &= x^q & &\text{if} \quad p \neq q, \\ f(x) &= x^p \ln x, & g(x) &= x^p & &\text{if} \quad p = q. \end{aligned} \tag{1}$$

Then the mean $M_{f,g;\mu}$ reduces to the following generalization of the so-called Gini means:

$$G_{p,q;\mu}(x,y) := \begin{cases} \left(\dfrac{\displaystyle\int_0^1 \big(tx+(1-t)y\big)^p d\mu(t)}{\displaystyle\int_0^1 \big(tx+(1-t)y\big)^q d\mu(t)} \right)^{\frac{1}{p-q}} & \text{if } p \neq q, \\ \exp\left(\dfrac{\displaystyle\int_0^1 \big(tx+(1-t)y\big)^p \ln\big(tx+(1-t)y\big) d\mu(t)}{\displaystyle\int_0^1 \big(tx+(1-t)y\big)^p d\mu(t)} \right) & \text{if } p = q. \end{cases}$$

In the particular case when $\mu = \frac{1}{2}(\delta_0 + \delta_1)$, the mean $G_{p,q;\mu}$ goes over into the standard Gini mean (cf. [Gin38]) defined as

$$G_{p,q;\mu}(x,y) = G_{p,q}(x,y) := \begin{cases} \left(\dfrac{x^p + y^p}{x^q + y^q} \right)^{\frac{1}{p-q}} & \text{if } p \neq q \\ \exp\left(\dfrac{x^p \ln x + y^p \ln y}{x^p + y^p} \right) & \text{if } p = q \end{cases} \qquad (x, y \in \mathbb{R}_+).$$

The other particular case of great importance is when μ is equal to the Lebesgue measure λ. Then

$$G_{p,q;\lambda}(x,y) = S_{p+1,q+1}(x,y) \qquad (x, y \in \mathbb{R}_+),$$

where $S_{p,q}$ is the so-called Stolarsky mean (cf. [Sto75]) given by

$$S_{p,q}(x,y) := \begin{cases} \left(\dfrac{q(x^p - y^p)}{p(x^q - y^q)} \right)^{\frac{1}{p-q}} & \text{if } (p-q)pq \neq 0 \\ \exp\left(-\dfrac{1}{p} + \dfrac{x^p \ln x - y^p \ln y}{x^p - y^p} \right) & \text{if } p = q \neq 0 \\ \left(\dfrac{x^p - y^p}{p(\ln x - \ln y)} \right)^{\frac{1}{p}} & \text{if } p \neq 0,\ q = 0 \\ \left(\dfrac{x^q - y^q}{q(\ln x - \ln y)} \right)^{\frac{1}{q}} & \text{if } p = 0,\ q \neq 0 \\ \sqrt{xy} & \text{if } p = q = 0 \end{cases} \qquad (x, y \in \mathbb{R}_+).$$

In [Los71], the first author obtained Minkowski-type inequalities for Bajraktarević means. Investigating Minkowski-type inequalities for the standard Gini means, Czinder and the second author obtained the following result (cf. [CP00, Theorem 5]).

Theorem A. *Let $n \geq 2$ and $p_0, p_1, \dots, p_n, q_0, q_1, \dots, q_n \in \mathbb{R}$. Then*

$$G_{p_0,q_0}(x_1 + \dots + x_n, y_1 + \dots + y_n) \leq G_{p_1,q_1}(x_1, y_1) + \dots + G_{p_n,q_n}(x_n, y_n)$$

holds for all $x_1, \dots, x_n, y_1, \dots, y_n > 0$ if and only if

(a) $0 \leq \min\{p_1, q_1, \dots, p_n, q_n\}$,

(b) $\min\{p_0, q_0\} \leq \min\{1, p_1, q_1, \dots, p_n, q_n\}$,

(c) $\max\{1, p_0 + q_0\} \leq \min\{p_1 + q_1, \dots, p_n + q_n\}$.

The particular case $p_0 = p_1 = \dots = p_n$, $q_0 = q_1 = \dots = q_n$, i.e., when all the Gini means are the same, was investigated by the authors in [LP96]. It is interesting to note that the characterization of the reversed Minkowski-type inequality even in this particular setting is *still unknown*.

In the context of Stolarsky means, in [LP98], we obtained the following result (formulated in the case $n = 2$ only).

Theorem B. *Let $n \geq 2$ and $p, q \in \mathbb{R}$. Then the inequality*

$$S_{p,q}(x_1 + \dots + x_n, y_1 + \dots + y_n) \underset{[\geq]}{\leq} S_{p,q}(x_1, y_1) + \dots + S_{p,q}(x_n, y_n)$$

holds for all $x_1, \dots, x_n, y_1, \dots, y_n > 0$ if and only if

$$3 \underset{[\geq]}{\leq} p + q \qquad \text{and} \qquad 1 \underset{[\geq]}{\leq} \min\{p, q\}.$$

In order that the more general inequality

$$S_{p_0,q_0}(x_1 + \dots + x_n, y_1 + \dots + y_n) \underset{[\geq]}{\leq} S_{p_1,q_1}(x_1, y_1) + \dots + S_{p_n,q_n}(x_n, y_n)$$

be valid for all $x_1, \dots, x_n, y_1, \dots, y_n > 0$, Czinder and the second author obtained necessary conditions and also sufficient conditions for the parameters $p_0, p_1, \dots, p_n$, $q_0, q_1, \dots, q_n \in \mathbb{R}$ in [CP03].

Motivated by the above preliminaries, the aim of this paper is to study Minkowski-type inequalities for the means $M_{f,g;\mu}$, i.e., our purpose is to find conditions for the generating functions $f_0, g_0 : I_0 \to \mathbb{R}$, $f_1, g_1 : I_1 \to \mathbb{R}$, $\dots$, $f_n, g_n : I_n \to \mathbb{R}$, and for the measure μ such that

$$\begin{aligned} &M_{f_0,g_0;\mu}(x_1 + \dots + x_n, y_1 + \dots + y_n) \\ &\qquad \underset{[\geq]}{\leq} M_{f_1,g_1;\mu}(x_1, y_1) + \dots + M_{f_n,g_n;\mu}(x_n, y_n) \end{aligned} \tag{2}$$

be valid for all $x_1, y_1 \in I_1$, $\dots$, $x_n, y_n \in I_n$ with $x_1 + \dots + x_n, y_1 + \dots + y_n \in I_0$. In the main results of the paper we give sufficient conditions (which, in a certain sense, are also necessary) for (2) to hold. As an important particular case, we also consider Minkowski-type inequalities involving the generalized Gini means.

2. Main results

In order to describe the regularity conditions related the two generating functions f, g of the mean $M_{f,g;\mu}$ in a convenient way, we say that the pair (f, g) of functions is in the class $\mathcal{C}_1(I)$ if f, g are continuously differentiable functions such that $g \in \mathcal{CP}(I)$ and the *Wronski determinant*

$$\begin{vmatrix} f'(x) & f(x) \\ g'(x) & g(x) \end{vmatrix} = g^2(x) \left(\frac{f(x)}{g(x)} \right)' \qquad (x \in I) \tag{3}$$

does not vanish on I. Obviously, the latter condition implies that f/g is strictly monotone, i.e., $f/g \in \mathcal{CM}(I)$. For $(f, g) \in \mathcal{C}_1(I)$, we define the *deviation* function $\mathcal{D}^*_{f,g} : I^2 \to \mathbb{R}$ by

$$\mathcal{D}^*_{f,g}(x, y) := \frac{\begin{vmatrix} f(x) & f(y) \\ g(x) & g(y) \end{vmatrix}}{\begin{vmatrix} f'(y) & f(y) \\ g'(y) & g(y) \end{vmatrix}} = \frac{g(x) \left(\dfrac{f(x)}{g(x)} - \dfrac{f(y)}{g(y)} \right)}{g(y) \left(\dfrac{f(y)}{g(y)} \right)'} \qquad (x, y \in I). \tag{4}$$

Clearly, we have that $\mathcal{D}^*_{f,g}(x, y) \lesseqqgtr 0$ if and only if $x \lesseqqgtr y$.

The next result characterizes the mean $M_{f,g;\mu}$ via an implicit equation and signifies the role of the function $\mathcal{D}^*_{f,g}$ (cf. [LP08]).

Lemma 1. *Let $(f,g) \in \mathcal{C}_1(I)$ and μ be a Borel probability measure on $[0,1]$. Then for all $x, y \in I$ and $u \in [x,y]$,*

$$M_{f,g;\mu}(x,y) \begin{smallmatrix} < \\ = \\ > \end{smallmatrix} u \quad \text{if and only if} \quad \int_0^1 \mathcal{D}^*_{f,g}\big(tx+(1-t)y, u\big)d\mu(t) \begin{smallmatrix} < \\ = \\ > \end{smallmatrix} 0. \tag{5}$$

As a consequence of (5), *we have the identity*

$$\int_0^1 \mathcal{D}^*_{f,g}\big(tx+(1-t)y, M_{f,g;\mu}(x,y)\big)d\mu(t) = 0 \quad (x,y \in I). \tag{6}$$

By Lemma 2 below, the function $\mathcal{D}^*_{f,g}$ is also connected to the sequence of means $M_{f,g;m_k}$, where (m_k) is the sequence of measures defined by

$$m_k := \left(1 - \frac{1}{k}\right)\delta_0 + \frac{1}{k}\delta_1 \quad (k \in \mathbb{N}). \tag{7}$$

For its proof, the reader should consult [LP08].

Lemma 2. *Let $(f,g) \in \mathcal{C}_1(I)$. Then*

$$\lim_{k\to\infty} k\big[M_{f,g;m_k}(x,y) - y\big] = \mathcal{D}^*_{f,g}(x,y) \qquad (x,y \in I). \tag{8}$$

Now we can formulate our main result which gives a sufficient condition for the general Minkowski-type inequality (2) which does not involve the measure μ.

Theorem 3. *Let $I_0, I_1, \ldots, I_n$ be open real intervals, and let $(f_i, g_i) \in \mathcal{C}_1(I_i)$ for $i = 0, 1, \ldots, n$. Then the following three assertions are equivalent:*

(i) *For all Borel probability measures μ on $[0,1]$,*

$$M_{f_0,g_0;\mu}(x_1+\cdots+x_n, y_1+\cdots+y_n) \underset{[\geq]}{\leq} M_{f_1,g_1;\mu}(x_1,y_1)+\cdots+M_{f_n,g_n;\mu}(x_n,y_n) \tag{9}$$

holds for all $x_1, y_1 \in I_1, \ldots, x_n, y_n \in I_n$ with $x_1+\cdots+x_n, y_1+\cdots+y_n \in I_0$.

(ii) *For all $k \in \mathbb{N}$,*

$$M_{f_0,g_0;m_k}(x_1+\cdots+x_n, y_1+\cdots+y_n) \underset{[\geq]}{\leq} M_{f_1,g_1;m_k}(x_1,y_1)+\cdots+M_{f_n,g_n;m_k}(x_n,y_n) \tag{10}$$

holds for all $x_1, y_1 \in I_1, \ldots, x_n, y_n \in I_n$ with $x_1+\cdots+x_n, y_1+\cdots+y_n \in I_0$ (where (m_k) is the sequence of measures defined by (7)).

(iii)

$$\mathcal{D}^*_{f_0,g_0}(x_1+\dots+x_n,y_1+\dots+y_n) \underset{[\geq]}{\leq} \mathcal{D}^*_{f_1,g_1}(x_1,y_1)+\dots+\mathcal{D}^*_{f_n,g_n}(x_n,y_n) \quad (11)$$

holds for all $x_1,y_1 \in I_1, \ldots, x_n,y_n \in I_n$ *with* $x_1+\dots+x_n, y_1+\dots+y_n \in I_0$.

Proof. The implication (i)$\Longrightarrow$(ii) is obvious. To prove (ii)$\Longrightarrow$(iii), for $x_1,y_1 \in I_1, \ldots, x_n,y_n \in I_n$ with $x_1+\dots+x_n, y_1+\dots+y_n \in I_0$, use (10) and Lemma 2 to get

$$\begin{aligned}
&\mathcal{D}^*_{f_0,g_0}(x_1+\dots+x_n,y_1+\dots+y_n)\\
&\quad= \lim_{k\to\infty} k\left[M_{f_0,g_0;m_k}(x_1+\dots+x_n,y_1+\dots+y_n)-(y_1+\dots+y_n)\right]\\
&\quad\underset{[\geq]}{\leq} \lim_{k\to\infty} k\left[\big(M_{f_1,g_1;m_k}(x_1,y_1)-y_1\big)+\dots+\big(M_{f_n,g_n;m_k}(x_n,y_n)-y_n\big)\right]\\
&\quad= \mathcal{D}^*_{f_1,g_1}(x_1,y_1)+\dots+\mathcal{D}^*_{f_n,g_n}(x_n,y_n),
\end{aligned}$$

which proves (11).

(iii)$\Longrightarrow$(i) Let $u_1,v_1 \in I_1, \ldots, u_n,v_n \in I_n$ with $u_1+\dots+u_n, v_1+\dots+v_n \in I_0$. Substituting

$$x_i := tu_i+(1-t)v_i, \quad y_i := M_{f_i,g_i;\mu}(u_i,v_i) \qquad (i=1,\ldots,n)$$

into (11) and integrating on $[0,1]$ with respect to t by the measure μ, we get

$$\begin{aligned}
&\int_0^1 \mathcal{D}^*_{f_0,g_0}\big(t(u_1+\dots+u_n)+(1-t)(v_1+\dots+v_n),y_1+\dots+y_n\big)d\mu(t)\\
&\qquad\underset{[\geq]}{\leq} \int_0^1 \mathcal{D}^*_{f_1,g_1}\big(tu_1+(1-t)v_1,y_1\big)d\mu(t)+\dots\\
&\qquad\quad+\int_0^1 \mathcal{D}^*_{f_n,g_n}\big(tu_n+(1-t)v_n,y_n\big)d\mu(t). \qquad (12)
\end{aligned}$$

By Lemma 1 and the choice of $y_1,\ldots,y_n$, the right hand side of this inequality is zero. Thus, we obtain from (12) that

$$\int_0^1 \mathcal{D}^*_{f_0,g_0}\big(t(u_1+\dots+u_n)+(1-t)(v_1+\dots+v_n),y_1+\dots+y_n\big)d\mu(t) \underset{[\geq]}{\leq} 0.$$

This inequality, by Lemma 1 again, yields that

$$\begin{aligned}
&M_{f_0,g_0;\mu}(u_1+\dots+u_n,v_1+\dots+v_n)\\
&\qquad\underset{[\geq]}{\leq} y_1+\dots+y_n = M_{f_1,g_1;\mu}(u_1,v_1)+\dots+M_{f_n,g_n;\mu}(u_n,v_n),
\end{aligned}$$

which proves (9) on the domain indicated. $\square$

The following result concerns generalized quasi-arithmetic means when a stronger statement can be obtained.

Theorem 4. *Let $I_0, I_1, \dots, I_n$ be open intervals with $I_1 + \cdots + I_n \subseteq I_0$. Assume that $f_i : I_i \to I_0$ are continuously differentiable functions such that $f_i'(x) \neq 0$ if $x \in I_i$ $i = 0, 1, \dots, n$ (the latter conditions ensure that $(f_i, 1) \in \mathcal{C}_1(I_i)$ for $i = 0, 1, \dots, n$). Then the following three assertions are equivalent:*

(i) *For all Borel probability measures μ on $[0, 1]$,*

$$M_{f_0,1;\mu}(x_1 + \cdots + x_n, y_1 + \cdots + y_n) \underset{[\geq]}{\leq} M_{f_1,1;\mu}(x_1, y_1) + \cdots + M_{f_n,1;\mu}(x_n, y_n)$$

holds for all $x_1, y_1 \in I_1, \dots, x_n, y_n \in I_n$.

(ii) *For all $k \in \mathbb{N}$,*

$$M_{f_0,1;m_k}(x_1 + \cdots + x_n, y_1 + \cdots + y_n) \underset{[\geq]}{\leq} M_{f_1,1;m_k}(x_1, y_1) + \cdots + M_{f_n,1;m_k}(x_n, y_n)$$

holds for all $x_1, y_1 \in I_1, \dots, x_n, y_n \in I_n$ (where m_k is the sequence of measures defined by (7)).

(iii)

$$\mathcal{D}^*_{f_0,1}(x_1 + \cdots + x_n, y_1 + \cdots + y_n) \underset{[\geq]}{\leq} \mathcal{D}^*_{f_1,1}(x_1, y_1) + \cdots + \mathcal{D}^*_{f_n,1}(x_n, y_n) \tag{13}$$

holds for all $x_1, y_1 \in I_1, \dots, x_n, y_n \in I_n$.

(iv) *The function $F : f_1(I_1) \times \cdots \times f_n(I_n) \to \mathbb{R}$ defined by*

$$F(u_1, \dots, u_n) := f_0\big(f_1^{-1}(u_1) + \cdots + f_n^{-1}(u_n)\big) \qquad (u_i \in f_i(I_i),\ i = 1, \dots, n)$$

is $\underset{[convex]}{concave}$ on its domain provided that f_0 is increasing and $\underset{[concave]}{convex}$ on its domain provided that f_0 is decreasing.

Proof. The equivalence (i)$\iff$(ii)$\iff$(iii) follows from the previous theorem. To complete the proof we show that (iii) and (iv) are equivalent too. Assume, for the sake of definiteness that f_0 is increasing and the upper inequality sign holds in (13). By known characterizations of differentiable concave functions (see [RV73, p. 98, Theorem A], or [NP06, p. 141, Theorem 3.9.1], F is concave if and only if

$$F(u) - F(v) \leq \sum_{i=1}^{n} \partial_i F(v)(u_i - v_i) \tag{14}$$

holds for all $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n)$ in the domain of F, where $\partial_i F$ denotes the partial derivative of F with respect to its ith variable. A simple calculation shows that

$$\partial_i F(v) = f_0'\big(f_1^{-1}(v_1) + \dots + f_n^{-1}(v_n)\big) \frac{1}{f_i'\big(f_i^{-1}(v_i)\big)}.$$

Dividing (14) by $f_0'(f_1^{-1}(v_1)+\dots+f_n^{-1}(v_n)) > 0$ and then substituting $f_i^{-1}(u_i) =: x_i, f_i^{-1}(v_i) =: y_i$, we obtain exactly (13), which proves the equivalence we claimed. □

3. Minkowski-type inequalities for generalized Gini means

Theorem 5. *Let $n \geq 2$ and $p_0, p_1, \dots, p_n, q_0, q_1, \dots, q_n \in \mathbb{R}$. Then the following three assertions are equivalent:*

(i) *For all Borel probability measures μ on $[0,1]$,*

$$G_{p_0,q_0;\mu}(x_1+\dots+x_n, y_1+\dots+y_n) \leq G_{p_1,q_1;\mu}(x_1,y_1)+\dots+G_{p_n,q_n;\mu}(x_n,y_n) \quad (15)$$

holds for all $x_1, y_1, \dots, x_n, y_n > 0$.

(ii) *For all $k \in \mathbb{N}$,*

$$G_{p_0,q_0;m_k}(x_1 + \dots + x_n, y_1 + \dots + y_n) \\ \leq G_{p_1,q_1;m_k}(x_1,y_1) + \dots + G_{p_n,q_n;m_k}(x_n,y_n) \quad (16)$$

holds for all $x_1, y_1, \dots, x_n, y_n > 0$.

(iii) (a) $0 \leq \min\{p_1, q_1, \dots, p_n, q_n\}$,

(b) $\min\{p_0, q_0\} \leq \min\{1, p_1, q_1, \dots, p_n, q_n\}$,

(c) $\max\{1, p_0, q_0\} \leq \max\{p_i, q_i\}, \qquad (i = 1, \dots, n). \quad (17)$

Concerning the reversed Minkowski inequality, we have the following result.

Theorem 6. *Let $n \geq 2$ and $p_0, p_1, \dots, p_n, q_0, q_1, \dots, q_n \in \mathbb{R}$. Then the following three assertions are equivalent:*

(i) *For all Borel probability measures μ on $[0,1]$,*

$$G_{p_0,q_0;\mu}(x_1+\dots+x_n, y_1+\dots+y_n) \geq G_{p_1,q_1;\mu}(x_1,y_1)+\dots+G_{p_n,q_n;\mu}(x_n,y_n) \quad (18)$$

holds for all $x_1, y_1, \dots, x_n, y_n > 0$.

(ii) *For all* $k \in \mathbb{N}$,

$$G_{p_0,q_0;m_k}(x_1+\cdots+x_n, y_1+\cdots+y_n) \geq G_{p_1,q_1;m_k}(x_1,y_1)+\cdots+G_{p_n,q_n;m_k}(x_n,y_n) \quad (19)$$

holds for all $x_1, y_1, \ldots, x_n, y_n > 0$.

(iii) (a) $1 \geq \max\{p_1, q_1, \ldots, p_n, q_n\}$,

(b) $\max\{p_0, q_0\} \geq \max\{0, p_1, q_1, \ldots, p_n, q_n\}$, (20)

(c) $\min\{0, p_0, q_0\} \geq \min\{p_i, q_i\}, \qquad (i = 1, \ldots, n)$.

Proof of Theorem 5 and Theorem 6. The equivalence of conditions (i) and (ii) in both theorems is a consequence of the equivalence of conditions (i) and (ii) of Theorem 3. To elaborate the third equivalent condition of Theorem 3, observe that if f, g are defined by (1), then the function $\mathcal{D}^*_{f,g}$ is of the form

$$\mathcal{D}^*_{f,g}(x,y) = y\delta_{p,q}\Big(\frac{x}{y}\Big) \qquad (x, y \in \mathbb{R}_+),$$

where

$$\delta_{p,q}(t) := \begin{cases} \dfrac{t^p - t^q}{p-q} & \text{if } p \neq q \\ t^p \ln t & \text{if } p = q \end{cases} \qquad (t \in \mathbb{R}_+). \quad (21)$$

Thus, by Theorem 3, inequalities (15) and (18) are satisfied if and only if

$$(y_1+\cdots+y_n)\delta_{p_0,q_0}\Big(\frac{x_1+\cdots+x_n}{y_1+\cdots+y_n}\Big) \underset{[\geq]}{\leq} y_1\delta_{p_1,q_1}\Big(\frac{x_1}{y_1}\Big)+\cdots+y_n\delta_{p_n,q_n}\Big(\frac{x_n}{y_n}\Big)$$

holds for all $x_1, y_1, \ldots, x_n, y_n > 0$. With the notation $u_i := x_i/y_i$ and $t_i := y_i/(y_1+\cdots+y_n)$, the above inequality is satisfied if and only if

$$\delta_{p_0,q_0}(t_1u_1+\cdots+t_nu_n) \underset{[\geq]}{\leq} t_1\delta_{p_1,q_1}(u_1)+\cdots+t_n\delta_{p_n,q_n}(u_n) \quad (22)$$

for all $u_1, \ldots, u_n, t_1, \ldots, t_n > 0$ with $t_1+\cdots+t_n = 1$.

The domain of parameters when (22) is valid on the indicated domain was characterized in the paper [Pál82]. As it was proved in [Pál82], (22) holds with $\leq$ and $\geq$ inequality signs if and only if condition (iii) of Theorem 5 and Theorem 6 holds, respectively. □

References

[Baj58] M. Bajraktarević, Sur une équation fonctionnelle aux valeurs moyennes, *Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske* Ser. II **13** (1958), 243–248.

[Baj69] M. Bajraktarević, Über die Vergleichbarkeit der mit Gewichtsfunktionen gebildeten Mittelwerte, *Studia Sci. Math. Hungar.* **4** (1969), 3–8.

[Ber98] L. R. Berrone, The mean value theorem: functional equations and Lagrangian means, *Epsilon* **14**, no. 1(40) (1998), 131–151.

[BM98] L. R. Berrone and J. Moro, Lagrangian means, *Aequationes Math.* **55**, no. 3 (1998), 217–226.

[BM00] L. R. Berrone and J. Moro, On means generated through the Cauchy mean value theorem, *Aequationes Math.* **60**, no. 1–2 (2000), 1–14.

[CP00] P. Czinder and Zs. Páles, A general Minkowski-type inequality for two variable Gini means, *Publ. Math. Debrecen* **57** (2000), 203–216.

[CP03] P. Czinder and Zs. Páles, Minkowski-type inequalities for two variable Stolarsky means, *Acta Sci. Math. (Szeged)* **69**, no. 1–2 (2003), 27–47.

[Gin38] C. Gini, Di una formula compressiva delle medie, *Metron* **13** (1938), 3–22.

[HLP34] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, *Cambridge University Press, Cambridge*, 1934, (first edition), 1952 (second edition).

[Los71] L. Losonczi, Subadditive Mittelwerte, *Arch. Math. (Basel)* **22** (1971), 168–174.

[Los00] L. Losonczi, Equality of Cauchy mean values, *Publ. Math. Debrecen* **57** (2000), 217–230.

[LP96] L. Losonczi and Zs. Páles, Minkowski's inequality for two variable Gini means, *Acta Sci. Math. (Szeged)* **62** (1996), 413–425.

[LP98] L. Losonczi and Zs. Páles, Minkowski's inequality for two variable difference means, *Proc. Amer. Math. Soc.* **126**, no. 3 (1998), 779–789.

[LP08] L. Losonczi and Zs. Páles, Comparison of means generated by two functions and a measure, *J. Math. Anal. Appl.* **345**, no. 1 (2008), 135–146.

[NP06] C. P. Niculescu and L.-E. Persson, Convex Functions and Their Applications, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 23, *Springer-Verlag, New York*, 2006, A contemporary approach.

[Pál82] Zs. Páles, A generalization of the Minkowski inequality, *J. Math. Anal. Appl.* **90**, no. 2 (1982), 456–462.

[RV73] A. W. Roberts and D. E. Varberg, Convex Functions, Pure and Applied Mathematics, vol. 57, *Academic Press*, *New York – London*, 1973.

[Sto75] K. B. Stolarsky, Generalizations of the logarithmic mean, *Math. Mag.* **48** (1975), 87–92.

LÁSZLÓ LOSONCZI
FACULTY OF ECONOMICS
UNIVERSITY OF DEBRECEN
KASSAI ÚT 26
H-4028 DEBRECEN
HUNGARY

E-mail: losi@math.klte.hu

ZSOLT PÁLES
INSTITUTE OF MATHEMATICS
UNIVERSITY OF DEBRECEN
H-4010 DEBRECEN, P. O. BOX 12
HUNGARY

E-mail: pales@math.klte.hu

(Received October 1, 2010)