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Resolution of a nontrivial diophantine equation without reduction methods

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Dedicated to Professors Kálmán Győry and András Sárközy on their 70th birthdays and Professors Attila Pethő and János Pintz on their 60th birthdays

Abstract. In this note a non-separable Runge-type diophantine equation related to the equal values of certain combinatorial numbers is solved. The novelty of our approach is avoiding the use of reduction methods, although as a first step we have a huge bound ($\approx 10^{50000}$) for the solutions.

1. Introduction

One of the most important methods of the modern Diophantine Analysis is Baker's theory, see [4]. Using this approach several upper bounds were proved for the solutions of certain large classes of diophantine equations including unit equations, Thue-equations, superelliptic equations and some multi-variables equations, see e.g. [9], [13]. However, these estimates are too large for the resolution of these problems thus we have to apply so-called reduction algorithms

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like Baker–Davenport Lemma [3] or LLL-algorithm [12] for reducing the original bounds.

In this note we solve a diophantine conjecture by Brindza, Pintér and Turjányi [7] without using the above mentioned reduction methods. Before stating our theorem we introduce the concepts of polygonal and pyramidal numbers. Let

$$\mathrm{Pol}_x^m = \frac{x}{2}\left((m-2)x + 4 - m\right)$$

and

$$\mathrm{Pyr}_y^n = \frac{y(y+1)}{6}\left((n-2)y + 5 - n\right)$$

be the polygonal and pyramidal numbers with integral parameters $x > m \geq 3$ and $y > n \geq 3$, respectively. In [7], the authors proved that apart from an effectively computable set of m and n, the equation

$$\mathrm{Pol}_x^m = \mathrm{Pyr}_y^n \tag{1}$$

possesses only finitely many solutions and $\max(x, y) < C$, where C is an effectively computable constant depending only on m and n. They conjectured that the cardinality of this exceptional set is one, namely it consists of the pair $(m, n) = (4, 5)$. Their proof is based on Runge-method and Baker's theory. The geometric interpretation of these combinatorial numbers is well-known and several special cases of (1) are solved, see [13]. For small values of the parameters m and n one can solve the corresponding elliptic equations by MAGMA [5].

We obtain the following

Theorem. *Apart from the pair $(m, n) = (4, 5)$ all the solutions x and y to (1) satisfy $\max(x, y) < C$ where C is an effectively computable constant depending only on m and n.*

One can transform (1) into an elliptic equation and using Baker's classical result concerning the solutions of elliptic equations (see Lemma 2) it is enough to guarantee that the discriminant of the corresponding cubic polynomial is nonzero except for the unique pair $(m, n) = (4, 5)$. We remark that in [11] our theorem is also stated without proof.

2. Auxiliary results

First we give a special case of a Runge-type result due to Grytczuk and Schinzel [10] (cf. [14]). Let

$$F(x,y)=\sum_{i=0}^{k}\sum_{j=0}^{l}a_{i,j}x^iy^j$$

be a polynomial with rational integer coefficients of degree $k>0$ in x and $l>0$ in y which is irreducible in $\mathbb{Q}[x,y]$.

Lemma 1. *Let x and y be solutions of the equation*

$$F(x,y)=0$$

and suppose that $a_{k,j}\neq 0$ for some nonzero j. Then we have

$$|x|\leq\Big((k+1)(l+1)(kl+1)^{2/l}h\Big)^{2l(kl+1)^3}$$

and

$$|y|\leq\Big((k+1)(l+1)(kl+1)^{2/l}h\Big)^{2(kl+1)^3},$$

where $h=\max_{i,j}|a_{i,j}|$.

Proof. See [10].

Our second lemma is a result by Baker [2]. For generalizations we refer the reader to [1], [6] and [8].

Lemma 2. *Let $f(x)$ be a cubic polynomial with rational integer coefficients and nonzero discriminant. The equation $f(x)=y^2$ implies $\max(|x|,|y|)<C_1$, where C_1 is an effectively computable constant depending only on the coefficients of f.*

Proof. See [2].

3. Proof of the Theorem

From (1) we get

$$f_{m,n}=z^3+6(m-2)z^2-4(n-5)(n-2)(m-2)^2z+6(n-2)^2(m-2)^2(m-4)^2=6t^2,$$

where

$$z = 2(n-2)(m-2)y$$

and

$$t = (n-2)(m-2)\left(2(m-2)x - (m-4)\right).$$

This equation has only finitely many solutions in z and t if and only if the discriminant $\mathrm{disc}(f_{m,n})$ is nonzero, i.e. $f_{m,n}$ has only simple zero. We obtain that

$$\begin{aligned}\mathrm{disc}(f_{m,n}) = -4(m-2)^4(n-2)^2\big[&243m^4(n^2-4n+4) - 648m^3(5n^2-17n+12)\\ &- 16m^2(4n^4-68n^3-624n^2+1807n+253)\\ &+ 64m(4n^4-68n^3-219n^2+430n+1225)\\ &- 64(4n^4-68n^3-219n^2+430n+1225)\big]. \qquad (2)\end{aligned}$$

One can check by the program package MAPLE that the polynomial in square brackets, denoted by $F(m,n)$ is irreducible in $\mathbb{Q}[m,n]$. By the linear transformations $m \mapsto m+4$ and $n \mapsto n+4$ we can reduce the height of the polynomial F from 78400 to 1344 and finally Lemma 1 yields that all the solutions m and n of the equation $F(m,n) = 0$ satisfy

$$m \le 1.05 \cdot 10^{202084} \qquad (3)$$

and

$$n \le 1.013 \cdot 10^{50521}. \qquad (4)$$

Of course, it is hopeless to check all the pairs (m,n) in the intervals above. Our argument is based on the fact that the solutions of a Pellian equation grow exponentially. If $f_{m,n}$ has a multiple zero α, say, then we have to distinguish two cases. If α is a triple zero of $f_{m,n}$ then

$$(z-\alpha)^3 = z^3 + 6(m-2)z^2 - 4(n-5)(n-2)(m-2)^2 z + 6(n-2)^2(m-2)^2(m-4)^2,$$

and

$$\alpha = 2(2-m) \quad \text{and} \quad n^2 - 7n + 13 = 0,$$

which is impossible for integer n. In the remaining case $f_{m,n}(z) = (z-\alpha)^2(z-\beta)$, where α and β are rational integers and

$$3\alpha^2 + 12(m-2)\alpha - 4(n-5)(n-2)(m-2)^2 = 0. \qquad (5)$$

Since α is a rational integer, we have that

$$3(n-5)(n-2) + 9 = A^2,$$

where A is a rational integer. From this equation we get

$$(2A_1)^2 - 3B^2 = 1,$$

where $A = 3A_1$ and $2n - 7 = 3B$. By a straightforward application of the theory of Pellian equation we obtain

$$A_1 = \frac{(2+\sqrt{3})^k + (2-\sqrt{3})^k}{4},\ k \text{ odd},$$

and

$$B = \frac{(2+\sqrt{3})^k - (2-\sqrt{3}\,)^k}{2\sqrt{3}}.$$

From estimate (4) we have $k \leq 88382$ and k is odd.

One can solve the quadratic equation (5) for α and this yields

$$\alpha_{1,2} = \frac{2}{3}(m-2)(-2 \pm A).$$

Substituting α_1 and α_2 into $f_{m,n}(z)$ and dividing by $(m-2)^2$ we have a quadratic equation for m with discriminant

$$16(A-3)(-3n+12+A)(3n-9+A)(A^2+3nA-3A+45n-108)(-3n+3+A)$$

and

$$16(A+3)(3n-12+A)(-3n+9+A)(A^2-3nA+3A+45n-108(3n-3+A),$$

respectively. Since m is a rational integer, A and n depend on k, for all possible values of k we check whether these numbers are squares or not. We implemented a short and simple MAPLE program to verify this property using the built-up function `issqr`. The CPU time is under 4 hours in a PC with quad-core processor. The computation shows that the first discriminant is square when $n = 5$ and the second one is square when $n = 26$. In the second case the corresponding quadratic polynomial is reducible over $\mathbb{Q}$, however, its zeros are not integers, while in the first case $m = 4$.

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