

On the diophantine equation $(a^n - 1)(b^n - 1) = x^2$

By PINGZHI YUAN (Guangzhou) and ZHONGFENG ZHANG (Zhaoqing)

Abstract. Let a and b be distinct positive integers. In this paper, we will present some new results on the positive integer solutions (n, x) of the equation of the title.

1. Introduction

Let $\mathbb{N}^+$ denote the set of all positive integers, and let $a, b \in \mathbb{N}^+$. There are some results on the following equation

$$(a^n - 1)(b^n - 1) = x^2. \quad (1.1)$$

SZALAY [7] has shown that equation (1.1) has no solution for $(a, b) = (2, 3)$, only the solution $n = 1$ for $(2, 5)$, and for $(2, 2^k)$ has no solutions with $k \geq 2$ except for $n = 3$ and $k = 2$. HAJDU and SZALAY [3] proved that (1.1) has no solution for $(2, 6)$ and for (a, a^k) , there are no solutions with $k \geq 2$ and $kn > 2$ except for the three cases $(a, n, k) = (2, 3, 2), (3, 1, 5)$ and $(7, 1, 4)$. Walsh [9] proved that equation $(2^n - 1)(3^m - 1) = z^2$ has no positive integer solutions (n, m, x) . COHN [2] obtained some general results for equation (1.1). He proved that there is no solution to (1.1) when $n = 4$, except for $(a, b) = (13, 239)$.

LUCA and WALSH [6] have shown that the equation

$$(a^k - 1)(b^k - 1) = x^n \quad (1.2)$$

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has finitely many solutions in positive integers (k, x, n) with $n > 1$. Moreover, they showed how one can determine all solutions of the equation (1.1) with $n > 1$, for almost all pairs (a, b) with $2 \leq b < a \leq 100$. Recently, LE [5] proved that if $3|b$, then the equation

$$(2^n - 1)(b^n - 1) = x^2 \quad (1.3)$$

has no solutions in positive integers n and x . Tang [8] showed that (1.1) has no solutions for (a, b) with $a \equiv 0 \pmod{2}$ and $b \equiv 15 \pmod{20}$ or $a \equiv 2 \pmod{6}$ and $b \equiv 0 \pmod{3}$. LI and SZALAY [4] proved that (1.1) has no solution if $a \equiv 2 \pmod{6}$ and $b \equiv 0 \pmod{3}$.

In this paper we prove the following result. This theorem generalizes the results in [7] (Theorem 1), in [3], in [5] and in [4] (Theorem 1).

Theorem. $a, b \in \mathbb{N}^+$. Suppose that one of the following properties is satisfied:

- i) $a \equiv 2 \pmod{3}$ and $b \equiv 0 \pmod{3}$,
- ii) $a \equiv 3 \pmod{4}$ and $b \equiv 0 \pmod{2}$,
- iii) $a \equiv -1 \pmod{5}$ and $b \equiv 0 \pmod{5}$.

Then equation (1.1) has no positive integer solutions (n, x) with $n > 2$.

To prove our result, beside combining some known tools from [2], [4], [6], we introduce a new one as well: a result of Bennett and Skinner concerning ternary equations of signature $(n, n, 2)$.

2. Lemmas and the proof of the Theorem

To prove the Theorem, we need some results on divisibility properties of the solutions of Pell equation and some known results.

Let D be a non-square positive integer. It is well-known that the Pell equation

$$u^2 - Dv^2 = 1, \quad u, v \in \mathbb{N}^+ \quad (2.1)$$

has infinitely many solutions (u, v) . If $(u, v) = (u_1, v_1)$ denotes the fundamental solution to equation (2.1), then every positive solution (u_k, v_k) ($k \in \mathbb{N}^+$) can be represented by

$$u_k + v_k\sqrt{D} = (u_1 + v_1\sqrt{D})^k, \quad k = 1, 2, \dots \quad (2.2)$$

First, we need the following simple lemma.

Lemma 1. (i) $u_1|u_k$ if and only if $2 \nmid k$.

(ii) If $q \in \{2, 3, 5\}$, then $q|u_k$ implies that k is odd and $q|u_1$.

(iii) $u_{2k} = 2u_k^2 - 1$.

For the proof of the above lemma, we refer to [4] Lemma 1 or the more general result of the first author [10].

The following lemma is Theorem 1.1 in [1]. It plays a key role in the proof of our Theorem.

Lemma 2. If $n \geq 3$, then the diophantine equation

$$x^n = 2y^2 - 1 \quad (2.3)$$

has no solution (x, y) with $x > 1$.

The following lemma is an immediate consequence of Result 2 of [2].

Lemma 3. The diophantine equation

$$(a^{4m} - 1)(b^{4n} - 1) = z^2 \quad (2.4)$$

has the only positive integer solution $(a, b, m, n, z) = (13, 239, 1, 1, 9653280)$.

PROOF OF THE THEOREM. We prove only part i) of the statement, the proof of the other parts are similar. Let $a \equiv 2 \pmod{3}$ and $b \equiv 0 \pmod{3}$, and suppose that (n, x) is a solution to equation (1.1). Put $D = \gcd(a^n - 1, b^n - 1)$. By (1.1), we get

$$a^n - 1 = Dy^2, \quad b^n - 1 = Dz^2, \quad x = Dyz, \quad D, y, z \in \mathbb{N}^+. \quad (2.5)$$

Since $3|b$, by $b^n - 1 = Dz^2$, it follows that

$$D \equiv -1 \pmod{3} \quad \text{and} \quad 3 \nmid z. \quad (2.6)$$

Now we distinguish two cases. Firstly, if $3 \nmid y$, then $y^2 \equiv 1 \pmod{3}$, and (2.5), together with (2.6) implies

$$a^n = Dy^2 + 1 \equiv D + 1 \equiv 0 \pmod{3}, \quad (2.7)$$

which contradicts $a \equiv 2 \pmod{3}$.

Assume now that $3|y$. Since $a \equiv 2 \pmod{3}$, by $a^n - 1 = Dy^2$ we obtain

$$2^n \equiv a^n \equiv Dy^2 + 1 \equiv 1 \pmod{3}, \quad (2.8)$$

which implies that n is even.

Put $n = 2m$. Therefore, by (2.5), D cannot be a square, and the corresponding Pell equation $u^2 - Dv^2 = 1$ has two solutions

$$(x, y) = (a^m, y), (b^m, z). \quad (2.9)$$

Since $a \neq b$, there exist distinct positive integers r and s such that

$$(a^m, y) = (u_r, v_r) \quad \text{and} \quad (b^m, z) = (u_s, v_s) \quad (2.10)$$

hold.

By Lemma 1 (ii) and $3|b$, we obtain that $2 \nmid s$ and $3|u_1$. On the other hand, $a \equiv 2 \pmod{3}$, which together with $3|u_1$ and Lemma 1 (i), shows that $2|r$.

Put $r = 2t$, then by Lemma 1 (iii),

$$u_{2t} = 2u_t^2 - 1 = a^m. \quad (2.11)$$

Now we distinguish two cases. Firstly, if $2|m$, then $4|n$, and so Lemma 2.3 implies that $(a, b) = (13, 239)$, which contradicts $3|b$. Now we assume that $2 \nmid m$ and $m > 1$, then Lemma 2 implies that (2.11) has no positive integer solutions, a contradiction. $\square$

Remark. By [2] Result 4 and the above proof, it is easy to see equation

$$(a^2 - 1)(b^2 - 1) = z^2, \quad a, b, z \in \mathbb{N}^+$$

has infinitely many solutions (a, b, x) with $a \equiv 5 \pmod{6}$ and $b \equiv 0 \pmod{3}$.

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PINGZHI YUAN
SCHOOL OF MATHEMATICS
SOUTH CHINA NORMAL UNIVERSITY
GUANGZHOU 510631
P.R. CHINA

E-mail: yuanpz@scnu.edu.cn

ZHONGFENG ZHANG
COLLEGE OF MATHEMATICS
AND INFORMATION SCIENCE
ZHAOQING UNIVERSITY
ZHAOQING, 526061
P.R. CHINA

E-mail: zh12zh31f@yahoo.com.cn

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