

A note on two conjectures associated to Goldbach's problem

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Abstract. Chen and Chen recently proposed two conjectures on the structure of multiplicative functions f for which $f(p) + f(q) = f(p + q)$ for all odd primes p and q . In this note, we show that the second conjecture is either true unconditionally or follows from the first conjecture, depending on whether or not there is an odd prime p_0 such that $f(p_0) \neq 0$.

Define $g_1(n) := n$ for $n \geq 1$ and define $g_2(n)$ by

$$g_2(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Note that both g_1 and g_2 are multiplicative functions.

CHEN and CHEN [1] gave the following result.

Theorem 1 (CHEN and CHEN [1]). *Let f be a multiplicative function for which*

$$f(p) + f(q) = f(p + q)$$

for all odd primes p and q . If there is an odd prime p_0 for which $f(p_0) \neq 0$, then either $f = g_1$ or $f = g_2$. Moreover, $f = g_1$ if and only if $f(3) = 3$.

As probable extensions of this theorem, CHEN and CHEN gave the following two conjectures (see [1, Conjectures 1 and 2]).

Mathematics Subject Classification: Primary 11A25.

Key words and phrases: multiplicative function, recurrence sequences.

The research of M. Coons is supported by a Fields–Ontario Fellowship and NSERC.

Conjecture 1 (CHEN and CHEN [1]). *If f is a multiplicative function such that $f(2) \neq 0$, $f(3) = 0$ and $f(p) + f(q) = f(p + q)$ for all odd primes p and q , then $f(n) = 0$ for all $n \geq 5$.*

Conjecture 2 (CHEN and CHEN [1]). *If f is a multiplicative function such that $f(2) = 2$ and $f(p) + f(q) = f(p + q)$ for all odd primes p and q , then for $n \geq 3$*

$$f(2n) = \frac{f(3)}{3} ((n-3)f(4) + 12 - 2n),$$

and

$$f(2n-1) = \frac{f(3)}{3} ((n-2)f(4) + 7 - 2n).$$

Concerning Conjecture 1, Chen and Chen remark that *if f satisfies the conditions of Conjecture 1, then $f(p) = 0$ for all primes $p \geq 5$. Thus by induction on n we can prove that the Goldbach conjecture implies Conjecture 1. This implies that if Conjecture 1 is false, then the Goldbach conjecture is false.*

It seems that the use of the recurrence relations in Conjecture 2 may be a bit misleading. Indeed, Conjecture 2 can be considered following two cases: if $f(p_0) \neq 0$ for some odd prime p_0 , then Conjecture 2 is true by Theorem 1, and if $f(p) = 0$ for all odd primes p , then Conjecture 2 is implied by Conjecture 1, which we will now show.

Proposition 1. *Let f be a function satisfying the assumptions of Conjecture 2 and suppose further that there is an odd prime p_0 such that $f(p_0) \neq 0$. Then the conclusion of Conjecture 2 holds unconditionally.*

PROOF. If there is an odd prime p_0 such that $f(p_0) \neq 0$, then by Theorem 1 f is one of g_1 or g_2 . If $f = g_1$, then for $n \geq 3$ we have that both

$$\frac{f(3)}{3} ((n-3)f(4) + 12 - 2n) = (n-3)4 + 12 - 2n = 2n = g_1(2n) = f(2n),$$

and

$$\frac{f(3)}{3} ((n-2)f(4) + 7 - 2n) = (n-2)4 + 7 - 2n = 2n - 1 = g_1(2n-1) = f(2n-1),$$

so that Conjecture 2 holds. Now if $f = g_2$, then for $n \geq 3$, we have that both

$$\frac{f(3)}{3} ((n-3)f(4) + 12 - 2n) = \frac{(n-3)2 + 12 - 2n}{3} = 2 = g_2(2n) = f(2n),$$

and

$$\frac{f(3)}{3} ((n-2)f(4) + 7 - 2n) = \frac{(n-2)2 + 7 - 2n}{3} = 1 = g_2(2n-1) = f(2n-1),$$

so that in either case since $f(p_0) \neq 0$ for some odd prime p_0 , Conjecture 2 holds. $\square$

Proposition 2. *Let f be a function satisfying the assumptions of Conjecture 2 and suppose further that $f(p) = 0$ for all odd primes p . Then the conclusion of Conjecture 2 follows from Conjecture 1.*

PROOF. Suppose that Conjecture 1 holds and let f be a multiplicative function such that $f(2) = 2$ and $f(p) + f(q) = f(p + q)$ for all odd primes p and q , and suppose further that $f(p) = 0$ for all odd primes p .

Since 3 is an odd prime $f(3) = 0$ and it is easy to see for $n \geq 3$ that

$$\frac{f(3)}{3} ((n-3)f(4) + 12 - 2n) = 0 = f(2n),$$

and

$$\frac{f(3)}{3} ((n-2)f(4) + 7 - 2n) = 0 = f(2n-1),$$

where in each of these equations the last equals sign is given by Conjecture 1. This finishes the proof. $\square$

References

- [1] KANG-KANG CHEN and YONG-GAO CHEN, On $f(p) + f(q) = f(p + q)$ for all odd primes p and q , *Publ. Math. Debrecen* **76** (2010), 425–430.

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(Received November 5, 2010; revised August 15, 2011)