

Fractional part integral representation for derivatives of a function related to $\ln \Gamma(x + 1)$

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Abstract. For $0 \neq x > -1$ let

$$\Delta(x) = \frac{\ln \Gamma(x + 1)}{x}.$$

Recently Adell and Alzer proved the complete monotonicity of Δ' on $(-1, \infty)$ by giving an integral representation of $(-1)^n \Delta^{(n+1)}(x)$ in terms of the Hurwitz zeta function $\zeta(s, a)$. We reprove this integral representation in different ways, and then re-express it in terms of fractional part integrals. Special cases then have explicit evaluations. Other relations for $\Delta^{(n+1)}(x)$ are presented, including its leading asymptotic form as $x \rightarrow \infty$.

Introduction and statement of results

For $0 \neq x > -1$ let

$$\Delta(x) = \frac{\ln \Gamma(x + 1)}{x}, \quad \Delta(0) = -\gamma, \tag{1.1}$$

where Γ is the Gamma function, $\gamma = -\psi(1)$ is the Euler constant, and $\psi(x) = \Gamma'/\Gamma$ is the digamma function. The study of the convexity and monotonicity of the functions Γ and Δ and of their derivatives is of interest [8], [13], [14], [17]. For instance, the paper [8] gave an analog of the well known Bohr–Mollerup theorem for the function $\Delta(x)$. Monotonicity and convexity are very useful properties for developing a variety of inequalities. Completely monotonic functions have

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applications in several branches, including complex analysis, potential theory, number theory, and probability (e.g., [5]). In [4], $-\Delta(x)$ was shown to be a Pick function, with integral representation

$$-\Delta(x) = -\frac{\pi}{4} + \int_{-\infty}^{-1} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) \frac{dt}{-t}.$$

I.e., this function is holomorphic in the upper half plane with nonnegative imaginary part.

Recently ADELL and ALZER [2] proved the complete monotonicity of Δ' on $(-1, \infty)$ by demonstrating the following integral representation.

Proposition 1 (Adell and Alzer). *For $x > -1$ and $n \geq 0$ an integer one has*

$$(-1)^n \Delta^{(n+1)}(x) = (n+1)! \int_0^1 u^{n+1} \zeta(n+2, xu+1) du, \quad (1.2)$$

where $\zeta(s, a)$ is the Hurwitz zeta function (e.g., [10]). The complete monotonicity of Δ' , the statement $(-1)^n \Delta^{(n+1)}(x) \geq 0$, then follows from $\zeta(n+2, xu+1) \geq 0$ for $x > -1$.

We reprove the result (1.2) in two other ways, and in so doing illustrate properties of the ζ function.

Corollary 1. *We have the following recurrence:*

$$\frac{(-1)^n}{(n+1)!} \Delta^{(n+1)}(x) = \frac{1}{x} \frac{(-1)^{n-1}}{n!} \Delta^{(n)}(x) - \frac{\zeta(n+1, x+1)}{(n+1)x}. \quad (1.3)$$

We then relate cases of (1.2) to fractional part integrals, including the following, wherein we let $\{x\} = x - [x]$ denote the fractional part of x .

Proposition 2. *Let $k \geq 1$ be an integer. Then we have*

$$\begin{aligned} \int_0^1 u^{n+1} \zeta(n+2, ku+1) du &= \frac{1}{k^{n+2}} \left[\int_1^\infty \frac{\{w\}^{n+1}}{w^{n+2}} dw \right. \\ &\quad \left. + \sum_{j=1}^{k-1} \int_0^\infty \frac{(\{x\}+j)^{n+1}}{(x+j+1)^{n+2}} dx \right]. \end{aligned} \quad (1.4)$$

As a further special case we have

Corollary 2. *We have*

$$\begin{aligned} (-1)^n \Delta^{(n+1)}(1) &= (n+1)! \int_0^1 y^{n+1} \zeta(n+2, y+1) dy = (n+1)! \int_0^\infty \frac{\{x\}^{n+1}}{(x+1)^{n+2}} dx \\ &= (n+1)! \left[1 - \gamma - \sum_{j=2}^{n+1} \frac{1}{j} [\zeta(j) - 1] \right], \end{aligned} \quad (1.5)$$

where $\zeta(s) = \zeta(s, 1)$ is the Riemann zeta function [7], [10], [15], [16].

More generally, we have the following, wherein we put $P_1(x) = \{x\} - 1/2$. Let ${}_2F_1$ be the Gauss hypergeometric function [3], [9].

Proposition 3. *We have for integers $n \geq 0$*

$$\begin{aligned} \int_0^1 u^{n+1} \zeta(n+2, xu+1) du &= \frac{1}{2(n+2)} \frac{1}{(x+1)^{n+2}} {}_2F_1 \left(1, n+2; n+3; \frac{x}{x+1} \right) \\ &+ \frac{1}{(n+1)(n+2)} \frac{1}{(x+1)^{n+1}} {}_2F_1 \left(1, n+1; n+3; \frac{x}{x+1} \right) \\ &- \int_0^\infty \frac{1}{(t+1)} \frac{P_1(t)}{(t+x+1)^{n+2}} dt. \end{aligned} \quad (1.6)$$

From this Proposition we may then determine the following asymptotic form:

Corollary 3. *We have*

$$\Delta^{(n+1)}(x) \sim (-1)^n \frac{n!}{(x+1)^{n+1}}, \quad x \rightarrow \infty, \quad (1.7)$$

in agreement with Corollary 1.2 of [2].

In fact, the proof shows how higher order terms may be systematically found.

Many expressions may be found for the ${}_2F_1$ functions in (1.6) and (2.20) below, and we present a sample of these in an Appendix.

A simple property of Δ is given in the following.

Proposition 4. *We have (a)*

$$\int_0^1 \Delta(x) dx = -\gamma + \sum_{k=2}^{\infty} \frac{(-1)^k}{k^2} \zeta(k), \quad (1.8a)$$

and (b)

$$\int_0^1 \Delta^2(x) dx = \gamma^2 - 2\gamma \sum_{k=2}^{\infty} \frac{(-1)^k}{k^2} \zeta(k) + \sum_{m=4}^{\infty} \frac{(-1)^m}{(m-1)} \sum_{\ell=2}^{m-2} \frac{\zeta(m-\ell)\zeta(\ell)}{(m-\ell)\ell}. \quad (1.8b)$$

Throughout we let $\psi^{(j)}$ denote the polygamma functions (e.g., [1]), and we note the relation for integers $n > 0$

$$\psi^{(n)}(x) = (-1)^{n+1} n! \zeta(n+1, x). \quad (1.9)$$

Therefore, as to be expected, (1.2) could equally well be written as an integral over $\psi^{(n+1)}(xu+1)$. The polygamma functions possess the functional equation

$$\psi^{(j)}(x+1) = \psi^{(j)}(x) + (-1)^j \frac{j!}{x^{j+1}}. \quad (1.10)$$

For a very recent development of single- and double-integral and series representations for the Gamma, digamma, and polygamma functions, [6] may be consulted.

Proof of Propositions

Proposition 1. *We provide two alternative proofs of this result. The result holds for $n = 0$, and for the first proof we proceed by induction. For the inductive step we have*

$$\begin{aligned} \Delta^{(n+2)}(x) &= \frac{d}{dx} \Delta^{(n+1)}(x) = (-1)^n (n+1)! \int_0^1 u^{n+1} \frac{d}{dx} \zeta(n+2, xu+1) du \\ &= (-1)^{n+1} (n+2)! \int_0^1 u^{n+2} \zeta(n+3, xu+1) du. \end{aligned} \quad (2.1)$$

In the last step, we used $\partial_a \zeta(s, a) = -s \zeta(s+1, a)$.

We remark that this first method shows that (1.2) may be evaluated by repeated integration by parts, for we have

$$(n+1)! \int_0^1 u^{n+1} \zeta(n+2, xu+1) du = \frac{(-1)^n}{x^n} \int_0^1 u^{n+1} \left(\frac{\partial}{\partial u} \right)^n \zeta(2, xu+1) du. \quad (2.2)$$

Second method. By the product rule we have

$$\begin{aligned} \Delta^{(n+1)}(x) &= \sum_{j=0}^{n+1} \binom{n}{j} [\ln \Gamma(x+1)]^{(n-j)} \frac{(-1)^j j!}{x^{j+1}} \\ &= \sum_{j=0}^{n+1} \binom{n+1}{j} \psi^{(n-j)}(x+1) \frac{(-1)^j j!}{x^{j+1}}. \end{aligned} \quad (2.3)$$

Here, it is understood that $\psi^{(-1)}(x) = \ln \Gamma(x)$. We now apply (1.9) and the integral representation

$$(n-j)!\zeta(n-j+1, x+1) = \int_0^\infty \frac{t^{n-j}e^{-xt}}{e^t-1} dt, \quad (2.4)$$

so that

$$\begin{aligned} \Delta^{(n+1)}(x) &= (-1)^{n+1} \sum_{j=0}^{n+1} \frac{j!}{x^{j+1}} \int_0^\infty \frac{t^{n-j}e^{-xt}}{e^t-1} dt \\ &= \frac{(-1)^{n+1}}{x^{n+2}} \int_0^\infty \frac{e^{-xt}}{(e^t-1)} [e^{xt}\Gamma(n+2, xt) - (n+1)!] \frac{dt}{t}, \end{aligned} \quad (2.5)$$

where the incomplete Gamma function $\Gamma(x, y)$ has the property [9] (p. 941)

$$\Gamma(n+1, x) = n!e^{-x} \sum_{m=0}^n \frac{x^m}{m!}. \quad (2.6)$$

Now we use a Laplace transform,

$$\int_0^1 u^{n+1} e^{-xut} du = \frac{1}{(xt)^{n+2}} [(n+1)! - \Gamma(n+2, xt)], \quad (2.7)$$

to write

$$\begin{aligned} \Delta^{(n+1)}(x) &= (-1)^n \int_0^\infty \frac{e^{-xt}}{(e^t-1)} t^{n+1} \int_0^1 u^{n+1} e^{-xtu} du dt \\ &= (-1)^n \int_0^1 u^{n+1} \int_0^\infty \frac{t^{n+1}}{e^t-1} e^{-xtu} dt du \\ &= -(-1)^n \int_0^1 u^{n+1} \zeta(n+2, xu+1) du. \end{aligned} \quad (2.8)$$

By absolute convergence and the Tonelli–Hobson theorem, the interchange of integrations is justified. In the last step, we applied the representation (2.4).

Corollary 1. *This is proved by integrating by parts in (1.2).*

Remark. It is possible to find explicit expressions for the values $\Delta^{(n+1)}(j+1/2)$ with half-integer argument. This is due to the functional equation (1.10) along with the values $\psi^{(-1)}(1/2) = \ln \sqrt{\pi}$, $\psi(1/2) = -\gamma - 2 \ln 2$, and [1] (p. 260)

$$\psi^{(n)}\left(\frac{1}{2}\right) = (-1)^{n+1} n! (2^{n+1} - 1) \zeta(n+1), \quad n \geq 1. \quad (2.9)$$

We then obtain, for instance, by using (2.3)

$$\frac{\Delta^{(n+1)}\left(-\frac{1}{2}\right)}{(n+1)!} = \sum_{j=0}^{n-1} \frac{(-1)^{n-j}}{(n-j+1)} (2^{n+2} - 2^{j+1}) \zeta(n-j+1) \\ + 2^{n+1}(\gamma + 2 \ln 2) - 2^{n+2} \ln \sqrt{\pi}. \quad (2.10)$$

Similarly, it is possible to find explicit expressions for the values $\Delta^{(n+1)}(j+1/4)$ and $\Delta^{(n+1)}(j+3/4)$ by using the corresponding values of $\psi^{(k)}$ [11].

Proposition 2. *We use two Lemmas.*

Lemma 1. *When the integrals involved are convergent, we have for integrable functions f and g*

$$\int_1^\infty f(\{x\})g(x)dx = \int_0^1 f(y) \sum_{\ell=1}^\infty g(y+\ell)dy. \quad (2.11)$$

Lemma 2. *For $b > 0$, $\lambda > 1$, and $c \geq 0$ we have for integrable functions f*

$$\int_0^\infty f\left(\left\{\frac{x}{b}\right\}\right) \frac{dx}{(x+c)^\lambda} = \frac{1}{b^{\lambda-1}} \int_0^1 f(y) \zeta(\lambda, y+c/b) dy. \quad (2.12)$$

This holds when the integrals are convergent.

PROOF. For Lemma 1 we have

$$\int_1^\infty f(\{x\})g(x)dx = \sum_{\ell=1}^\infty \int_\ell^{\ell+1} f(\{x\})g(x)dx \\ = \sum_{\ell=1}^\infty \int_\ell^{\ell+1} f(x-\ell)g(x)dx = \sum_{\ell=1}^\infty \int_0^1 f(y)g(y+\ell)dy. \quad (2.13)$$

For Lemma 2 we first have

$$\int_0^\infty f\left(\left\{\frac{x}{b}\right\}\right) g(x)dx = b \int_0^\infty f(\{v\})g(bv)dv \\ = b \sum_{\ell=0}^\infty \int_\ell^{\ell+1} f(v-\ell)g(bv)dv = b \sum_{\ell=0}^\infty \int_0^1 f(y)g[b(y+\ell)]dy. \quad (2.14)$$

We now put $g(x) = 1/(x+c)^\lambda$, so that

$$\sum_{\ell=0}^\infty g[b(y+\ell)] = \frac{1}{b^\lambda} \sum_{\ell=0}^\infty \frac{1}{(y+\ell+c/b)^\lambda} = \frac{1}{b^\lambda} \zeta(\lambda, y+c/b). \quad (2.15)$$

□

PROOF OF PROPOSITION 2. We have for integers $k \geq 1$

$$\begin{aligned}
 \int_0^1 u^{n+1} \zeta(n+2, ku+1) du &= \frac{1}{k^{n+2}} \int_0^k v^{n+1} \zeta(n+2, v+1) dv \\
 &= \frac{1}{k^{n+2}} \sum_{\ell=0}^{k-1} \int_{\ell}^{\ell+1} v^{n+1} \zeta(n+2, v+1) dv \\
 &= \frac{1}{k^{n+2}} \sum_{\ell=0}^{k-1} \int_0^1 (w+\ell)^{n+1} \zeta(n+2, w+\ell+1) dw. \tag{2.16}
 \end{aligned}$$

We now apply Lemma 2 with $b = 1$, $c = \ell + 1$, and $f(w) = (w + \ell)^{n+1}$, giving

$$\begin{aligned}
 \int_0^1 u^{n+1} \zeta(n+2, ku+1) du &= \frac{1}{k^{n+2}} \sum_{\ell=0}^{k-1} \int_0^{\infty} \frac{(\{x\} + \ell)^{n+1}}{(x + \ell + 1)^{n+2}} dx \\
 &= \frac{1}{k^{n+2}} \left[\int_1^{\infty} \frac{\{w\}^{n+1}}{w^{n+2}} dw + \sum_{\ell=1}^{k-1} \int_0^{\infty} \frac{(\{x\} + \ell)^{n+1}}{(x + \ell + 1)^{n+2}} dx \right]. \tag{2.17}
 \end{aligned}$$

In the last step we used the periodicity $\{w-1\} = \{w\}$ for $w \geq 1$. $\square$

For Corollary 2, we apply Lemma 2 of [6].

Proposition 3. *We start from the integral representation*

$$\zeta(s, a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} - s \int_0^{\infty} \frac{P_1(x)}{(x+a)^{s+1}} dx, \quad \operatorname{Re} s > -1. \tag{2.18}$$

Then

$$\begin{aligned}
 \int_0^1 u^{n+1} \zeta(n+2, xu+1) du &= \int_0^1 u^{n+1} \left[\frac{1}{2(xu+1)^{n+2}} + \frac{1}{(n+1)(xu+1)^{n+1}} \right. \\
 &\quad \left. - (n+2) \int_0^{\infty} \frac{P_1(t) dt}{(t+xu+1)^{n+3}} \right] du. \tag{2.19}
 \end{aligned}$$

By using a standard integral representation of ${}_2F_1$ (e.g., [9], p. 1040 or [3] p. 65) we have

$$\begin{aligned}
 \int_0^1 u^{n+1} \zeta(n+2, xu+1) du &= \frac{1}{2(n+2)} {}_2F_1(n+2, n+2; n+3; -x) \\
 &+ \frac{1}{(n+1)(n+2)} {}_2F_1(n+1, n+2; n+3; -x) - \int_0^{\infty} \frac{1}{(t+1)} \frac{P_1(t)}{(t+x+1)^{n+2}} dt. \tag{2.20}
 \end{aligned}$$

By applying a standard transformation rule [9] (p. 1043) to the ${}_2F_1$ functions, we obtain the Proposition.

Corollary 2. We give the detailed asymptotic forms as $x \rightarrow \infty$ of the hypergeometric functions in (1.6). We easily have that ${}_2F_1(1, n+1; n+3; 1) = n+2$ and these forms will then show that the corresponding term in (1.6) gives the leading term as $x \rightarrow \infty$. We let $(a)_j = \Gamma(a+j)/\Gamma(a)$ be the Pochhammer symbol. The following expansions are valid for $|z-1| < 1$ and $|\arg(1-z)| < \pi$:

$${}_2F_1(1, y; 1+y; z) = y \sum_{k=0}^{\infty} \frac{(y)_k}{k!} [\psi(k+1) - \psi(k+y) - \ln(1-z)] (1-z)^k, \quad (2.21a)$$

and

$${}_2F_1(1, y; 2+y; z) = y+1 - y(y+1) \sum_{k=0}^{\infty} \frac{(y+1)_k}{k!} [\psi(k+1) - \psi(k+y+1) - \ln(1-z)] (1-z)^{k+1}, \quad (2.21b)$$

where $(y)_0 = 1$. These expansions are the $n=0$ and $n=1$ cases of (9.7.5) in [12] (p. 257), respectively. We put $y = n+1$, $z = x/(x+1)$, $\ln(1-z) = -\ln(x+1)$ and then find

$$\begin{aligned} & {}_2F_1\left(1, n+1; n+3; \frac{x}{x+1}\right) \\ &= n+2 + (n+1)(n+2)[\gamma - \ln x + \psi(n+2)] \frac{1}{x} + O\left(\frac{\ln x}{x^2}\right), \end{aligned} \quad (2.22a)$$

and

$${}_2F_1\left(1, n+2; n+3; \frac{x}{x+1}\right) = -(n+2)[\gamma - \ln x + \psi(n+2)] + O\left(\frac{\ln x}{x}\right). \quad (2.22b)$$

The integral term in (1.6) is at most $O[(x+1)^{-(n+2)}]$, and is actually much smaller due to cancellation within the integrand, and the Corollary then follows.

Remarks. Of course we have from (1.2) $\Delta^{(n+1)}(0) = (-1)^n(n+1)!\zeta(n+2)/(n+2)$, in agreement with the expansion (2.27). This special case is recovered from Proposition 3 in the following way. We have ${}_2F_1(a, b; c; 0) = 1$ and the representation [16] (p. 14)

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s \int_1^{\infty} \frac{P_1(x)}{x^{s+1}} dx, \quad (2.23)$$

the $a=1$ case of (2.18), giving the identity $\int_0^1 u^{n+1} \zeta(n+2, 1) du = \zeta(n+2)/(n+2)$.

In connection with Propositions 2 and 3, another representation that might be employed is [7]

$$\ln \Gamma(x+1) = \left(x + \frac{1}{2}\right) \ln x - x + \frac{1}{2} \ln 2\pi - \int_0^\infty \frac{P_1(t)}{t+x} dt. \quad (2.24)$$

We may note that representations (2.19) or (2.20), for instance, provide another basis for proving integral representations for $(-1)^n \Delta^{(n+1)}(x)$ by induction. When using (2.20), we use the derivative property

$$\frac{d}{dx} {}_2F_1(a, b; c; -x) = -\frac{ab}{c} {}_2F_1(a+1, b+1; c+1; -x). \quad (2.25)$$

The ${}_2F_1$ function in (1.6) can be written in other ways, including using a transformation formula [9] (p. 1043), so that

$${}_2F_1\left(1, n+2; n+3; \frac{x}{x+1}\right) = (x+1) {}_2F_1(1, 2; n+3; -x). \quad (2.26)$$

Proposition 4. *The result uses the expansion [9] (p. 939)*

$$\ln \Gamma(x+1) = -\gamma x + \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \zeta(k) x^k. \quad (2.27)$$

Remark. Let $\text{Ei}(x)$ be the exponential integral function (e.g., [9], p. 925). Given the relations [9] (pp. 927, 942)

$$\Gamma(0, x) = -\text{Ei}(-x) = -\left(\gamma + \ln x + \sum_{k=1}^{\infty} \frac{(-x)^k}{kk!}\right), \quad (2.28)$$

it is possible to write

$$\int_0^1 \Delta(x) dx = -\gamma - \int_0^\infty \frac{[\gamma - t + \Gamma(0, t) + \ln t]}{t(e^t - 1)} dt. \quad (2.29)$$

This follows by inserting a standard integral representation for the values $\zeta(k)$ into the right side of (1.8a).

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Appendix

Here we present illustrative relations for the sort of hypergeometric functions appearing in (1.6) and (2.20).

The contiguous relations [9] (pp. 1044–45) may be readily applied. As well, we have for instance [9] (p. 1043)

$${}_2F_1(n+2, n+2; n+3; -x) = (1+x)^{-(n+1)} {}_2F_1(1, 1; n+3; -x). \quad (\text{A.1})$$

The next result provides a type of recurrence relation in the first parameter of the ${}_2F_1$ function.

Proposition A1. *For integers $n \geq -1$ we have*

$$\begin{aligned} \int_0^1 \frac{u^{n+1}}{(xu+1)^{n+2}} du &= \frac{1}{(n+2)} {}_2F_1(n+2, n+2; n+3; -x) \\ &= \frac{1}{(n+1)} \left[\frac{1}{(x+1)^{n+1}} - \frac{1}{(n+2)} {}_2F_1(n+1, n+2; n+3; -x) \right]. \end{aligned} \quad (\text{A.2})$$

PROOF. With $v = xu$ in (A.2), we have

$$\begin{aligned} \frac{1}{x^{n+2}} \int_0^x \frac{v^{n+1}}{(v+1)^{n+2}} dv &= \frac{1}{x^{n+2}} \int_0^x [(v+1) - v] \frac{v^{n+1}}{(v+1)^{n+2}} dv \\ &= \frac{1}{x^{n+2}} \left[\int_0^x \frac{v^{n+1}}{(v+1)^{n+1}} dv - \int_0^x \frac{v^{n+2}}{(v+1)^{n+2}} dv \right] \\ &= \frac{1}{x^{n+2}} \left[\int_0^x \frac{v^{n+1}}{(v+1)^{n+1}} dv - \frac{(n+2)}{(n+1)} \int_0^x \frac{v^{n+1}}{(v+1)^{n+1}} dv + \frac{1}{(n+1)} \frac{x^{n+2}}{(x+1)^{n+1}} \right], \end{aligned} \quad (\text{A.3})$$

where we integrated by parts. Using a standard integral representation for ${}_2F_1$ [9] (p. 1040) leads to the Proposition. $\square$

Proposition A1 may be iterated in the first parameter of the ${}_2F_1$ function. Then the following relation may be applied:

$$\int_0^1 \frac{u^{n+1}}{(xu+1)^2} du = \frac{1}{1+x} - \left(\frac{n+1}{n+2} \right) {}_2F_1(1, n+2; n+3; -x). \quad (\text{A.4})$$

The ${}_2F_1$ functions of concern here may be written with one or more terms containing $\ln(x+1)$. One way to see this is the following. We have for the function of (A.4), first integrating by parts,

$$\int_0^1 \frac{u^{n+1}}{(xu+1)^2} du = -\frac{1}{x} \left[(n+1) \int_0^1 \frac{u^n}{xu+1} du + \frac{1}{x+1} \right]$$

$$\begin{aligned}
 &= -\frac{1}{x} \left[\frac{(n+1)}{x^{n+1}} \int_0^x \frac{v^n}{v+1} dv + \frac{1}{x+1} \right] \\
 &= -\frac{1}{x} \left[\frac{(n+1)}{x^{n+1}} \int_0^x \frac{[1 - (1-v^n)]}{v+1} dv + \frac{1}{x+1} \right] \\
 &= -\frac{1}{x} \left\{ \frac{(n+1)}{x^{n+1}} \left[\ln(x+1) - \int_0^x \frac{(1-v^n)}{v+1} dv \right] + \frac{1}{x+1} \right\}. \quad (\text{A.5})
 \end{aligned}$$

For $0 \leq x \leq 1$ we may note the following simple inequality for the integral of (A.5):

$$\int_0^x \frac{(1-v^n)}{v+1} dv \leq \int_0^x (1-v^n) dv \leq x. \quad (\text{A.6})$$

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