

## Additive irreducibles in $\alpha$ -expansions

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**Abstract.** The Bergman number system uses the base  $\alpha = \frac{1+\sqrt{5}}{2}$ , the digits 0 and 1, and the condition that adjacent ones are forbidden. We are interested in those positive integers such that replacing one or more of the ones never results again in a positive integer; they are called (additively) irreducible. These numbers are characterised in terms of the positions of their ones. Further, the number of irreducible positive integers below a given bound is considered and evaluated asymptotically, as the bound goes to infinity. The periodic function that appears is analysed in detail.

### 1. Introduction

Let  $\alpha = \frac{1+\sqrt{5}}{2}$  be the golden ratio. It is well known that each natural number  $n$  has a unique  $\alpha$ -expansion

$$n = \sum_{k \in \mathbb{Z}} \varepsilon_k \alpha^k \quad (1)$$

with the digits  $\varepsilon_k \in \{0, 1\}$ , such that no nonzero digits are adjacent to each other, and only a finite number of digits is different from 0 (cf. [1], [4]). It is sometimes called Bergman's number system.

If one considers a standard  $q$ -ary expansion

$$\sum_{k \geq 0} \varepsilon_k q^k \quad (2)$$

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of a natural number, with digits  $\varepsilon_k$  in the set  $\{0, \dots, q-1\}$  then one still gets a natural number if one replaces some of the digits by zero.

We are interested in positive integers

$$I = \sum_{k=-K}^L \varepsilon_k \alpha^k,$$

such that all the numbers

$$\sum_{k=-K'}^{L'} \varepsilon_k \alpha^k \text{ for } K' < K \text{ and } L' < L$$

are no integers (if they are not zero). The integers  $I$  with this property can be seen as the additive building blocks of the integers in the Bergman-representation.

Positive integers that cannot be decomposed any further in the above sense will be called *additive irreducibles*, or  $\alpha$ -irreducibles for short. It is easy to see that the *decomposition* into  $\alpha$ -irreducibles is unique. Here is a short list of the first few natural numbers, decomposed into additive irreducibles:

$$\begin{aligned} 1, 2, 3, 4 &= 1 + 3, 5, 6, 7, 8 = 1 + 7, 9 = 2 + 7, 10 = 3 + 7, 11 = 4 + 7, 12, \\ 13, 14, 15 &= 1 + 14, 16, 17, 18, 19 = 1 + 18, 20 = 2 + 18, 21 = 3 + 18, \\ 22 = 4 + 18, 23 &= 5 + 18, 24 = 6 + 18, 25 = 7 + 18, 26 = 1 + 7 + 18, \dots, \\ 7.000.000 &= 16 + 767 + 267.872 + 1.860.498 + 4.870.847, \dots \end{aligned}$$

The original motivation for studying these integers was that every positive integer  $n$  can be decomposed uniquely into a sum of  $I$ s;  $n = I_1 + \dots + I_k$ . The sum-of-digits function in Bergman-representation of integers

$$s_B \left( \sum_{\ell=-K}^L \varepsilon_\ell \alpha^\ell \right) = \sum_{\ell=-K}^L \varepsilon_\ell$$

is additive with respect to this decomposition

$$s_B(I_1 + \dots + I_k) = s_B(I_1) + \dots + s_B(I_k).$$

The sum-of-digits function  $s_B$  has been studied in [2].

In this paper, we will prove the following theorems, a characterisation of the  $\alpha$ -irreducibles  $m$  in terms of the  $\alpha$ -expansion, and an *irreducible number (counting) theorem*, giving an asymptotic formula for the number of  $\alpha$ -irreducibles below a given number  $n$ .

**Theorem 1.** *An integer  $m > 1$  with  $\alpha$ -expansion  $m = \alpha^{e_1} + \dots + \alpha^{e_\ell} + \alpha^{f_1} + \dots + \alpha^{f_h}$  with  $e_1 > \dots > e_\ell \geq 1$ , and  $0 > f_1 > \dots > f_h$  is an  $\alpha$ -irreducible, iff  $e_1, \dots, e_{\ell-1}$  are odd. Equivalently,  $m$  is an  $\alpha$ -irreducible, iff  $f_1, \dots, f_{h-1}$  are odd and  $f_h$  is even.*

(In words, the positive exponents must all be odd, with the possible exception of the smallest one. Since  $m = 1$  is  $\alpha$ -irreducible by definition,  $\alpha^0$  cannot occur in the Bergman-expansion of an  $\alpha$ -irreducible.)

**Theorem 2.** *The number  $A(n)$  of  $\alpha$ -irreducibles among the numbers  $1, \dots, n$  satisfies for  $n \rightarrow \infty$  an asymptotic formula*

$$A(n) = \Phi\left(\frac{1}{2} \log_\alpha n\right) n^\rho + \mathcal{O}(\log n),$$

with  $\rho = \frac{\log 2}{2 \log \alpha} = 0.72021 \dots$  and  $\Phi$  a continuous periodic function of period 1.

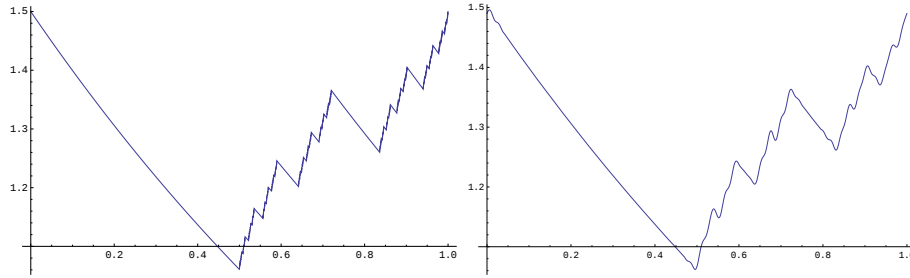


Figure 1. Plot of the function  $\Phi$  compared to the first 50 terms of its Fourier series.

Fibo-Primes-Fourier.eps

## 2. Proofs

**Lemma 1.** *Let  $\varepsilon_{-K}, \dots, \varepsilon_L \in \{0, 1\}$  be a sequence of digits satisfying the condition that no two consecutive digits are both 1 and assume that*

$$n = \sum_{\ell=-K}^L \varepsilon_\ell \alpha^\ell \in \mathbb{N} \quad (\text{with } \varepsilon_{-K} = \varepsilon_L = 1).$$

*Then  $K = 2\lceil \frac{L}{2} \rceil$  holds.*

PROOF. Since  $n$  is an integer, it is not changed by applying the conjugation  $\alpha \mapsto -\frac{1}{\alpha}$  in the ring  $\mathbb{Z}[\alpha]$ . This gives

$$n = \sum_{\ell=-K}^L \varepsilon_\ell (-\alpha)^{-\ell} = \sum_{\ell=-L}^K (-1)^\ell \varepsilon_\ell \alpha^\ell.$$

The highest occurring power in this sum has to be even, since otherwise the number represented by the sum would be negative. For even  $K$  we have

$$\alpha^K - \alpha^{K-3} - \alpha^{K-5} - \dots = \alpha^{K-1} < n < \alpha^K + \alpha^{K-2} + \dots = \alpha^{K+1}.$$

On the other hand  $\alpha^L < n < \alpha^{L+1}$ , which gives the inequalities  $L < K+1$  and  $L+1 > K-1$ . Thus we have  $L \leq K \leq L+1$ , which together with the fact that  $K$  is even implies  $K = 2\lceil \frac{L}{2} \rceil$ .  $\square$

The following Lemma explains how the positive powers in the expansion of an integer determine the negative powers.

**Lemma 2.** *Let  $\varepsilon_0, \dots, \varepsilon_L \in \{0, 1\}$  be a sequence of digits satisfying the condition that no two consecutive digits are both 1. Let  $0 < 2\ell_1 < 2\ell_2 < \dots < 2\ell_m$  be the strictly positive even indices of non-zero digits (set  $m = 0$ , if there are none) and set  $\ell_0 = 0$  and  $\ell_{m+1} = \infty$ . Furthermore, let  $2\ell_j < 2k_j + 1 < 2\ell_{j+1}$  ( $0 \leq j \leq m$ ) be the largest odd index of a non-zero digit between the two consecutive even indices  $2\ell_j$  and  $2\ell_{j+1}$ . Set  $k_j = \ell_j - 1$ , if all digits  $\varepsilon_{2\ell_j+1}, \dots, \varepsilon_{2\ell_{j+1}-1}$  are zero.*

*Then*

$$\left\lceil \sum_{\ell=0}^L \varepsilon_\ell \alpha^\ell \right\rceil = \sum_{\ell=0}^L \varepsilon_\ell \alpha^\ell + \sum_{j=0}^m \left( \sum_{\ell=\ell_j}^{k_j} (1 - \varepsilon_{2\ell+1}) \alpha^{-2\ell-1} + \alpha^{-2k_j-2} \right) - \begin{cases} 1 & \text{for } k_0 = -1, \\ 0 & \text{otherwise,} \end{cases} \quad (3)$$

where the sum in the first parenthesis is 0 if  $k_0 = -1$ .

PROOF. We first notice that the sum over the negative powers of  $\alpha$  is an admissible expansion (no two consecutive digits are 1) and therefore less than 1. Let  $L_k = \alpha^k + (-\alpha)^{-k}$  denote the  $k$ -th Lucas-number. Then we have

$$\varepsilon_{2\ell_j} \alpha^{2\ell_j} + \sum_{\ell=\ell_j}^{k_j} \varepsilon_{2\ell+1} \alpha^{2\ell+1} + \sum_{\ell=\ell_j}^{k_j} (1 - \varepsilon_{2\ell+1}) \alpha^{-2\ell-1} + \alpha^{-2k_j-2}$$

$$\begin{aligned}
&= \varepsilon_{2\ell_j} \alpha^{2\ell_j} + \sum_{\ell=\ell_j}^{k_j} \varepsilon_{2\ell+1} (\alpha^{2\ell+1} - \alpha^{-2\ell-1}) + \sum_{\ell=\ell_j}^{k_j} \alpha^{-2\ell-1} + \alpha^{-2k_j-2} \\
&= \varepsilon_{2\ell_j} L_{2\ell_j} + \sum_{\ell=\ell_j}^{k_j} \varepsilon_{2\ell+1} L_{2\ell+1} \in \mathbb{N}
\end{aligned}$$

for  $j > 0$ ; a slight modification is necessary for  $j = 0$ . Thus the number in (3) is an integer which is at most 1 larger than  $\sum_{\ell=0}^L \varepsilon_\ell \alpha^\ell$ , which proves the lemma.  $\square$

PROOF OF THEOREM 1. The decomposition of an integer given in Lemma 2 shows that the only possibility for an integer to be additively irreducible is that there is only one summand in the decomposition (3). Notice that by Lemma 1 every expansion of an integer has to end with the digit 1 in an even position. Thus the summands in (3) cannot be decomposed any further.

The only possibilities to have only one summand in the decomposition (3) is to either have  $m = 0$  and therefore no non-zero digit in an even position, or  $m = 1$  and  $k_0 = -1$ . The first possibility is the case when  $e_1, \dots, e_\ell$  are all odd, the second possibility is the case when  $e_1, \dots, e_{\ell-1}$  are odd and  $e_\ell$  is even.  $\square$

PROOF OF THEOREM 2. Let

$$A' = \left\{ \sum_{\ell=0}^L \varepsilon_{2\ell+1} \alpha^{2\ell+1} \mid \forall \ell : \varepsilon_{2\ell+1} \in \{0, 1\} \right\}.$$

Then define the set  $A''$  by

$$A'' = \bigcup_{k=1}^{\infty} \alpha^{2k} (\alpha^2 A' + 1) \cup \{1\}, \quad (4)$$

where the union is disjoint. Furthermore, we define  $A = (A' \cup A'') \setminus \{0\}$ . The set of  $\alpha$ -irreducibles is then formed by completing  $A$  with appropriate digits after the comma by Theorem 1 and Lemma 2.

We now count the number of elements of  $A$  less than

$$X = \sum_{\ell=0}^{2L+1} \varepsilon_\ell \alpha^\ell, \quad (5)$$

where we assume that either  $\varepsilon_{2L+1} = 1$  or  $\varepsilon_{2L} = 1$ . We denote

$$X_k = \sum_{\ell=k}^{2L+1} \varepsilon_\ell \alpha^\ell$$

and define

$$M = \max \{ \ell \geq 1 \mid \varepsilon_{2\ell} = 1 \}$$

(set  $M = 0$  if the set is empty). Then we have

$$\# \{a \in A \mid a < X\} = \sum_{k=2M}^{2L+1} \# \{a \in A \mid X_{k+1} \leq a < X_k\}.$$

The condition  $X_{k+1} \leq a < X_k$  implies that

$$a = \sum_{\ell=0}^k \delta_{\ell} \alpha^{\ell} + \sum_{\ell=k+1}^{2L+1} \varepsilon_{\ell} \alpha^{\ell}$$

and

$$\sum_{\ell=0}^k \delta_{\ell} \alpha^{\ell} < \varepsilon_k \alpha^k.$$

Thus  $a \in A$  can only hold if the digits  $\varepsilon_{2L+1}, \dots, \varepsilon_{k+1}$  have the property that all non-zero digits have odd index, except possibly for the index  $k+1$ . This explains why the sum starts with  $k = 2M$ .

For  $k \geq 2M$  all non-zero digits in  $\varepsilon_{2L+1}, \dots, \varepsilon_{k+1}$  have odd index; then the digits  $\delta_k, \dots, \delta_0$  can be chosen so that all non-zero digits have odd index, except possibly for the index of the last non-zero digit. For  $k \geq 1$  the number of possible choices for  $\delta_k, \dots, \delta_0$  is then

$$\varepsilon_k \left( \sum_{\ell=1}^{\lfloor \frac{k}{2} \rfloor - 1} 2^{\frac{k}{2} - \ell - 1} + 2^{\lfloor \frac{k}{2} \rfloor} \right) = \varepsilon_k \left( 3 \cdot 2^{\lfloor \frac{k}{2} \rfloor - 1} - 1 \right),$$

where each summand in the sum corresponds to those choices of digits, which end with a non-zero digit at the even position  $2\ell$ , whereas the second term counts the choices which only have non-zero digits in odd positions. For  $k = 0$  the number of possible choices is  $\varepsilon_0$ ; this can only occur, if  $M = 0$ .

For

$$x = \sum_{k=1}^{\infty} \varepsilon_k \alpha^{-k} \in [\alpha^{-2}, 1]$$

define

$$p(x) = \min \{k \mid \varepsilon_k = 1 \text{ and } k \text{ even}\}$$

and

$$\phi(x) = \sum_{k=1}^{p(x)} \varepsilon_k 2^{-\lfloor \frac{k+1}{2} \rfloor}.$$

Then the number of  $\alpha$ -irreducibles less than  $X = \alpha^{2L+2}y$  (with  $y \in [\alpha^{-2}, 1]$ ) equals

$$\sum_{k=1}^{\min\{p(y), 2L\}} \varepsilon_{2L+1-k} \left( 3 \cdot 2^{L-\lfloor \frac{k}{2} \rfloor - 1} - 1 \right) + \varepsilon_{2L+1} = 2^L 3 \phi(y) + \mathcal{O}(L).$$

Defining

$$\Phi(t) = 2^{-t} 3 \phi(\alpha^{2t-2})$$

for  $0 \leq t \leq 1$  and writing

$$\begin{aligned} \#\{a \in A \mid a < X\} &= 2^L 3 \phi(X \alpha^{-2L-2}) + \mathcal{O}(L) \\ &= X^\rho (X \alpha^{-2L-2})^{-\rho} 3 \phi(X \alpha^{-2L-2}) + \mathcal{O}(L) \\ &= X^\rho \Phi\left(\frac{1}{2} \log_\alpha X - L\right) + \mathcal{O}(\log X) \end{aligned}$$

gives the number of (truncated) irreducibles.

The function  $\phi$  is continuous on the interval  $[\alpha^{-2}, 1]$ : we only have to check that two different representations of a number  $x$  give the same value for  $\phi(x)$ . For that let

$$x = \sum_{\ell=1}^L \varepsilon_\ell \alpha^{-\ell} = \sum_{\ell=1}^{L-1} \varepsilon_\ell \alpha^{-\ell} + \sum_{k=1}^{\infty} \alpha^{-(L+2k-1)},$$

where  $\varepsilon_L = 1$ . Then we have

$p(x) < L$ :  $\phi(x) = \sum_{\ell=1}^{p(x)} \varepsilon_\ell 2^{-\lfloor \frac{\ell+1}{2} \rfloor}$ , since the digits with index  $< L$  are the same for both representations

$p(x) = L$ : in this case we have

$$\phi(x) = \sum_{\ell=1}^L \varepsilon_\ell 2^{-\lfloor \frac{\ell+1}{2} \rfloor} = \sum_{\ell=1}^{L-1} \varepsilon_\ell 2^{-\lfloor \frac{\ell+1}{2} \rfloor} + \sum_{k=1}^{\infty} 2^{-\lfloor \frac{L+2k}{2} \rfloor},$$

since the last sum equals  $2^{-\lfloor \frac{L}{2} \rfloor} = 2^{-\lfloor \frac{L+1}{2} \rfloor}$ , because  $L$  is even.

$p(x) > L$ : in this case we have

$$\phi(x) = \sum_{\ell=1}^L \varepsilon_\ell 2^{-\lfloor \frac{\ell+1}{2} \rfloor} = \sum_{\ell=1}^{L-1} \varepsilon_\ell 2^{-\lfloor \frac{\ell+1}{2} \rfloor} + 2^{-\lfloor \frac{L+1}{2} \rfloor},$$

because  $L+1$  is even.

Let  $x, y$  be two real numbers with  $\alpha^{-k-1} \leq |x - y| < \alpha^{-k}$ . Then either the digital expansions of  $x$  and  $y$  to base  $\alpha$  agree up to the  $k$ -th digit, which yields  $|\phi(x) - \phi(y)| < 2^{-\frac{k}{2}}$ ; or the two numbers have less than  $k$  digits in common. In this case there exists a number  $\xi$  between  $x$  and  $y$  which has two different representations to base  $\alpha$ , which have  $k$  digits in common with  $x$  and  $y$  respectively. Then

$$|\phi(x) - \phi(y)| \leq |\phi(x) - \phi(\xi)| + |\phi(\xi) - \phi(y)| \leq 2^{-\frac{k}{2}+1}$$

and we have

$$|\phi(x) - \phi(y)| \leq 2|x - y|^\rho. \quad (6)$$

We now have for  $N = \lceil X \rceil$  and  $X$  as in (5)

$$\begin{aligned} A(N) &= \#\{n < N \mid n \text{ irreducible}\} = \#\{a \in A \mid a < X\} + \mathcal{O}(1) \\ &= X^\rho \Phi\left(\frac{1}{2} \log_\alpha X\right) + \mathcal{O}(\log X) = (N + (X - N))^\rho \Phi\left(\frac{1}{2} \log_\alpha N\right) \\ &\quad + \mathcal{O}\left(N^\rho \cdot \left|\Phi\left(\frac{1}{2} \log_\alpha N\right) - \Phi\left(\frac{1}{2} \log_\alpha X\right)\right|\right) + \mathcal{O}(\log N) \\ &= N^\rho \Phi\left(\frac{1}{2} \log_\alpha N\right) + \mathcal{O}(\log N), \end{aligned} \quad (7)$$

where we have used (6) to bound  $|\Phi(\frac{1}{2} \log_\alpha N) - \Phi(\frac{1}{2} \log_\alpha X)|$  by  $\mathcal{O}(N^{-\rho})$ . This proves the theorem.  $\square$

### 3. Dirichlet series and Fourier coefficients

We will explain now how to compute the Fourier coefficients of  $\Phi$  numerically to high precision. For this purpose we study the Dirichlet series

$$\zeta_A(s) = \sum_{a \in A} a^{-s}, \quad \zeta_{A'}(s) = \sum_{\substack{a \in A' \\ a \neq 0}} a^{-s}, \quad \text{and} \quad \zeta_{A''}(s) = \sum_{a \in A''} a^{-s}. \quad (8)$$

Clearly, we have  $\zeta_A(s) = \zeta_{A'}(s) + \zeta_{A''}(s)$  and

$$\begin{aligned} \zeta_{A''}(s) &= \sum_{k=1}^{\infty} \alpha^{-2ks} \left(1 + \sum_{\substack{a \in A' \\ a \neq 0}} (1 + \alpha^2 a)^{-s}\right) \\ &= \frac{1}{\alpha^{2s} - 1} \left(1 + \alpha^{-2s} \zeta_{A'}(s) + \alpha^{-2s} \sum_{\ell=1}^{\infty} \binom{-s}{\ell} \alpha^{-2\ell} \zeta_{A'}(s + \ell)\right) \end{aligned} \quad (9)$$

by (4). Furthermore, by the decomposition

$$A' \setminus \{0\} = \alpha^2(A' \setminus \{0\}) \uplus (\alpha + \alpha^2(A' \setminus \{0\})) \uplus \{\alpha\}$$

we have

$$\begin{aligned} \zeta_{A'}(s) &= \alpha^{-2s} \zeta_{A'}(s) + \sum_{\substack{a \in A' \\ a \neq 0}} (\alpha + \alpha^2 a)^{-s} + \alpha^{-s} \\ &= 2\alpha^{-2s} \zeta_{A'}(s) + \alpha^{-s} + \alpha^{-2s} \sum_{\ell=1}^{\infty} \binom{-s}{\ell} \alpha^{-\ell} \zeta_{A'}(s + \ell). \end{aligned}$$

This gives the functional equation for  $\zeta_{A'}(s)$

$$\zeta_{A'}(s) = \frac{1}{\alpha^{2s} - 2} \left( \alpha^s + \sum_{\ell=1}^{\infty} \binom{-s}{\ell} \alpha^{-\ell} \zeta_{A'}(s + \ell) \right). \quad (10)$$

This functional equation provides the analytic continuation of  $\zeta_{A'}(s)$  to the left of  $\Re s = \rho$ . It is similar in its form to the trivial functional equation of the Riemann zeta function and also in the way it is derived.

This functional equation can be used to compute values of  $\zeta_{A'}(s)$  numerically to high precision by the observation that for large  $\ell$ ,  $|\zeta_{A'}(s + \ell) - \alpha^{-s-\ell}| = \mathcal{O}(\alpha^{-3\ell})$  (with an explicit  $\mathcal{O}$ -constant). Thus the series in (10) is geometrically convergent and the error can be controlled. Starting with  $\zeta_{A'}(s + k)$  with  $k$  large enough to meet the prescribed error bound, we can use the functional equation to compute  $\zeta_{A'}(s + k - 1), \dots, \zeta_{A'}(s)$  by rapidly convergent series, where we can control the error bounds. For a detailed description in a similar instance we refer to [5].

From (10) we read off that  $\zeta_{A'}(s)$  has simple poles at the points  $s = \rho + \frac{k\pi i}{\log \alpha}$  with residues

$$\operatorname{Res}_{s=\rho+\frac{k\pi i}{\log \alpha}} \zeta_{A'}(s) = \frac{1}{4 \log \alpha} \left( (-1)^k \sqrt{2} + \sum_{\ell=1}^{\infty} \binom{-\rho - \frac{k\pi i}{\log \alpha}}{\ell} \alpha^{-\ell} \zeta_{A'}\left(\rho + \ell + \frac{k\pi i}{\log \alpha}\right) \right),$$

which can be computed numerically by the procedure described above. By (9)  $\zeta_{A''}(s)$  has the same poles with residues

$$\operatorname{Res}_{s=\rho+\frac{k\pi i}{\log \alpha}} \zeta_{A''}(s) = \frac{1}{2} \operatorname{Res}_{s=\rho+\frac{k\pi i}{\log \alpha}} \zeta_{A'}(s).$$

We now introduce the Dirichlet generating function of all irreducible elements:

$$\eta(s) = \sum_{\substack{n \in \mathbb{N} \\ n \text{ irreducible}}} n^{-s}.$$

Since the irreducible integers differ at most by 1 from the elements of  $A$  by Lemma 2, this function has the same poles and residues as  $\zeta_A(s)$ .

The counting function of irreducible elements can now be expressed by the Mellin–Perron formula from classical analytic number theory (cf. [6]):

$$B(N) = \sum_{\substack{n < N \\ n \text{ irreducible}}} \left(1 - \frac{n}{N}\right) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \eta(s) N^s \frac{ds}{s(s+1)}.$$

Shifting the line of integration to  $\Re s = \rho - \varepsilon$ , where the integral is still convergent, we obtain by residue calculus

$$\begin{aligned} B(N) &= N^\rho \sum_{k \in \mathbb{Z}} \frac{1}{(\rho + \frac{k\pi i}{\log \alpha})(\rho + 1 + \frac{k\pi i}{\log \alpha})} \operatorname{Res}_{s=\rho+\frac{k\pi i}{\log \alpha}} \zeta_A(s) e^{k\pi i \log_\alpha N} + \mathcal{O}(N^{\rho-\varepsilon}) \\ &= N^\rho \Psi\left(\frac{1}{2} \log_\alpha N\right) + \mathcal{O}(N^{\rho-\varepsilon}) \end{aligned}$$

with a continuous periodic function  $\Psi$ . For more details on the application of this technique to digital functions we refer to [3], [5].

The counting functions  $A(N)$  defined in (7) and  $B(N)$  are related by partial (Abel) summation:

$$B(N) = \frac{1}{N} \sum_{n \leq N} A(n). \quad (11)$$

The periodic function  $\Phi$  in Theorem 2 is Hölder continuous with exponent  $\rho > \frac{1}{2}$  by (6). Therefore its Fourier series is absolutely convergent by Bernstein’s theorem (cf. [7]). Thus the Fourier coefficients of  $\Phi$  can be related to those of  $\Psi$  by (11)

$$\widehat{\Phi}(k) = \left(\rho + 1 + \frac{k\pi i}{\log \alpha}\right) \widehat{\Psi}(k) = \frac{1}{\rho + \frac{k\pi i}{\log \alpha}} \operatorname{Res}_{s=\rho+\frac{k\pi i}{\log \alpha}} \zeta_A(s)$$

by [3] or [5, Proposition 5].

We have used the numerical method for the computation of  $\zeta_A(s)$  as described above to compute the first Fourier coefficients

$$\begin{aligned} \widehat{\Phi}(0) &= 1.2805518195386460216946037033568563347660801437933 \dots, \\ \widehat{\Phi}(1) &= 0.072680149457538598589977133015533730235053618517778 \dots \\ &\quad + 0.010398381210825974099196428902369009784854020134648 \dots i, \\ \widehat{\Phi}(2) &= -0.001564808148052238888496969303490183402915232587181 \dots \\ &\quad - 0.017356976045800856810743168654268946461607279485855 \dots i. \end{aligned}$$

The plot of the first 50 terms of the Fourier series compared to the function  $\Phi$  is shown in Figure 1.

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### References

- [1] G. M. BERGMAN, A number system with an irrational base, *Math. Mag.* **31** (1957/58), 98–110.
- [2] C. COOPER and R. E. KENNEDY, The first moment of the number of 1's function in the  $\beta$ -expansion of the positive integers, *J. Inst. Math. Comput. Sci. Math. Ser.* **14** (2001), 69–77.
- [3] P. FLAJOLET, P. J. GRABNER, P. KIRSCHENHOFER, H. PRODINGER and R. F. TICHY, Mellin transforms and asymptotics: digital sums, *Theor. Comput. Sci.* **123** (1994), 291–314.
- [4] C. FROUGNY, How to write integers in a non-integer base, Vol. 583, LATIN 92, São Paulo, Lecture Notes in Computer Science, *Springer, Berlin*, 1992.
- [5] P. J. GRABNER and H.-K. HWANG, Digital sums and divide-and-conquer recurrences: Fourier expansions and absolute convergence, *Constr. Approx.* **21** (2005), 149–179.
- [6] G. TENENBAUM, Introduction to analytic and probabilistic number theory, Cambridge Studies in Advanced Mathematics, no. 46, *Cambridge University Press*, 1995.
- [7] A. ZYGMUND, Trigonometric Series, Vol. I, II, *Cambridge University Press, Cambridge*, 1988.

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