

Super-paracompactness and continuous sections

By DAVID BUHAGIAR (Msida) and VALENTIN GUTEV (Durban)

Abstract. It is demonstrated that a space X is super-paracompact if and only if for every completely metrizable Y , every l.s.c. mapping from X into the nonempty closed subsets of Y has a compact-valued continuous section.

1. Introduction

For a space Y , we will use 2^Y to denote the *power set* of Y , i.e. the set of all subsets of Y . Also, we will use $\mathcal{F}(Y)$ to denote the set of all nonempty closed subsets of Y , and $\mathcal{C}(Y)$ – that of all compact members of $\mathcal{F}(Y)$. For a set-valued mapping $\varphi : X \rightarrow 2^Y$ and $B \subset Y$, let $\varphi^{-1}[B] = \{x \in X : \varphi(x) \cap B \neq \emptyset\}$. The mapping φ is *lower semi-continuous*, or l.s.c., if the set $\varphi^{-1}[U]$ is open in X for every open $U \subset Y$. The mapping φ is *upper semi-continuous*, or u.s.c., if the set

$$\varphi^\# [U] = X \setminus \varphi^{-1}[Y \setminus U] = \{x \in X : \varphi(x) \subset U\}$$

is open in X for every open $U \subset Y$. For convenience, we say that φ is *usco* if it is u.s.c. and nonempty-compact-valued, and that φ is *continuous* if it is both l.s.c. and u.s.c.

A mapping $\varphi : X \rightarrow 2^Y$ is a *multi-selection* (or, a *set-valued selection*) for $\Phi : X \rightarrow 2^Y$ if $\varphi(x) \subset \Phi(x)$ for every $x \in X$; and $\varphi : X \rightarrow 2^Y$ is a *section* for $\Phi : X \rightarrow 2^Y$ if $\varphi(x) \cap \Phi(x) \neq \emptyset$ for every $x \in X$. If φ is a section for Φ , then both φ and Φ must be nonempty-valued. Of course, every nonempty-valued multi-selection for Φ is also a section for Φ .

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There is a natural relationship between covering properties and multi-selections. Namely, such familiar properties of topological spaces as paracompactness, metacompactness, collectionwise normality, etc., were transformed and thus essentially generalised in terms of multi-selections for l.s.c. mappings in completely metrizable spaces, see, for instance, [4], [5], [6], [8], [10], [12], [14], [17]. The present paper deals with a similar characterisation of another covering property, but now in terms of sections. Let $\mathcal{W}$ be a collection of subsets of a set X . If $U, V \in \mathcal{W}$, then a finite sequence $W_1, W_2, \dots, W_k$ of elements of $\mathcal{W}$ is called a *chain* from U to V if $U = W_1$, $V = W_k$ and $W_i \cap W_{i+1} \neq \emptyset$ for every $i = 1, \dots, k-1$. A subset $\mathcal{P} \subset \mathcal{W}$ is called *connected* if every pair of elements of $\mathcal{P}$ is connected by a chain. The components of $\mathcal{W}$ are defined as the maximal connected subsets of $\mathcal{W}$. A space X is called *super-paracompact* (Pasynkov, see [13]) if every open cover of X has an open finite component (i.e., having finite components) refinement. The purpose of this paper is to prove the following theorem.

Theorem 1.1. *A space X is super-paracompact if and only if for every completely metrizable space Y , every l.s.c. mapping $\Phi : X \rightarrow \mathcal{F}(Y)$ has a continuous section $\varphi : X \rightarrow \mathcal{C}(Y)$.*

Theorem 1.1 can be compared with [7, Proposition 1.1] that a regular space X is paracompact if and only if for every completely metrizable Y , every l.s.c. mapping $\Phi : X \rightarrow \mathcal{F}(Y)$ has an usco section $\psi : X \rightarrow \mathcal{C}(Y)$.

A word should be said also about the paper itself. Theorem 1.1 is proved in Section 3; the preparation for this proof is done in the next section. The technique developed to prove Theorem 1.1 allows to generalise it for a non-metrizable range, see Theorem 3.2.

2. Completeness and special sieves

A partially ordered set $(T, \preceq)$ is a *tree* if $\{s \in T : s \preceq t\}$ is well-ordered for every $t \in T$. For a tree $(T, \preceq)$, we use $T(0)$ to denote the set of the minimal elements of T . Given an ordinal α , if $T(\beta)$ is defined for every $\beta < \alpha$, then $T(\alpha)$ denotes the minimal elements of $T \setminus (T \restriction \alpha)$, where $T \restriction \alpha = \bigcup \{T(\beta) : \beta < \alpha\}$. The set $T(\alpha)$ is called the α^{th} -level of T , while the *height* of T is the least ordinal α such that $T \restriction \alpha = T$. We say that $(T, \preceq)$ is an α -tree if its height is α . A maximal linearly ordered subset of a tree $(T, \preceq)$ is called a *branch*, and $\mathcal{B}(T)$ is used to denote the set of all branches of T . A tree $(T, \preceq)$ is *pruned* if every element of T has a successor in T , i.e. if for every $s \in T$ there exists $t \in T$, with $s \prec t$. In these

terms, an ω -tree $(T, \preceq)$ is pruned if each branch $\beta \in \mathcal{B}(T)$ is infinite. Following NYIKOS [15], for every $t \in T$, set

$$\mathcal{O}(t) = \{\beta \in \mathcal{B}(T) : t \in \beta\}. \quad (2.1)$$

For a pruned ω -tree $(T, \preceq)$, the family $\{\mathcal{O}(t) : t \in T\}$ is a base for a completely metrizable non-Archimedean topology on $\mathcal{B}(T)$. We will refer to this topology on $\mathcal{B}(T)$ as the *branch topology*, and to the resulting topological space as the *branch space*. It is well known that $\mathcal{B}(T)$ is compact if and only if all levels of T are finite.

For a tree $(T, \preceq)$ and $t \in T$, the *node* of t in T is the subset $\text{node}(t) \subset T$ of all immediate successors of t . For convenience, let $\text{node}(\emptyset) = T(0)$. Finally, for a mapping $\Psi : Z \rightarrow 2^Y$ and $A \subset Z$, let

$$\Psi[A] = \bigcup \{\Psi(z) : z \in A\}.$$

Given a set Y and a pruned ω -tree $(T, \preceq)$, a set-valued mapping $\mathcal{S} : T \rightarrow 2^Y$ is a *sieve* on Y if

- (i) $Y = \mathcal{S}[\text{node}(\emptyset)]$, and
- (ii) $\mathcal{S}(t) = \mathcal{S}[\text{node}(t)]$ for every $t \in T$.

A sieve $\mathcal{S} : T \rightarrow 2^Y$ on a space Y is *complete* [3], [11] if for every branch $\beta \in \mathcal{B}(T)$ and every nonempty centred (i.e., with the finite intersection property) family $\mathcal{F} \subset 2^Y$ which refines $\{\mathcal{S}(t) : t \in \beta\}$ it follows that $\bigcap \{\overline{F} : F \in \mathcal{F}\} \neq \emptyset$. In other words, a sieve $\mathcal{S} : T \rightarrow 2^Y$ on Y is complete if each family $\{\mathcal{S}(t) : t \in \beta\}$, $\beta \in \mathcal{B}(T)$, is a *compact* filter base (i.e., each ultrafilter containing it is convergent) [16].

For a tree $(T, \preceq)$ and $\mathcal{S} : T \rightarrow 2^Y$, the *polar mapping* $\Omega_{\mathcal{S}} : \mathcal{B}(T) \rightarrow 2^Y$, associated to $\mathcal{S}$, is defined by $\Omega_{\mathcal{S}}(\beta) = \bigcap \{\mathcal{S}(t) : t \in \beta\}$, $\beta \in \mathcal{B}(T)$. Also, to the mapping $\mathcal{S} : T \rightarrow 2^Y$ we associate the pointwise-closure $\overline{\mathcal{S}} : T \rightarrow 2^Y$ of $\mathcal{S}$ by $\overline{\mathcal{S}}(t) = \overline{\mathcal{S}(t)}$, $t \in T$. If $\mathcal{S} : T \rightarrow 2^Y$ is a nonempty-valued complete sieve on a space Y , then for every branch $\beta \in \mathcal{B}(T)$, the polar $\Omega_{\overline{\mathcal{S}}}(\beta)$ is a nonempty compact subset of Y , and every open $V \supset \Omega_{\overline{\mathcal{S}}}(\beta)$ contains some $\overline{\mathcal{S}}(t)$ for $t \in \beta$, see, e.g., [3, Proposition 2.10]. In terms of set-valued mappings, this means that the polar mapping $\Omega_{\overline{\mathcal{S}}} : \mathcal{B}(T) \rightarrow \mathcal{C}(Y)$ is usco. In this section, we show that every completely metrizable space Y has a special complete sieve $\mathcal{S} : T \rightarrow 2^Y$ such that the polar mapping $\Omega_{\overline{\mathcal{S}}}$ is continuous. To this end, recall that a sieve $\mathcal{S} : T \rightarrow 2^Y$ is *finitely-additive* if each collection $\{\mathcal{S}(t) : t \in T(n)\}$, $n < \omega$, as well as each collection of the form $\{\mathcal{S}(s) : s \in \text{node}(t)\}$, $t \in T$, is closed under finite unions.

Lemma 2.1. *Every completely metrizable space Y has a nonempty-open-valued, finitely-additive and complete sieve $\mathcal{S} : T \rightarrow 2^Y$ such that the polar mapping $\Omega_{\mathcal{S}} : \mathcal{B}(T) \rightarrow \mathcal{C}(Y)$ is continuous.*

PROOF. Let d be a complete metric on Y compatible with the topology of Y , and let $\mathcal{R} : D \rightarrow 2^Y$ be a nonempty-open-valued sieve on Y with $\text{diam}_d(\mathcal{R}(s)) < 2^{-n}$ for every $s \in D(n)$ and $n < \omega$. According to Cantor's intersection theorem, $\mathcal{R}$ is a complete sieve on Y . In fact, each $\Omega_{\mathcal{R}}(\delta)$, $\delta \in \mathcal{B}(D)$, is a singleton, and hence the polar mapping $\Omega_{\mathcal{R}} : \mathcal{B}(D) \rightarrow \mathcal{C}(Y)$ is singleton-valued and usco (thus, continuous as well). Keeping this in mind, let Σ_D be the set of all nonempty finite subsets of D . By [11, Lemma 2.3], there is a finitely-additive sieve $\mathcal{S} : T \rightarrow 2^Y$ on Y generated by the sieve $\mathcal{R}$, where $T \subset \Sigma_D$ and for each $\sigma \in T$, the value of $\mathcal{S}$ in σ is $\mathcal{S}(\sigma) = \mathcal{R}[\sigma] = \bigcup \{\mathcal{R}(s) : s \in \sigma\}$. The order on T is defined in a natural way so that each branch $\beta = \{\sigma_n : n < \omega\} \in \mathcal{B}(T)$ corresponds to a pruned subtree $\bigcup \beta = \bigcup \{\sigma_n : n < \omega\}$ of D such that $\sigma_n \subset D(n)$, for every $n < \omega$, see the proof of [11, Lemma 2.3]. In particular, if $\beta \in \mathcal{B}(T)$, then $\mathcal{B}(\bigcup \beta) \subset \mathcal{B}(D)$ and, in fact,

$$\Omega_{\mathcal{S}}(\beta) = \Omega_{\mathcal{R}} \left[\mathcal{B} \left(\bigcup \beta \right) \right]. \quad (2.2)$$

Indeed, the inclusion $\Omega_{\mathcal{R}}[\mathcal{B}(\bigcup \beta)] \subset \Omega_{\mathcal{S}}(\beta)$ is obvious. For the converse, take a point $y \in \Omega_{\mathcal{S}}(\beta)$ and let $K(y) = \{s \in \bigcup \beta : y \in \overline{\mathcal{R}(s)}\}$. Because every sieve is order-preserving with respect to the inverse inclusion, $K(y)$ is a subtree of $\bigcup \beta$ such that each $K(y) \cap D(n)$, $n < \omega$, is nonempty and finite. Hence, by König's lemma (see Lemma 5.7 in Chapter II of [9]), $K(y)$ contains an infinite branch

$$\delta \in \mathcal{B}(K(y)) \subset \mathcal{B} \left(\bigcup \beta \right) \subset \mathcal{B}(D).$$

Therefore, $y \in \Omega_{\mathcal{R}}(\delta) \subset \Omega_{\mathcal{R}}[\mathcal{B}(\bigcup \beta)]$.

We are now ready to show that the sieve $\mathcal{S}$ is as required. By [11, Lemma 2.3], $\mathcal{S}$ remains nonempty-open-valued and complete, hence the polar mapping $\Omega_{\mathcal{S}} : \mathcal{B}(T) \rightarrow \mathcal{C}(Y)$ is usco. To show that $\Omega_{\mathcal{S}}$ is also l.s.c., take an open set $U \subset Y$ and a branch $\beta \in \mathcal{B}(T)$ such that $\Omega_{\mathcal{S}}(\beta) \cap U \neq \emptyset$. According to (2.2), there is a branch $\delta \in \mathcal{B}(\bigcup \beta) \subset \mathcal{B}(D)$ with $\Omega_{\mathcal{R}}(\delta) \cap U \neq \emptyset$. Since $\Omega_{\mathcal{R}}(\delta)$ is a singleton, we have that $\Omega_{\mathcal{R}}(\delta) \subset U$, and, by the completeness of $\mathcal{R}$, we also have that $\overline{\mathcal{R}(s)} \subset U$ for some $s \in \delta$. Then, $s \in \sigma_s$ for some $\sigma_s \in \beta$, and the neighbourhood $\mathcal{O}(\sigma_s)$ of β in $\mathcal{B}(T)$ is such that

$$\emptyset \neq \Omega_{\mathcal{R}}(\eta) \subset \Omega_{\mathcal{S}}(\gamma) \cap \overline{\mathcal{R}(s)} \subset U$$

for every $\gamma \in \mathcal{O}(\sigma_s)$ and every $\eta \in \mathcal{B}(\bigcup \gamma)$ with $s \in \eta$, see (2.1). The proof is completed. $\square$

3. Proof of Theorem 1.1

Let X be a super-paracompact space, Y be a completely metrizable space, and let $\Phi : X \rightarrow \mathcal{F}(Y)$ be an l.s.c. mapping. According to Lemma 2.1, Y has a nonempty-open-valued, finitely-additive and complete sieve $\mathcal{S} : T \rightarrow 2^Y$ such that the polar mapping $\Omega_{\mathcal{S}} : \mathcal{B}(T) \rightarrow \mathcal{C}(Y)$ is continuous. Consider the composite mapping $\Phi^{-1}[\mathcal{S}(t)]$, $t \in T$, which defines an open-valued finitely-additive sieve $\Phi^{-1} \circ \mathcal{S} : T \rightarrow 2^X$ on X because Φ is l.s.c. Since $\{\Phi^{-1}[\mathcal{S}(t)] : t \in T(0)\}$ is a finitely-additive open cover of X and X is super-paracompact, by [1, Proposition 2.3] (see, also, [2, Theorem 2.2]), X has a pairwise disjoint open cover $\{\mathcal{L}(t) : t \in T(0)\}$ such that $\mathcal{L}(t) \subset \Phi^{-1}[\mathcal{S}(t)]$, $t \in T(0)$. Take an element $s \in T(0)$. Then, $\mathcal{L}(s)$ is itself super-paracompact (being a clopen subset of X), while $\{\Phi^{-1}[\mathcal{S}(t)] : t \in \text{node}(s)\}$ is a finitely-additive open cover of $\mathcal{L}(s)$. Hence, just like before, $\mathcal{L}(s)$ has a pairwise disjoint open cover $\{\mathcal{L}(t) : t \in \text{node}(s)\}$ such that $\mathcal{L}(t) \subset \Phi^{-1}[\mathcal{S}(t)]$, $t \in \text{node}(s)$. Proceeding by induction on the levels of the tree T , there exists a clopen-valued sieve $\mathcal{L} : T \rightarrow 2^X$ on X such that each family $\{\mathcal{L}(t) : t \in T(n)\}$, $n < \omega$, is discrete and $\mathcal{L}(t) \subset \Phi^{-1}[\mathcal{S}(t)]$, $t \in T$. Consider now the mapping $\mathcal{U}_{\mathcal{L}} : X \rightarrow 2^{\mathcal{B}(T)}$ defined by $\mathcal{U}_{\mathcal{L}}(x) = \Omega_{\mathcal{L}}^{-1}[\{x\}]$, $x \in X$. By [5, Proposition 5.2 and Lemma 5.3], $\mathcal{U}_{\mathcal{L}} : X \rightarrow \mathcal{C}(\mathcal{B}(T))$ and is continuous. Finally, define $\varphi : X \rightarrow \mathcal{C}(Y)$ by $\varphi = \Omega_{\mathcal{S}} \circ \mathcal{U}_{\mathcal{L}}$. By Lemma 2.1, φ is continuous as a composition of continuous set-valued mappings, while, by [5, Lemma 7.1], φ is also a section for Φ .

To show the converse, suppose that X has the section property in Theorem 1.1. Take an open cover $\mathcal{V}$ of X , and let $\mathcal{W}$ be the cover of X consisting of all finite unions of elements of $\mathcal{V}$. Endow $\mathcal{V}$ with the discrete topology, and define a mapping $\Phi : X \rightarrow \mathcal{F}(\mathcal{V})$ by $\Phi(x) = \{V \in \mathcal{V} : x \in V\}$, $x \in X$. Since Φ is l.s.c., by assumption, it has a continuous section $\varphi : X \rightarrow \mathcal{C}(\mathcal{V})$. Then, $\mathcal{F} = \{\varphi(x) : x \in X\}$ is a family of nonempty finite subsets of $\mathcal{V}$. Set

$$U_F = \{x \in X : \varphi(x) = F\}, \quad F \in \mathcal{F}.$$

Since φ is continuous, U_F is a clopen subset of X , and clearly $\{U_F : F \in \mathcal{F}\}$ is a pairwise disjoint cover of X . Finally, observe that $W_F = \Phi^{-1}[F] \in \mathcal{W}$ for every $F \in \mathcal{F}$ because $\mathcal{W}$ consists of finite union of elements of $\mathcal{V}$. If $x \in U_F$ for some $F \in \mathcal{F}$, then $\varphi(x) = F$ and $\varphi(x) \cap \Phi(x) \neq \emptyset$. Hence, it follows that $\Phi(x) \cap F \neq \emptyset$ and, therefore, $x \in W_F$. Thus, $\{U_F : F \in \mathcal{F}\}$ is a refinement of $\{W_F : F \in \mathcal{F}\}$ and, by [1, Proposition 2.3] (see, also, [2, Theorem 2.2]), X is super-paracompact.

Remark 3.1. The metrizability of Y in Theorem 1.1 was used only in terms

of Lemma 2.1, while Lemma 2.1 remains valid as far as Y has a nonempty-open-valued sieve $\mathcal{S} : D \rightarrow 2^Y$ for which the polar mapping $\Omega_{\mathcal{S}} : \mathcal{B}(D) \rightarrow 2^Y$ is singleton-valued and continuous. Spaces with this property were said to have a λ -base [3], they are also known as *monotonically developable sieve complete* spaces. Monotonically developable spaces are a natural generalisation of Moore spaces, hence not necessarily metrizable. This gives the following generalisation of Theorem 1.1 to the case of a non-metrizable range.

Theorem 3.2. *If X is super-paracompact and Y is monotonically developable and sieve complete, then every l.s.c. mapping $\Phi : X \rightarrow \mathcal{F}(Y)$ has a continuous section $\varphi : X \rightarrow \mathcal{C}(Y)$.*

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DAVID BUHAGIAR
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
UNIVERSITY OF MALTA
MSIDA MSD.06
MALTA

E-mail: david.buhagiar@um.edu.mt

VALENTIN GUTEV
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF KWAZULU-NATAL
WESTVILLE CAMPUS
PRIVATE BAG X54001
DURBAN 4000
SOUTH AFRICA

E-mail: gudev@ukzn.ac.za

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