

Distribution functions of ratio sequences, III

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Dedicated to Professor Kálmán Györy on the occasion of his 70th birthday

Abstract. In this paper we study the distribution functions $g(x)$ of the sequence of blocks $X_n = (\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n})$, $n = 1, 2, \dots$, where x_n is an increasing sequence of positive integers. Assuming that the lower asymptotic density $\underline{d}$ of x_n is positive, we find the optimal lower and upper bounds of $g(x)$. As an application, we also get the optimal bounds of limit points of $\frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n}$, $n = 1, 2, \dots$

1. Introduction

Let $x_n, n = 1, 2, \dots$, be an increasing sequence of positive integers (by “increasing” we mean strictly increasing). The double sequence $x_m/x_n, m, n = 1, 2, \dots$ is called *the ratio sequence* of x_n , which has been introduced by T. ŠALÁT [Sa]. He studied its everywhere density. For further study of ratio sequences, O. STRAUCH and J. T. TÓTH [ST] introduced the sequence X_n of blocks

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n} \right), \quad n = 1, 2, \dots$$

and they studied the uniform distribution of X_n in the sense of the monographs [KN] and [DT]. The authors in [ST] further studied the distribution functions

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$g(x)$ of X_n . The motivation for this is that the existence of strictly increasing $g(x)$ implies everywhere density of x_m/x_n , which is the primary problem of ŠALÁT in [Sa].

In what follows, we will use the following definitions, and basic properties, see [SP, p. 1–28, 1.8.23].

- Denote by $F(X_n, x)$ the step distribution function

$$F(X_n, x) = \frac{\#\{i \leq n; \frac{x_i}{x_n} < x\}}{n},$$

for $x \in [0, 1)$, and $F(X_n, 1) = 1$. Directly from definition we have

$$F(X_m, x) = \frac{n}{m} F\left(X_n, x \frac{x_m}{x_n}\right) \quad (1)$$

for all pairs of integers $m < n$, and every $x \in [0, 1)$.

- For any increasing sequence of positive integers x_n , $n = 1, 2, \dots$, we define a counting function $A(t)$ as

$$A(t) = \#\{n \in \mathbb{N}; x_n < t\}.$$

Then for every $x \in (0, 1]$ we have the equality

$$\frac{nF(X_n, x)}{xx_n} = \frac{A(xx_n)}{xx_n}, \quad (2)$$

which we shall use to compute the asymptotic density of x_n . Here, the lower asymptotic density $\underline{d}$, and the upper asymptotic density $\bar{d}$ of x_n , $n = 1, 2, \dots$ are defined as

$$\underline{d} = \liminf_{t \rightarrow \infty} \frac{A(t)}{t} = \liminf_{n \rightarrow \infty} \frac{n}{x_n}, \quad \bar{d} = \limsup_{t \rightarrow \infty} \frac{A(t)}{t} = \limsup_{n \rightarrow \infty} \frac{n}{x_n}.$$

- A non-decreasing function $g : [0, 1] \rightarrow [0, 1]$, $g(0) = 0$, $g(1) = 1$ is called a distribution function. We shall identify any two distribution functions coinciding at common points of continuity.
- A distribution function $g(x)$ is a distribution function of the sequence of blocks X_n , $n = 1, 2, \dots$, if there exists an increasing sequence $n_1, n_2, \dots$ of positive integers such that

$$\lim_{k \rightarrow \infty} F(X_{n_k}, x) = g(x)$$

almost everywhere in $[0, 1]$. This is equivalent to weak convergence; it means that the preceding limit holds for every point $x \in [0, 1]$ of continuity of $g(x)$.

In this paper we frequently use the following two theorems of Helly (see First and Second Helly theorem [SP, Theorem 4.1.0.10 and Theorem 4.1.0.11, p. 4–5]).

- *Helly's selection principle*: For any sequence $g_n(x)$, $n = 1, 2, \dots$, of distribution functions in $[0, 1]$ there exists a subsequence $g_{n_k}(x)$, $k = 1, 2, \dots$, and a distribution function $g(x)$ such that $\lim_{k \rightarrow \infty} g_{n_k}(x) = g(x)$ almost everywhere.
- *Second Helly theorem*: If we have $\lim_{n \rightarrow \infty} g_n(x) = g(x)$ almost everywhere in $[0, 1]$, then for every continuous function $f : [0, 1] \rightarrow \mathbb{R}$ we have $\lim_{n \rightarrow \infty} \int_0^1 f(x) dg_n(x) = \int_0^1 f(x) dg(x)$.
- Note that applying Helly's selection principle, from the sequence $F(X_n, x)$, $n = 1, 2, \dots$, one can select a subsequence $F(X_{n_k}, x)$, $k = 1, 2, \dots$, such that $\lim_{k \rightarrow \infty} F(X_{n_k}, x) = g(x)$ holds not only for the continuity points x of $g(x)$, but also for all $x \in [0, 1]$.
- Denote by $G(X_n)$ the set of all distribution functions of X_n , $n = 1, 2, \dots$. For a singleton $G(X_n) = \{g(x)\}$, the distribution function $g(x)$ is also called asymptotic distribution function of X_n .
- We will use the one-step distribution function $c_\alpha(x)$ with the step 1 at α defined on $[0, 1]$ via

$$c_\alpha(x) = \begin{cases} 0, & \text{if } x \leq \alpha; \\ 1, & \text{if } x > \alpha, \end{cases} \quad (3)$$

while always $c_\alpha(0) = 0$ and $c_\alpha(1) = 1$.

- The lower distribution function $\underline{g}(x)$, and the upper distribution function $\overline{g}(x)$ of a sequence x_n , $n = 1, 2, \dots$ are defined as

$$\underline{g}(x) = \inf_{g \in G(X_n)} g(x), \quad \overline{g}(x) = \sup_{g \in G(X_n)} g(x).$$

In Section 2 of this paper we recall some known theorems, which we shall use and extend. In Section 3 (Theorem 5) we solve Open problem no. 7 from [SN, 1.9. Block sequence] stating that every sequence of blocks X_n has a distribution function $g(x)$ such that $g(x) \geq x$ for all $x \in [0, 1]$. Then, assuming $\underline{d} > 0$, we find (Theorem 6) boundaries $h_1(x) \leq g(x) \leq h_2(x)$, which hold for every distribution function $g(x) \in G(X_n)$, and which are, in a sense, optimal. As a consequence, we produce boundaries (Theorem 7) for $\frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n}$. In the last Section 4 (Example 3), we find the exact values of the $\liminf$ and $\limsup$ of $\frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n}$ for integers x_n from the intervals $(\gamma a^k, \delta a^k)$, $k = 1, 2, \dots$.

2. Basic known results

For an increasing sequence x_n , $n = 1, 2, \dots$ of positive integers the following theorems are known.

Theorem 1 ([ST, Theorem 7.1]). *For every sequence of positive integers x_n there exists $g(x) \in G(X_n)$ such that*

$$\int_0^1 g(x) dx \geq \frac{1}{2}. \quad (4)$$

Theorem 2 ([ST, Theorem 6.2 (ii),(iii)]). *If $\underline{d} > 0$, then there exists $g \in G(X_n)$ such that $g(x) \geq x$ for every $x \in [0, 1]$. Furthermore, for every $g(x) \in G(X_n)$, and $x \in [0, 1]$ we have*

$$x \frac{\underline{d}}{d} \leq g(x) \leq x \frac{\overline{d}}{\underline{d}}.$$

Theorem 3 ([ST, Propozicion 6.1]). *Assume for a sequence n_k , $k = 1, 2, \dots$ that*

- (i) $\lim_{k \rightarrow \infty} F(X_{n_k}, x) = g(x)$,
- (ii) $\lim_{k \rightarrow \infty} \frac{n_k}{x_{n_k}} = d_g$.

Then there exists

- (iii) $\lim_{k \rightarrow \infty} \frac{A(xx_{n_k})}{xx_{n_k}} = d_g(x)$ and

$$\frac{g(x)}{x} d_g = d_g(x). \quad (5)$$

Here the limits (i), and (iii) can be considered for all $x \in (0, 1]$, or all continuity points $x \in (0, 1]$ of $g(x)$.

Theorem 4 ([ST, Theorem 4.1, Theorem 6.2]). *Assume that every distribution function in $G(X_n)$ is continuous at 1. Then each distribution function in $G(X_n)$ is continuous in $(0, 1]$, i.e. the only point of discontinuity is possibly 0. Furthermore, if $\underline{d} > 0$, then all distribution functions in $G(X_n)$ are continuous in $[0, 1]$.*

3. Main results

We start with an extension of Theorem 1, and the first part of Theorem 2.

Theorem 5. *For every increasing sequence of positive integers x_n , $n = 1, 2, \dots$, there exists $g(x) \in G(X_n)$ such that $g(x) \geq x$ for all $x \in [0, 1]$.*

PROOF. If $\underline{d} > 0$, select n_k so that $\frac{n_k}{x_{n_k}} \rightarrow \underline{d} > 0$, and $F(X_{n_k}, x) \rightarrow g(x)$. For such $g(x)$, (5) implies

$$\frac{g(x)}{x} \underline{d} \geq \underline{d}.$$

Now, let $\underline{d} = 0$. Select n_k such that

$$\frac{n_k}{x_{n_k}} = \min_{i \leq n_k} \frac{i}{x_i},$$

and $F(X_{n_k}, x) \rightarrow g(x)$. Then for every $x \in (0, 1]$,

$$\frac{A(xx_{n_k})}{xx_{n_k}} \geq \frac{n_k - 1}{x_{n_k}}.$$

Applying (2) yields

$$\frac{F(X_{n_k}, x)}{x} \frac{n_k}{x_{n_k}} \geq \frac{n_k - 1}{x_{n_k}},$$

and taking the limit, as $k \rightarrow \infty$, we obtain $g(x) \geq x$ for all $x \in [0, 1]$. $\square$

Now we are going to study in more detail the second part of Theorem 2.

Theorem 6. Let $x_1 < x_2 < \dots$ be a sequence of positive integers with positive lower asymptotic density $\underline{d} > 0$ and upper asymptotic density $\bar{d}$. Then all distribution functions $g(x) \in G(X_n)$ are continuous, non-singular and bounded by $h_1(x) \leq g(x) \leq h_2(x)$, where

$$h_1(x) = \begin{cases} x \frac{\underline{d}}{\bar{d}} & \text{if } x \in \left[0, \frac{1 - \bar{d}}{1 - \underline{d}}\right]; \\ \frac{\underline{d}}{\frac{1}{x} - (1 - \underline{d})} & \text{otherwise,} \end{cases} \quad (6)$$

$$h_2(x) = \min \left(x \frac{\bar{d}}{\underline{d}}, 1 \right). \quad (7)$$

Moreover, $h_1(x)$ and $h_2(x)$ are the best possible in the following sense: for given $0 < \underline{d} \leq \bar{d}$, there exists $x_1 < x_2 < \dots$ with lower and upper asymptotic densities $\underline{d}$, $\bar{d}$, such that $\underline{g}(x) = h_1(x)$ for $x \in [\frac{1 - \bar{d}}{1 - \underline{d}}, 1]$; also, there exists $x_1 < x_2 < \dots$ with given $0 < \underline{d} \leq \bar{d}$ such that $\bar{g}(x) = h_2(x) \in G(X_n)$.

PROOF. For $g(x) \in G(X_n)$, let $n_k, k = 1, 2, \dots$, be an increasing sequence of indices such that $F(X_{n_k}, x) \rightarrow g(x)$. From n_k we can select a subsequence (for simplicity written as the original n_k)¹ such that

$$\frac{n_k}{x_{n_k}} \rightarrow d_g > 0. \quad (8)$$

¹We call d_g a local asymptotic density related to $g(x)$.

Then, by (5), we have

$$g(x) = x \frac{d_g(x)}{d_g}, \quad \text{where} \quad \frac{A(xx_{n_k})}{xx_{n_k}} \rightarrow d_g(x) \quad (9)$$

for arbitrary $x \in (0, 1]$.

We will continue in six steps 1^0 – 6^0 .

1^0 . We prove the continuity of $g(x)$ at $x = 1$ (improving (iv) in [ST, Theorem 6.2]) for each $g(x) \in G(X_n)$.

In view of the definition of the counting function $A(t)$

$$0 \leq A(x_{n_k}) - A(xx_{n_k}) \leq x_{n_k} - xx_{n_k};$$

thus,

$$0 \leq \frac{A(x_{n_k})}{x_{n_k}} - \frac{A(xx_{n_k})}{xx_{n_k}} = \frac{n_k - 1}{x_{n_k}} - \frac{A(xx_{n_k})}{xx_{n_k}} x \leq 1 - x,$$

and, as $k \rightarrow \infty$, we have $0 \leq d_g - d_g(x)x \leq 1 - x$, which implies

$$0 \leq d_g - d_g(x) + d_g(x)(1 - x) \leq 1 - x.$$

Consequently, $\lim_{x \rightarrow 1} d_g(x) = d_g$, and so $\lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} x \frac{d_g(x)}{d_g} = 1$. Since $g(x) \in G(X_n)$ is arbitrary, Theorem 4 gives continuity of $g(x)$ in the whole unit interval $[0, 1]$.

2^0 . We prove that $g(x)$ has a bounded right derivative for every $x \in (0, 1)$, and for each $g(x) \in G(X_n)$.

For $0 < x < y < 1$ again

$$0 \leq A(yx_{n_k}) - A(xx_{n_k}) \leq (y - x)x_{n_k},$$

which implies

$$0 \leq \frac{A(yx_{n_k})}{yx_{n_k}} y - \frac{A(xx_{n_k})}{xx_{n_k}} x \leq y - x.$$

Letting $k \rightarrow \infty$, we get $0 \leq d_g(y)y - d_g(x)x \leq y - x$, hence

$$0 \leq g(y) - g(x) = \frac{d_g(y)y - d_g(x)x}{d_g} \leq \frac{y - x}{d_g}.$$

Consequently,

$$0 \leq \frac{g(y) - g(x)}{y - x} \leq \frac{1}{d_g} \quad (10)$$

for all $x, y \in (0, 1)$, $x < y$, which gives the upper bound of the right derivatives of $g(x)$ for every $x \in (0, 1)$. Note that a singular distribution function (continuous, strictly increasing, having zero derivative almost everywhere) has infinite right Dini derivatives in a dense subset of $(0, 1)$.

3⁰. We prove a local form of Theorem 5.

As $\underline{d} \leq d_g \leq \bar{d}$, (9) implies

$$x \frac{\underline{d}}{d_g} \leq g(x) \leq x \frac{\bar{d}}{d_g} \quad (11)$$

for every $x \in [0, 1]$. It follows from (10), that there exists an extreme point $A_g = (x_g, y_g)$ on the line $y = x \frac{\underline{d}}{d_g}$ such that $g(x)$ has no common point with this line for $x > x_g$. This point A_g is the intersection of the lines

$$y = x \frac{\underline{d}}{d_g} \text{ and } y = x \frac{1}{d_g} + 1 - \frac{1}{d_g} \quad (12)$$

therefore,

$$A_g = (x_g, y_g) = \left(\frac{1 - d_g}{1 - \underline{d}}, \frac{\underline{d}}{d_g} \frac{1 - d_g}{1 - \underline{d}} \right). \quad (13)$$

It means that for a given $g(x) \in G(X_n)$, $h_{1,g}(x) \leq g(x) \leq h_{2,g}(x)$, where

$$h_{1,g}(x) = \begin{cases} x \frac{\underline{d}}{d_g} & \text{if } x < y_0 = \frac{1 - d_g}{1 - \underline{d}}; \\ x \frac{1}{d_g} + 1 - \frac{1}{d_g} & \text{if } y_0 \leq x \leq 1, \end{cases} \quad (14)$$

$$h_{2,g}(x) = \min \left(x \frac{\bar{d}}{d_g}, 1 \right). \quad (15)$$

4⁰. Now we find $h_1(x)$, and $h_2(x)$ such that

$$h_1(x) \leq h_{1,g}(x) \leq h_{2,g}(x) \leq h_2(x)$$

for every $g \in G(X_n)$.

In the parametric expression (13) of A_g , the local asymptotic density d_g defined by (8) belongs to the interval $[\underline{d}, \bar{d}]$. The well-known Darboux property of the asymptotic density implies that for an arbitrary $d \in [\underline{d}, \bar{d}]$ there exists an increasing n_k , $k = 1, 2, \dots$, such that $\frac{n_k}{x_{n_k}} \rightarrow d$ ², and then the Helly selection

²A simple proof follows from the fact that for every $d \in (\underline{d}, \bar{d})$ there exist infinitely many $n \in \mathbb{N}$ such that $A(n)/n \leq d \leq A(n+1)/(n+1)$. These n we denote as n_k .

principle implies the existence of a subsequence of n_k such that $F(X_{n_k}, x) \rightarrow g(x)$ for some $g(x) \in G(X_n)$. Thus, if $g(x)$ runs over $G(X_n)$, then d_g runs over the entire interval $[\underline{d}, \overline{d}]$. Substituting $d_g = 1 - x_g(1 - \underline{d})$ in $A_g = (x_g, y_g)$ we get

$$y_g = y_g(x_g) = \frac{\underline{d}}{\frac{1}{x_g} - (1 - \underline{d})},$$

where $x_g = \frac{1-d_g}{1-\underline{d}}$ runs through the interval $I = [\frac{1-\overline{d}}{1-\underline{d}}, 1]$ for $d_g \in [\underline{d}, \overline{d}]$. By putting $x_g = x$, and $y_g = h_1$ we find a part of $h_1(x)$ for $x \in I$ in (6). The remaining part of $h_1(x)$, and also the whole $h_2(x)$, follow from the basic inequality (11), see Figure 1.

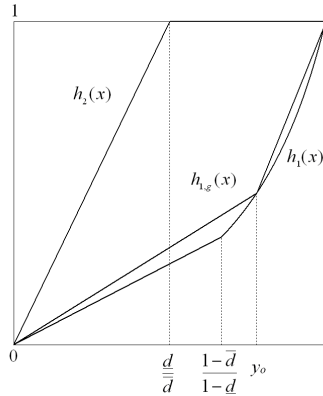


Figure 1. Boundaries of $g(x) \in G(X_n)$

⁵⁰. The optimality of $h_1(x)$ follows from the following example.

The increasing sequence x_n of the integers lying in the intervals

$$(\gamma, \delta), (\gamma a, \delta a), \dots, (\gamma a^k, \delta a^k), \dots,$$

where $1 \leq \gamma < \delta \leq a$, has been used in [ST, pp. 774–777, Example 11.2]. For its lower, and upper asymptotic densities $\underline{d}$, and $\overline{d}$, it has been shown that

$$\underline{d} = \frac{(\delta - \gamma)}{\gamma(a - 1)}, \quad \overline{d} = \frac{(\delta - \gamma)a}{\delta(a - 1)}, \quad (16)$$

and that the graph of every $g \in G(X_n)$ lies in the intervals

$$[1/a, 1] \times [1/a, 1] \cup [1/a^2, 1/a] \times [1/a^2, 1/a] \cup \dots$$

Moreover, the part of the graph in $[1/a^k, 1/a^{k-1}] \times [1/a^k, 1/a^{k-1}]$ is similar to the part of the graph in $[1/a^{k+1}, 1/a^k] \times [1/a^{k+1}, 1/a^k]$ with the scale a . It is also proved in [ST], that the graph of $\underline{g}(x)$ in $[1/a, 1] \times [1/a, 1]$ has the form $\underline{g}(x) = (1 + \frac{1}{\underline{d}}(\frac{1}{x} - 1))^{-1}$ for $x \in [\frac{\gamma}{\underline{d}}, 1]$, and it coincides with the graph of $h_1(x)$ in the interval $I = [\frac{1-\underline{d}}{1-\underline{d}}, 1]$, since $\frac{1-\underline{d}}{1-\underline{d}} = \frac{\gamma}{\underline{d}}$.

6⁰. Finally, we prove the optimality of $h_2(x)$. Before proving it in several substeps, note that in 5⁰ the graph of the upper distribution function $\overline{g}(x)$ in $[1/a, 1] \times [1/a, 1]$ is a straight line which intersects the line $y = 1$ at $x = \frac{\delta}{\gamma a} = \frac{\underline{d}}{\underline{d}}$. Thus, $\overline{g}(\frac{\underline{d}}{\underline{d}}) = h_2(\frac{\underline{d}}{\underline{d}}) = 1$ proving that the point $(\frac{\underline{d}}{\underline{d}}, 1)$ is optimal.

To complete the proof of 6⁰, in the following steps a)- f) we shall construct a sequence of positive integers $x_1 < x_2 < \dots$ with $0 < \underline{d} \leq \overline{d}$ such that $h_2(x) \in G(X_n)$. This implies $h_2(x) = \overline{g}(x)$.

a) The condition $h_2(x) \in G(X_n)$ for a sequence of positive integers $x_1 < x_2 < \dots$ is equivalent to the existence of an increasing sequence of indices n_k such that $F(X_{n_k}, x) \rightarrow h_2(x)$ for $x \in [0, 1]$, and $\frac{n_k}{n_{k+1}} \rightarrow 0$. An application of (1) yields that this is equivalent (see Fig. 2) to the existence of $m'_k < m_k < n_k$ such that the values $x_{m'_k} < x_{m_k} < x_{n_k}$ satisfy

$$(i) \quad \frac{x_{m_k}}{x_{n_k}} \rightarrow \frac{\underline{d}}{\underline{d}},$$

$$(ii) \quad \frac{m_k}{n_k} \rightarrow 1,$$

$$(iii) \quad \frac{x_{m'_k}}{x_{n_k}} \rightarrow 0,$$

$$(iv) \quad \frac{m'_k}{n_k} \rightarrow 0.$$

Moreover, because the sequence of positive integers x_n increases, we have (see Figure 3)

$$(v) \quad x_{m_k} - x_{m'_k} \geq m_k - m'_k,$$

$$(vi) \quad x_{n_k} - x_{m_k} \geq n_k - m_k,$$

$$(vii) \quad x_{m'_{k+1}} - x_{n_k} \geq m'_{k+1} - n_k,$$

$$(viii) \quad n_k < m'_{k+1},$$

$$(ix) \quad m'_1 \leq x_{m'_1}.$$

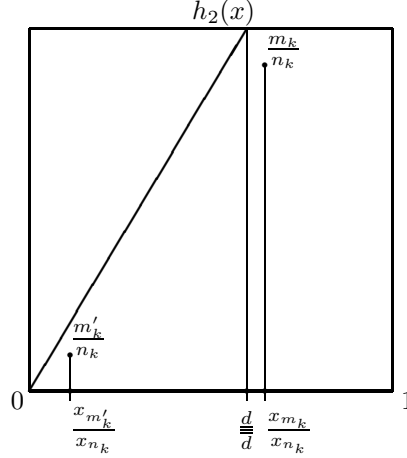


Figure 2: $F(X_{n_k}, x) \rightarrow h_2(x)$, the properties (i)–(iv).

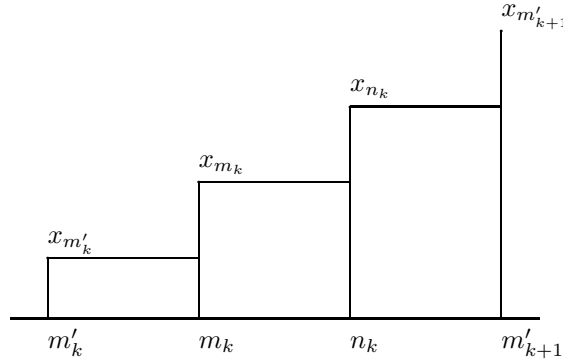


Figure 3: The (v)–(viii) properties.

b) Before solving (i)–(ix) we must capture a role of $\underline{d}$ and $\bar{d}$. By (i) and, (ii) we have the limit

$$\frac{n_k}{m_k} \frac{x_{m_k}}{x_{n_k}} \rightarrow \frac{\underline{d}}{\bar{d}}.$$

Selecting a subsequence of (m_k, n_k) , $k = 1, 2, \dots$, we can assume the existence of the limits $\frac{n_k}{x_{n_k}} \rightarrow d_{h_2}$, and $\frac{m_k}{x_{m_k}} \rightarrow d_g$ (for simplicity, also assume $F(X_{m_k}, x) \rightarrow g(x)$). Then

$$\frac{n_k}{m_k} \frac{x_{m_k}}{x_{n_k}} = \frac{\frac{x_{m_k}}{m_k}}{\frac{x_{n_k}}{n_k}} \rightarrow \frac{\frac{1}{d_g}}{\frac{1}{d_{h_2}}} = \frac{\underline{d}}{\bar{d}},$$

and since

$$\frac{\underline{d}}{\bar{d}} = \min_{d_1, d_2 \in [\underline{d}, \bar{d}]} \frac{d_1}{d_2},$$

we have $d_{h_2} = \underline{d}$, and $d_g = \bar{d}$. This yields the additional conditions

$$(x) \quad \frac{n_k}{x_{n_k}} \rightarrow \underline{d},$$

$$(xi) \quad \frac{m_k}{x_{m_k}} \rightarrow \bar{d}.$$

c) In what follows, we assume $\underline{d} < \bar{d}$, because from Theorem 2, by $0 < \underline{d} = \bar{d}$, we get $G(X_n) = \{x\}$, and also $h_2(x) = x$.

d) To find a sequence x_n which satisfies (i)–(xi), we define x_{n_k} , x_{m_k} , m_k , $x_{m'_k}$, m'_k by using n_k (for a simplifying the definitions, the integer part will be omitted):

$$x_{n_k} = \frac{n_k}{\underline{d}},$$

$$x_{m_k} = x_{n_k} \frac{\underline{d}}{\bar{d}} = \frac{n_k}{\bar{d}},$$

$$m_k = x_{m_k} \bar{d} - o(n_k) = n_k - \sqrt{n_k},$$

$$x_{m'_k} = \sqrt{x_{m_k}} = \sqrt{\frac{n_k}{\bar{d}}},$$

$$m'_k = d' x_{m'_k} = d' \sqrt{\frac{n_k}{\bar{d}}},$$

for some $d' \in (\underline{d}, \bar{d})$.

These $x_{n_k}, x_{m_k}, x_{m'_k}, m_k, m'_k$ satisfy (i)–(vii), (x), (xi). For (viii) we need

$$n_{k+1} > \frac{\bar{d}}{d'^2} \frac{1}{\underline{d}^2} n_k^2$$

for $k = 2, 3, \dots$, and for (ix) the n_1 must be large.

e) For linearity of $h_2(x)$ in $[0, \frac{\underline{d}}{\bar{d}}]$, and to guarantee the asymptotic densities $\underline{d}, \bar{d}$, define

$$(xii) \quad x_n = x_a + (n - a) \frac{x_b - x_a}{b - a} \text{ for } n \in (a, b), \text{ where } (a, b) \text{ coincides successively with } (m'_k, m_k), (m_k, n_k), \text{ or } (n_k, m'_{k+1}).$$

Then

$$\frac{n}{x_n} = \frac{a + (n - a)}{x_a + (n - a) \frac{x_b - x_a}{b - a}},$$

and because the derivative $\left(\frac{Ax+B}{Cx+D}\right)' = \frac{AD-BC}{(Cx+D)^2}$, the minimum and maximum of $\frac{n}{x_n}$ for $n \in (a, b)$ are attained at the endpoints $n = a$, and $n = b$, i.e., for $n = m'_k, m_k, n_k$. Since the limits of $\frac{n_k}{x_{n_k}}, \frac{m_k}{x_{m_k}}, \frac{m'_k}{x_{m'_k}}$ are from $[\underline{d}, \bar{d}]$, and the boundary points are attained, $\liminf_{n \rightarrow \infty} \frac{n}{x_n} = \underline{d}$, and $\limsup_{n \rightarrow \infty} \frac{n}{x_n} = \bar{d}$.

f) For such $x_1 < x_2 < \dots$ we have $\underline{d}, \bar{d}$, and $F(X_{n_k}, x) \rightarrow h_2(x)$ for $x \in [0, 1]$; hence, the proof of Theorem 6 is finished. $\square$

Remark 1. In a sharp contrast to $h_2(x) \in G(X_n)$ in 6⁰ we note that for every sequence of integers $x_1 < x_2 < \dots$, $0 < \underline{d} < \bar{d}$, we have $h_1(x) \notin G(X_n)$, because for every $g(x) \in G(X_n)$, $h_{1,g}(x) \leq g(x) \leq h_{2,g}(x)$, and $h_{1,g}(x) \neq h_1(x)$.

Theorem 6 implies the following best possible boundaries of the sum

$$\frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n}.$$

Theorem 7. For every increasing sequence $x_1 < x_2 < \dots$ of positive integers with $0 < \underline{d} \leq \bar{d}$ we have

$$\frac{1}{2} \frac{\underline{d}}{\bar{d}} \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n}, \quad (17)$$

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} \leq \frac{1}{2} + \frac{1}{2} \left(\frac{1 - \min(\sqrt{\underline{d}}, \bar{d})}{1 - \underline{d}} \right) \left(1 - \frac{\underline{d}}{\min(\sqrt{\underline{d}}, \bar{d})} \right). \quad (18)$$

Here the equality in both (17) and (18)³ can be attained.

PROOF. By the Helly theorem, $F(X_{n_k}, x) \rightarrow g(x)$ forces

$$\int_0^1 x dF(X_{n_k}, x) = \frac{1}{n_k} \sum_{i=1}^{n_k} \frac{x_i}{x_{n_k}} \rightarrow \int_0^1 x dg(x) = 1 - \int_0^1 g(x) dx; \quad (19)$$

thus,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} = 1 - \max_{g \in G(X_n)} \int_0^1 g(x) dx, \quad (20)$$

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} = 1 - \min_{g \in G(X_n)} \int_0^1 g(x) dx. \quad (21)$$

³If $\sqrt{\underline{d}} \leq \bar{d}$ then the right-hand side in (18) is $\frac{1}{1+\sqrt{\underline{d}}}$.

If $\underline{d} > 0$, then by Theorem 6, $h_1(x) \leq g(x) \leq h_2(x)$, which implies

$$1 - \int_0^1 h_2(x) dx \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} \leq 1 - \int_0^1 h_1(x) dx. \quad (22)$$

For $x_1 < x_2 < \dots$ in step 6⁰, where $h_2(x) \in G(X_n)$, we have equality on the left hand side of (22). On the other hand, Remark 1 implies a sharp inequality on the right hand side, therefore,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} < 1 - \frac{1}{2} \frac{\underline{d}}{\bar{d}} \left(\frac{1 - \bar{d}}{1 - \underline{d}} \right)^2 - \frac{\underline{d}}{(1 - \underline{d})^2} \left(\log \frac{\underline{d}}{\bar{d}} - (\bar{d} - \underline{d}) \right) \quad (23)$$

holds for an arbitrary sequence of integers $x_1 < x_2 < \dots$ with $0 < \underline{d} < \bar{d}$.

Applying the inequality $h_{1,g}(x) \leq g(x) \leq h_{2,g}(x)$ for every $g \in G(X_n)$ from step 3⁰ to (19), we obtain

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} \leq \max_{g(x) \in G(X_n)} \left(1 - \int_0^1 h_{1,g}(x) dx \right). \quad (24)$$

If the maximum in (24) is attained for $g_0(x) \in G(X_n)$, and $h_{1,g_0}(x) \in G(X_n)$, then $g_0(x) = h_{1,g_0}(x)$, and

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} = 1 - \int_0^1 h_{1,g_0}(x) dx. \quad (25)$$

Using (14) we get

$$\int_0^1 h_{1,g}(x) dx = \frac{1}{2} \left(1 + \frac{1 - d_g}{1 - \underline{d}} \left(\frac{\underline{d}}{d_g} - 1 \right) \right),$$

and taking derivative with respect to $d_g \in [\underline{d}, \bar{d}]$

$$\left(\int_0^1 h_{1,g}(x) dx \right)' = \frac{1}{2(1 - \underline{d})} \left(1 - \frac{\underline{d}}{(d_g)^2} \right)$$

shows that $\min \int_0^1 h_{1,g}(x) dx$ is attained for $d_{g_0} = \min(\sqrt{\underline{d}}, \bar{d})$.

Now, to prove (25) we shall construct integers $x_1 < x_2 < \dots$ with $0 < \underline{d} \leq \bar{d}$ such that $h_{1,g_0}(x) \in G(X_n)$. We start with the sequence of indices n_k , and then

by (14) we shall find indices $m'_k < m_k < n_k$, and integers $x_{m'_k} < x_{m_k} < x_{n_k}$ such that

- (i) $\frac{n_k}{x_{n_k}} \rightarrow d_{g_0}$,
- (ii) $\frac{m_k}{n_k} \rightarrow \frac{\underline{d}}{d_{g_0}} \frac{1-d_{g_0}}{1-\underline{d}}$,
- (iii) $\frac{x_{m_k}}{x_{n_k}} \rightarrow \frac{1-d_{g_0}}{1-\underline{d}}$,
- (iv) $\frac{x_{m'_k}}{x_{n_k}} \rightarrow 0$,
- (v) $\frac{m'_k}{n_k} \rightarrow 0$,
- (vi) $\frac{m'_k}{x_{m'_k}} \rightarrow \bar{d}$.

Then from (i), (ii), and (iii) it follows that $\frac{m_k}{x_{m_k}} \rightarrow \underline{d}$. Furthermore, assume

- (v) $x_{m_k} - x_{m'_k} \geq m_k - m'_k$,
- (vi) $x_{n_k} - x_{m_k} \geq n_k - m_k$,
- (vii) $x_{m'_{k+1}} - x_{n_k} \geq m'_{k+1} - n_k$,
- (viii) $n_k < m'_{k+1}$,
- (ix) $m'_1 \leq x_{m'_1}$.

For these (i)–(ix) a sequence of integers x_n can be found similarly to 6⁰d). The rest of the terms of x_n define linearly as in e). $\square$

4. Examples

Example 1. a) If $0 < \underline{d} = \bar{d}$, then the bounds in both (17), and (18) equal to $\frac{1}{2}$, which implies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} = \frac{1}{2}.$$

This also follows from the fact that $G(X_n) = \{x\}$, see Theorem 2.

b) If $\underline{d} = \frac{1}{2}$, and $\bar{d} = 1$, then by (23), $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} < 2 - \log 4 < 1$. Using (18) we have an even better estimate $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} \leq 2 - \sqrt{2}$.

Example 2. Omitting $\underline{d} > 0$, we can find a sequence of positive integers $x_1 < x_2 < \dots$ such that $c_0(x), c_1(x) \in G(X_n)$, where $c_0(x), c_1(x)$ are one-steps

distribution functions defined by (3) in the Introduction. In this case

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} = 0 = 1 - \int_0^1 c_0(x) dx,$$

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} = 1 = 1 - \int_0^1 c_1(x) dx.$$

We shall construct such x_n by applying [GS, Theorem 5]. For the index sequences $m'_k < m_k < n'_k < n_k$ we shall find sequences of positive integers $x_{m'_k} < x_{m_k} < x_{n'_k} < x_{n_k}$ such that

- (i) $\frac{n'_k}{n_k} \rightarrow 0$,
- (ii) $\frac{m'_k}{m_k} \rightarrow 1$,
- (iii) $\frac{x_{n'_k}}{x_{n_k}} \rightarrow 1$,
- (iv) $\frac{x_{m'_k}}{x_{m_k}} \rightarrow 0$.

Furthermore,

- (v) $x_{n_k} - x_{n'_k} \geq n_k - n'_k$,
- (vi) $x_{n'_k} - x_{m_k} \geq n'_k - m_k$,
- (vii) $x_{m_k} - x_{m'_k} \geq m_k - m'_k$,
- (viii) $m'_{k+1} > n_k$,
- (ix) $x_{m'_1} \geq m'_1$,
- (x) $x_{m'_{k+1}} - x_{n_k} \geq m'_{k+1} - n_k$.

Then (i)–(x) will be satisfied, if for a given n_k we put $x_{n_k} = n_k^2$, $x_{n'_k} = n_k^2 - n_k$, $x_{m_k} = n_k^2 - 2n_k$, $x_{m'_k} = n_k$, $n'_k = \sqrt{n_k}$, $m_k = \sqrt{n_k} - \sqrt[4]{n_k}$, $m'_k = \sqrt{n_k} - 2\sqrt[4]{n_k}$; further, (viii) holds if $n_{k+1} \geq n_k^4$. For the other n 's in the intervals (m'_k, m_k) , (m_k, n'_k) , (n'_k, n_k) , and (n_k, m'_{k+1}) define x_n linearly.

Now, by (i), and (iii) we have $F(X_{n_k}, x) \rightarrow c_1(x)$, and (ii), (iv) imply $F(X_{m_k}, x) \rightarrow c_0(x)$.

Example 3. In this example we extend a characterization of distribution functions of the sequence $x_1 < x_2 < \dots$ in [ST, Example 11.2]. This sequence was used in the proof of Theorem 6, part 5⁰.

Let x_n , $n = 1, 2, \dots$, be the increasing sequence of all integer points in the sequence of intervals $(\gamma a^k, \delta a^k)$ (in short $a^k(\gamma, \delta)$), $k = 0, 1, 2, \dots$, where $1 \leq \gamma < \delta \leq a$ are real numbers.

It is proved in [ST, Ex. 11.2] that

1⁰. The set of all distribution functions can be expressed in parametric form as $G(X_n) = \{g_t(x); t \in [0, 1]\}$, where

$$F(X_{n_k}, x) \rightarrow g_t(x) \text{ for } n_k \text{ such that } x_{n_k} = [a^k \gamma + t a^k (\delta - \gamma)] \quad (26)$$

2⁰. The function $g_t(x)$ has constant values $g_t(x) = \frac{1}{a^i(1+t(a-1))}$ for $x \in \frac{(\delta, a\gamma)}{a^{i+1}(\gamma+t(\delta-\gamma))}$, $i = 0, 1, 2, \dots$, and in the component intervals it has a constant derivative $g'_t(x) = \frac{(a-1)(\gamma+t(\delta-\gamma))}{(\delta-\gamma)(1+t(a-1))}$ for $x \in \frac{(\gamma, \delta)}{a^{i+1}(\gamma+t(\delta-\gamma))}$, $i = 0, 1, 2, \dots$, and $x \in (\frac{\gamma}{\gamma+t(\delta-\gamma)}, 1)$.⁴

3⁰. The graph of every $g \in G(X_n)$ lies in the intervals

$$[1/a, 1] \times [1/a, 1] \cup [1/a^2, 1/a] \times [1/a^2, 1/a] \cup \dots,$$

and the graph of g in $[1/a^k, 1/a^{k-1}] \times [1/a^k, 1/a^{k-1}]$ is similar to the graph of g in $[1/a^{k+1}, 1/a^k] \times [1/a^{k+1}, 1/a^k]$ with coefficient a .

4⁰. We have $g_0(x) = \bar{g}(x)$, $\underline{g}(x) \notin G(X_n)$, and the asymptotic densities $\underline{d}$, $\bar{d}$ are

$$\underline{d} = \frac{(\delta - \gamma)}{\gamma(a - 1)}, \quad \bar{d} = \frac{(\delta - \gamma)a}{\delta(a - 1)}.$$

We can add the following new properties 5⁰–8⁰:

5⁰. By definition (8) of the local asymptotic density d_g , along with (26) for $g(x) = g_t(x)$ we get

$$\begin{aligned} d_{g_t} &= \lim_{k \rightarrow \infty} \frac{n_k}{x_{n_k}} = \lim_{k \rightarrow \infty} \frac{\sum_{i=0}^{k-1} a^i (\delta - \gamma) + t a^k (\delta - \gamma)}{a^k \gamma + t a^k (\delta - \gamma)} \\ &= \frac{(\delta - \gamma)(1 + t(a - 1))}{(a - 1)(\gamma + t(\delta - \gamma))}, \end{aligned} \quad (27)$$

for $t = 0$, $d_{g_0} = \underline{d}$, for $t = 1$, $d_{g_1} = \bar{d}$, and we have

$$g'_t(x) = \frac{1}{d_{g_t}} \quad (28)$$

for x in intervals where the derivative of $g_t(x)$ is constant.

⁴Here, as above, we write $(xz, yz) = (x, y)z$, and $(x/z, y/z) = (x, y)/z$.

6⁰. For the function $h_{1,g}(x)$ defined in (14), putting $g(x) = g_t(x)$, we have

$$\frac{\underline{d}}{d_{g_t}} = \frac{\gamma + t(\delta - \gamma)}{\gamma(1 + t(a - 1))}, \quad \frac{1 - d_{g_t}}{1 - \underline{d}} = \frac{\gamma}{\gamma + t(\delta - \gamma)}, \quad \frac{\underline{d}}{d_{g_t}} \frac{1 - d_{g_t}}{1 - \underline{d}} = \frac{1}{1 + t(a - 1)}.$$

Then

$$h_{1,g_t}(x) = \begin{cases} g_t(x) = x \frac{1}{d_{g_t}} + 1 - \frac{1}{d_{g_t}}, & \text{for } x \in \left(\frac{\gamma}{\gamma + t(\delta - \gamma)}, 1 \right); \\ g_t(x) = \frac{1}{a^i(1 + t(a - 1))}, & \text{for } x = \frac{\gamma}{a^i(\gamma + t(\delta - \gamma))}, \\ & i = 0, 1, 2, \dots, \end{cases} \quad (29)$$

see Figure 4.

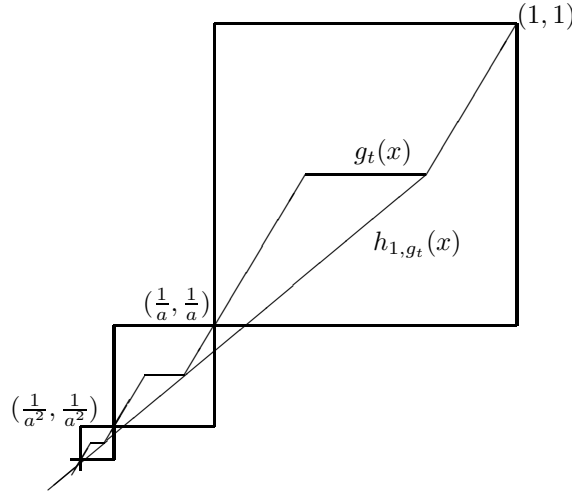


Figure 4: $g_t(x)$ and $h_{1,g_t}(x)$.

7⁰. In the proof of the upper bound (18) we have proved that $1 - \int_0^1 h_{1,g}(x) dx$ is maximal for $d_g = \min(\sqrt{\underline{d}}, \bar{d})$. Let $t_0 \in [0, 1]$ be such that $d_{g_{t_0}} = \min(\sqrt{\underline{d}}, \bar{d})$. This $t = t_0$ we shall find from (27) as

$$t = \frac{d_{g_t}(a - 1)\gamma - (\delta - \gamma)}{(\delta - \gamma)(a - 1)(1 - d_{g_t})}. \quad (30)$$

8⁰. Let $P(t)$ be the area in $[\frac{1}{a}, 1] \times [\frac{1}{a}, 1]$ bounded by the graph of $g_t(x)$. Then

$$\int_0^1 g_t(x) dx = P(t) \frac{1}{1 - \frac{1}{a^2}} + \frac{1}{a + 1} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{(a + 1)} \cdot \frac{(\gamma a - \delta)}{(1 + t(a - 1))(\gamma + t(\delta - \gamma))}$$

$$+ \frac{1}{2} \cdot \frac{t(\delta - \gamma a)}{(1 + t(a - 1))(\gamma + t(\delta - \gamma))}, \quad (31)$$

and, since $g_0(x) = \bar{g}(x)$, $\max_{t \in [0,1]} \int_0^1 g_t(x) dx$ is attained at $t = 0$. Putting $P'(t) = 0$ it follows that $\min_{t \in [0,1]} \int_0^1 g_t(x) dx$ is attained at $t = 1$. This can be derived also from the fact that for $x_{n+1} = x_n + 1$,

$$\begin{aligned} \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{x_i}{x_{n+1}} - \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} \\ = \frac{1}{n+1} - \left(\frac{1}{x_n+1} + \frac{1}{n+1} \cdot \frac{1}{1 + \frac{1}{x_n}} \right) \left(\frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} \right) > 0, \end{aligned}$$

and, because $c_1(x) \notin G(X_n)$, $\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} < 1$. Denoting the index n_k for $x_{n_k} = [a^k \delta]$, the limsup of $\frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n}$ is attained over $n = n_k$, $k = 0, 1, 2, \dots$, and for such n_k (see (26)) we have $F(X_{n_k}, x) \rightarrow g_1(x)$ for $x \in [0, 1]$.

It follows, by (20), and (21) that

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} = 1 - \int_0^1 g_0(x) dx = \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{(a+1)} \left(\frac{\gamma a - \delta}{\gamma} \right), \quad (32)$$

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{x_i}{x_n} = 1 - \int_0^1 g_1(x) dx = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{(a+1)} \left(\frac{\gamma a - \delta}{\delta} \right). \quad (33)$$

The upper bound in (18) coincides with the maximal value of $1 - \int_0^1 h_{1,g}(x) dx$ attained for $d_g = \min(\sqrt{d}, \bar{d})$. Since $1 - \int_0^1 g_1(x) dx$ is maximal for all $1 - \int_0^1 g_t(x) dx$, $t \in [0, 1]$, and $1 - \int_0^1 g_1(x) dx \leq 1 - \int_0^1 h_{1,g_1}(x) dx$, the upper bound (33) satisfies (18).

Using the explicit formulas (16) for asymptotic densities, we see again that (32), and (33) satisfy (17), and (18), respectively, in Theorem 7.

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