

A normality relationship between two families and its applications

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Abstract. Let k be a positive integer, and let $\mathcal{F}$ be a family of meromorphic functions defined in a domain $D \subset \mathbb{C}$, all of whose zeros have multiplicity at least k , and there exists $M > 0$ such that $|f^{(k)}(z)| \leq M$ whenever $f(z) = 0$ for $f \in \mathcal{F}$. If $\mathcal{F}_k = \{f^{(k)} : f \in \mathcal{F}\}$ is normal, then $\mathcal{F}$ is also normal in D . Some applications of this result are given.

1. Introduction

Let D be a domain in $\mathbb{C}$, and $\mathcal{F}$ be a family of meromorphic functions defined on D . $\mathcal{F}$ is said to be normal on D , in the sense of Montel, if for any sequence $\{f_n\} \in \mathcal{F}$ there exists a subsequence $\{f_{n_j}\}$, such that $\{f_{n_j}\}$ converges spherically locally uniformly on D , to a meromorphic function or ∞ (see [6], [9], [12]).

Let k be a positive integer. Consider the family $\mathcal{F}_k$ consisting of k th derivative functions of all $f \in \mathcal{F}$, that is, $\mathcal{F}_k = \{f^{(k)} : f \in \mathcal{F}, z \in D\}$. It is natural to consider the normality relation between these two families. However, the following examples show that there seems no direct relation between $\mathcal{F}$ and $\mathcal{F}_k$.

Example 1. Let $\Delta = \{z : |z| < 1\}$, and $\mathcal{F} = \{f_n(z) = n(z^2 - n^2) : n = 1, 2, \dots\}$. Then $\mathcal{F}_1 = \{f'_n(z) = 2nz : n = 1, 2, \dots\}$. For each $z \in \Delta$,

$$f_n^\#(z) = \frac{|2nz|}{1 + |n(z^2 - n^2)|^2} \leq \frac{2n}{1 + (n^3 - n)^2} \rightarrow 0$$

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as $n \rightarrow \infty$, where $f_n^\#(z) = |f_n'(z)|/(1 + |f_n(z)|^2)$ is the spherical derivative of f_n . By Marty's criterion, $\mathcal{F}$ is normal in Δ . But it is easy to see that $\mathcal{F}_1$ is not normal in Δ .

Example 2. Let $\Delta = \{z : |z| < 1\}$, and $\mathcal{F} = \{f_n(z) = nz : n = 1, 2, \dots\}$. Then $\mathcal{F}_1 = \{f_n'(z) = n : n = 1, 2, \dots\}$. Clearly, $\mathcal{F}_1$ is normal in Δ ; but $\mathcal{F}$ is not normal in Δ .

In 1996, CHEN and LAPPAN [2] first gave an interesting normality relation between $\mathcal{F}$ and $\mathcal{F}_k$ under an additional condition, as follows.

Theorem A ([2, Corollary 4]). *Let k be a positive integer, and let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D , all of whose zeros have multiplicity at least $k+1$. If $\mathcal{F}_k = \{f^{(k)} : f \in \mathcal{F}\}$ is normal, then $\mathcal{F}$ is also normal in D .*

In this paper, by using a different method from that in [2], we first give an extension to the above result, as follows.

Theorem 1. *Let k be a positive integer, and let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D , all of whose zeros have multiplicity at least k , and there exists $M > 0$ such that $|f^{(k)}(z)| \leq M$ whenever $f(z) = 0$ for $f \in \mathcal{F}$. If $\mathcal{F}_k = \{f^{(k)} : f \in \mathcal{F}\}$ is normal, then $\mathcal{F}$ is also normal in D .*

Remark 1. Theorem 1 is sharp, which can also be shown by Example 2.

The above normality relation between $\mathcal{F}$ and $\mathcal{F}_k$ is indeed useful to study normal families. In section 3, we shall give some applications of Theorem 1.

2. Proof of Theorem 1

We need the following well-known PANG–ZALCMAN lemma, which is the local version of [8, Lemma 2](cf. [13, pp. 216–217]).

Lemma 1. *Let k be a positive integer and let $\mathcal{F}$ be a family of functions meromorphic in a domain D , all of whose zeros have multiplicity at least k , and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in \mathcal{F}$. Then if $\mathcal{F}$ is not normal at $z_0 \in D$, there exist, for each $\alpha, 0 \leq \alpha \leq k$,*

- (a) *points $z_n \in D$, $z_n \rightarrow z_0$,*
- (b) *positive numbers $\rho_n \rightarrow 0$, and*
- (c) *functions $f_n \in \mathcal{F}$*

such that $g_n(\zeta) = \rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) \rightarrow g(\zeta)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function in $\mathbb{C}$, all of whose zeros have multiplicity at least k , such that $g^\#(\zeta) \leq g^\#(0) = kA + 1$.

PROOF OF THEOREM 1. Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. By Lemma 1, there exist functions $f_n \in \mathcal{F}$, points $z_n \rightarrow z_0$ and positive numbers $\rho_n \rightarrow 0$, such that

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n \zeta)}{\rho_n^k} \rightarrow g(\zeta) \quad (1)$$

converges spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\zeta)$ is a nonconstant meromorphic function in $\mathbb{C}$, all of whose zeros have multiplicity at least k , and $g^\#(\zeta) \leq g^\#(0) = kM + 1$. (Without loss of generality, we assume that $M > 1$).

From (1), we have

$$g_n^{(k)}(\zeta) = f_n^{(k)}(z_n + \rho_n \zeta) \rightarrow g^{(k)}(\zeta) \quad (2)$$

converges uniformly on compact subsets of $\mathbb{C}$ disjoint from the poles of g . Suppose that $g(\zeta_0) = 0$, by Hurwitz's theorem, there exist $\zeta_n, \zeta_n \rightarrow \zeta_0$, such that $f_n(z_n + \rho_n \zeta_n) = 0$. By the assumption of Theorem 1, we have $|f_n^{(k)}(z_n + \rho_n \zeta_n)| \leq M$. Now, it follows from (2) that $|g^{(k)}(\zeta_0)| \leq M$. This proves that $|g^{(k)}| \leq M$ whenever $g = 0$.

We claim that g can not be a polynomial of degree less than $k + 1$. Indeed, g can not be a polynomial of degree less than k since all zeros of g have multiplicity at least k . Now assume that g is a polynomial of degree k . It follows that g has the form

$$g(\zeta) = \frac{A}{k!} (\zeta - \alpha)^k \quad (3)$$

where A, α are complex numbers. Since $g = 0 \Rightarrow |g^{(k)}| \leq M$, we see that $|A| \leq M$. Calculating $g^\#(0)$, we get

$$g^\#(0) = \frac{\frac{|A| |\alpha|^{k-1}}{(k-1)!}}{1 + \left(\frac{|A| |\alpha|^k}{k!}\right)^2} = \frac{k}{|\alpha|} \cdot \frac{\frac{|A| |\alpha|^k}{k!}}{1 + \left(\frac{|A| |\alpha|^k}{k!}\right)^2}.$$

From the middle expression, we see that $g^\#(0) \leq |A|$ if $|\alpha| \leq 1$, and from the expression on the right we see that $g^\#(0) < k/2$ if $|\alpha| > 1$. But these contradict the fact that $g^\#(0) = kM + 1$ and $|A| \leq M$.

Hence, there exist a point ζ_0 and $M_1 > 0$ such that

$$M_1^{-1} \leq |g^{(j)}(\zeta_0)| \leq M_1, \quad \text{for } j = k, k + 1.$$

It follows that $(2M_1)^{-1} \leq |g_n^{(j)}(\zeta_0)| \leq 2M_1$ ($j = k, k+1$) for sufficiently large n . From (2), $g_n^{(k)}(\zeta_0) = f_n^{(k)}(z_n + \rho_n \zeta_0)$, and then $|f_n^{(k)}(z_n + \rho_n \zeta_0)| \leq 2M_1$ for sufficiently large n . So we have

$$\begin{aligned} (2M_1)^{-1} \leq |g_n^{(k+1)}(\zeta_0)| &= \rho_n |f_n^{(k+1)}(z_n + \rho_n \zeta_0)| \\ &\leq \rho_n (1 + 4M_1^2) \frac{|f_n^{(k+1)}(z_n + \rho_n \zeta_0)|}{1 + |f_n^{(k)}(z_n + \rho_n \zeta_0)|^2}, \end{aligned} \quad (4)$$

for sufficiently large n .

On the other hand, by Marty's criterion, the normality of the family $\mathcal{F}_k$ implies that for each compact subset $K \subset D$, there exists a positive number M_2 such that

$$\frac{|f^{(k+1)}(z)|}{1 + |f^{(k)}(z)|^2} \leq M_2$$

for each $f \in \mathcal{F}$ and $z \in K$. Then, for sufficiently large n , we have

$$\frac{|f_n^{(k+1)}(z_n + \rho_n \zeta_0)|}{1 + |f_n^{(k)}(z_n + \rho_n \zeta_0)|^2} \leq M_2. \quad (5)$$

Substituting (5) in (4), we obtain

$$(2M_1)^{-1} \leq |g_n^{(k+1)}(\zeta_0)| \leq \rho_n (1 + 4M_1^2) M_2 \rightarrow 0,$$

as $n \rightarrow \infty$, a contradiction. Theorem 1 is thus proved. $\square$

3. Some applications of Theorem 1

In this section, we shall give some applications of Theorem 1.

Recently, CHANG [1] proved the following result, which improve and generalize the related results due to PANG and ZALCMAN [8], FANG and ZALCMAN [5].

Theorem B ([1, Theorem 1]). *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D , let a, b be two nonzero complex numbers such that $a/b \notin \mathbb{N} \setminus \{1\}$. If, for each $f \in \mathcal{F}$, $f = a \Rightarrow f'(z) = a$, and $f'(z) = b \Rightarrow f''(z) = b$ in D , then $\mathcal{F}$ is normal.*

There is an example [1, Example 1], which shows that the condition ' $a/b \notin \mathbb{N} \setminus \{1\}$ ' in Theorem B is necessary. Chang proved another result without the condition ' $a/b \notin \mathbb{N} \setminus \{1\}$ ', as follows.

Theorem C ([1, Theorem 2]). *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D , let a, b be two nonzero complex numbers. If, for each $f \in \mathcal{F}$, $f = a \Rightarrow f'(z) = a$, $f'(z) \neq b$ and $f''(z) \neq b$ in D , then $\mathcal{F}$ is normal.*

Remark 2. CHANG also gave another example [1, Example 2] to show that the condition ' $f''(z) \neq b$ ' in Theorem C can not be omitted. However, it is easy to see that ' $f''(z) \neq b$ ' in Theorem C is not necessary for the case $a = b (\neq 0)$. Indeed, $f = a \Rightarrow f'(z) = a$ and $f'(z) \neq b$ yield that $f \neq a$ and $f' \neq a$ since $a = b$, then GU's normal criterion [3] implies that $\mathcal{F}$ is normal. We also find that ' a is nonzero' in Theorem C can be removed. In fact, if $a = 0$ and $b \neq 0$, noting that $f' \neq b$ and $f'' \neq b$, GU's normal criterion asserts that $\mathcal{F}_1 = \{f' : f \in \mathcal{F}\}$ is normal in D . Since $f = 0 \Rightarrow f' = 0$, we conclude from Theorem 1 that $\mathcal{F}$ is also normal in D .

Here, by using Theorem 1 and some known results, we can prove the following results, which improve and generalize Theorem C much more.

Theorem 2. *Let a, b, c be three complex numbers with $c \neq 0$, k, l be two positive integers, and let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D . Suppose that, for each $f \in \mathcal{F}$ and $z \in D$,*

- (1) *all zeros of $f - a$ have multiplicity at least k , and there exists $M > 0$ such that $f = a \Rightarrow |f^{(k)}| \leq M$;*
- (2) *all zeros of $f^{(k)} - b$ have multiplicity at least $l + 1$, and $f^{(k+l)} \neq c$.*

Then $\mathcal{F}$ is normal in D .

Let $k = l = 1$ and $b = c$ in Theorem 2, we have

Corollary 1. *Let a, b be two complex numbers with $b \neq 0$, and let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D . Suppose that, for each $f \in \mathcal{F}$ and $z \in D$,*

- (1) *there exists $M > 0$ such that $f = a \Rightarrow |f'| \leq M$;*
- (2) *all zeros of $f' - b$ have multiplicity at least 2, and $f'' \neq b$.*

Then $\mathcal{F}$ is normal in D .

Obviously, the above results improve and generalize Theorem C.

Next we give some more general extensions of Theorem C by extending constants ' a, b, c ' in Theorem 2 to functions ' $a(z), b(z), c(z)$ '.

Theorem 3. *Let k, l be two positive integers, D be a domain in $\mathbb{C}$, let $a(z), b(z)$ be two holomorphic functions in D , and $c(z)$ be a meromorphic function in D such that $c(z) \not\equiv \infty$ and $c(z) \not\equiv b'(z)$, and let $\mathcal{F}$ be a family of meromorphic functions defined in D . Suppose that, for each $f \in \mathcal{F}$ and $z \in D$,*

- (i) all zeros of $f(z) - a(z)$ have multiplicity at least k , and there exists $M > 0$ such that $f(z) = a(z) \Rightarrow |f^{(k)}| \leq M$;
- (ii) all zeros of $f^{(k)}(z) - b(z)$ have multiplicity at least 3, and $f^{(k+1)}(z) \neq c(z)$.

Then $\mathcal{F}$ is normal in D .

Theorem 4. Let $k, l (\geq 2)$ be two positive integers, D be a domain in $\mathbb{C}$, let $a(z), b(z)$ be two holomorphic functions in D , and $c(z)$ be a meromorphic function in D such that $c(z) \not\equiv \infty$ and $c(z) \not\equiv b^{(l)}(z)$, and let $\mathcal{F}$ be a family of meromorphic functions defined in D . Suppose that, for each $f \in \mathcal{F}$ and $z \in D$,

- (i) all zeros of $f(z) - a(z)$ have multiplicity at least k , and there exists $M > 0$ such that $f(z) = a(z) \Rightarrow |f^{(k)}| \leq M$;
- (ii) all zeros of $f^{(k)}(z) - b(z)$ have multiplicity at least $l+1$, and $f^{(k+l)}(z) \neq c(z)$.

Then $\mathcal{F}$ is normal in D .

Remark 3. If $k = 1$, the condition ‘all zeros of $f - a$ or $(f - a(z))$ have multiplicity at least k ’ in Theorem 2–4 holds naturally, and then can be removed.

Remark 4. The condition $c \neq 0$ in Theorem 2 ($b \neq 0$ in Corollary 1), $c(z) \neq b'(z)$ in Theorem 3, and $c(z) \neq b^{(l)}(z)$ in Theorem 4 can not be omitted, as is shown by the following examples.

Example 3. Let $\Delta = \{z : |z| < 1\}$, $a \neq 0$ and $b = c = 0$, and let $\mathcal{F} = \{f_n(z) = e^{nz} + a : n = 1, 2, \dots; z \in \Delta\}$. Obviously, $f_n(z) \neq a$, thus $f(z) = a \Rightarrow f'(z) = a$; $f'_n(z) = ne^{nz} \neq 0$, and $f''_n(z) = n^2e^{nz} \neq 0$. Then all conditions excepting $c \neq 0$ (or $c \neq 0$) of Theorem 2 (Corollary 1) are satisfied. But $\mathcal{F}$ is not normal in Δ .

Example 4. Let $\Delta = \{z : |z| < 1\}$, $a(z) = b(z) = c(z) = e^z$, and let $\mathcal{F} = \{f_n(z) = e^{nz} + e^z : n = 1, 2, \dots; z \in \Delta\}$. It is easy to see that all conditions excepting $c(z) \neq b'(z)$ ($c(z) \neq b^{(l)}(z)$) of Theorem 3–4 are satisfied. But $\mathcal{F}$ is not normal in Δ .

Remark 5. Example 4 also shows that ‘nonzero constants a, b ’ in Theorem B can not be replaced two nonconstant functions (even for non-vanishing holomorphic functions).

To prove the above theorems, we need some known results.

Lemma 2 ([10, Theorem 5]). Let k be a positive integer, and let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D , all of whose poles are multiple and whose zeros all have multiplicity at least $k+1$. If, for each $f \in \mathcal{F}$, $f^{(k)}(z) \neq 1$ in D , then $\mathcal{F}$ is normal in D .

Lemma 3 ([7, Theorem 1.3], cf. [11, Theorem 2]). *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D , all of whose poles are multiple and whose zeros all have multiplicity at least 3, and let $\psi(z) (\neq 0, \infty)$ be a function meromorphic in D . If, for each $f \in \mathcal{F}$ and for each $z \in D$, $f'(z) \neq \psi(z)$, then $\mathcal{F}$ is normal in D .*

Lemma 4 ([14, Theorem 2]). *Let $k \geq 2$ be an integer, $\mathcal{F}$ be a family of meromorphic functions defined in a domain D , all of whose poles are multiple and whose zeros all have multiplicity at least $k + 1$, and let $\psi(z) (\neq 0, \infty)$ be a function meromorphic in D . If, for each $f \in \mathcal{F}$ and for each $z \in D$, $f^{(k)}(z) \neq \psi(z)$, then $\mathcal{F}$ is normal in D .*

PROOF OF THEOREM 2. Let $\mathcal{G} = \{g = f^{(k)} - b : f \in \mathcal{F}\}$. Obviously, the poles of g have multiplicity at least $k + 1 \geq 2$. By the assumptions of theorem, for each $g \in \mathcal{G}$, all zeros of g have multiplicity at least $l + 1$, and $g^{(l)} = f^{(k+l)} \neq c$. Lemma 2 implies that $\mathcal{G}$ is normal in D . Hence, the family $\mathcal{H}_k = \{(f - a)^{(k)} : f \in \mathcal{F}, z \in D\}$ is also normal in D , where $\mathcal{H} = \{f - a : f \in \mathcal{F}\}$. Noting condition (1), by Theorem 1, we get that $\mathcal{H}$ is normal, and then $\mathcal{F}$ is normal in D . Theorem 2 is proved. $\square$

PROOF OF THEOREM 3. Since normality is a locally property, we only need to prove $\mathcal{F}$ is normal at each point in D .

Let $z_0 \in D$, then there exists $\delta > 0$ such that $\bar{D}_\delta(z_0) \subset D$, where $\bar{D}_\delta(z_0) = \{z : |z - z_0| \leq \delta\}$. Let $\mathcal{G} = \{g(z) = f^{(k)}(z) - b(z) : f \in \mathcal{F}\}$. Clearly, all poles of $g \in \mathcal{G}$ are multiple. By the hypotheses of the theorem, for each $g \in \mathcal{G}$, all zeros of g have multiplicity at least 3. Noting that $b(z)$ is holomorphic and $f^{(k+1)}(z) \neq c(z)$, we have $g' = f^{(k+1)}(z) - b'(z) \neq c(z) - b'(z) (\neq 0)$. Then, by Lemma 3, $\mathcal{G}$ is normal in D , and then in $D_\delta(z_0) = \{z : |z - z_0| < \delta\}$. It follows that the family $\mathcal{H}_k = \{(f(z) - a(z))^{(k)} : f \in \mathcal{F}\}$ is normal in $D_\delta(z_0)$, where $\mathcal{H} = \{h = f(z) - a(z) : f \in \mathcal{F}\}$. By the hypotheses of the theorem, for each $h \in \mathcal{H}$, all zeros of h have multiplicity at least k . Moreover, if $h(z) = 0$, that is, $f(z) = a(z)$, then $|f^{(k)}(z)| \leq M$, and thus

$$|h^{(k)}(z)| \leq M + |a^{(k)}(z)|.$$

Noting that $a(z)$ is holomorphic in D , there exists $M_1 > 0$ such that $|a^{(k)}(z)| \leq M_1$ in $\bar{D}_\delta(z_0)$, and then in $D_\delta(z_0)$. We get that $h(z) = 0 \Rightarrow |h^{(k)}(z)| \leq M_2$ for $z \in D_\delta(z_0)$, where $M_2 = M + M_1$. By Theorem 1, $\mathcal{H}$ is normal in $D_\delta(z_0)$. It follows that $\mathcal{F}$ is normal in $D_\delta(z_0)$, and this means that $\mathcal{F}$ is normal at z_0 . Theorem 3 is thus proved. $\square$

PROOF OF THEOREM 4. Using the same argument as in Theorem 3 and Lemma 4, we can prove Theorem 4. We here omit the details. $\square$

Next we give another application of Theorem 1. In [4], FANG and CHANG gave an extension to Gu's normal criterion in some sense, by allowing $f^{(k)} - 1$ have zeros but restricting the zeros of $f^{(k)}$, as follows.

Theorem D ([4, Theorem 1]). *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D , and let k be a positive integer. If, for each $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} \neq 0$ and the zeros of $f^{(k)} - 1$ have multiplicity at least $(k+2)/k$, then $\mathcal{F}$ is normal.*

Here, we can prove the following extension of Theorem D.

Theorem 5. *Let k, l_1, l_2 be three positive integers (l_1, l_2 can be ∞) with $1/l_1 + 1/l_2 < k/(k+1)$, and let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D . Suppose that, for each $f \in \mathcal{F}$ and $z \in D$,*

- (1) *all zeros of f have multiplicity at least k and there exists $M > 0$ such that $|f^{(k)}(z)| \leq M$ whenever $f(z) = 0$;*
- (2) *all zeros of $f^{(k)}$ have multiplicity at least l_1 ; and*
- (3) *all zeros of $f^{(k)} - 1$ have multiplicity at least l_2 .*

Then $\mathcal{F}$ is normal in D .

Remark 6. We should indicate that Theorem 5 can be followed from [4, Theorem 2] if condition (1) is replaced by a stronger condition “all zeros of f have multiplicity at least $k+1$ ”. However, the method in [4] does not work here, and our proof is very simple.

To prove Theorem 5, we need the following classical result due to BLOCH and VALIRON, which can be found in [6], [9], [12].

Lemma 5. *Let $a_1, a_2, \dots, a_q$ be q distinct complex numbers, and $l_1, l_2, \dots, l_q$ be positive integers (may equal to ∞) with $\sum_{i=1}^q (1 - 1/l_i) > 2$. Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D . If, for each $f \in \mathcal{F}$, the zeros of $f - a_i$ have multiplicity at least l_i ($i = 1, 2, \dots, q$) in D , then $\mathcal{F}$ is normal in D .*

PROOF OF THEOREM 5. Obviously, the poles of $f^{(k)}$ have multiplicity at least $k+1$. Since

$$\frac{1}{l_1} + \frac{1}{l_2} < \frac{k}{k+1},$$

we have

$$\left(1 - \frac{1}{l_1}\right) + \left(1 - \frac{1}{l_2}\right) + \left(1 - \frac{1}{k+1}\right) > 2.$$

Let $q = 3$, $a_1 = 0$, $a_2 = 1$ and $a_3 = \infty$, applying Lemma 6 for $\mathcal{F}_k = \{f^{(k)} : f \in \mathcal{F}\}$, we know that $\mathcal{F}_k$ is normal in D . Noting condition (1), Theorem 1 implies that $\mathcal{F}$ is also normal in D . Theorem 5 is proved. $\square$

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