

On the counting function of Stanley sequences

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Abstract. For a finite sequence $A = \{a_1 < a_2 < \cdots < a_t\}$ of nonnegative integers which contains no 3-term arithmetic progression, the Stanley sequence S generated by A is defined as follows: for $k \geq t$, a_{k+1} is the least integer $a > a_k$ such that $\{a_1, a_2, \dots, a_k, a\}$ contains no 3-term arithmetic progression. Recently, Moy proved that $\liminf S(x)/\sqrt{x} \geq \sqrt{2}$, which solves a problem posed by Erdős et al., where $S(x)$ is the counting function of S . In this note we show that $\limsup S(x)/\sqrt{x} \geq 1.77$.

1. Introduction

Let $\mathbb{N}_0$ denote the set of nonnegative integers. For a finite set $A = \{a_1 < a_2 < \cdots < a_t\} \subset \mathbb{N}_0$ which contains no 3-term arithmetic progression, we denote by $S = \{a_1, a_2, \dots\}$ the sequence defined by the following recursion: if $k \geq t$ and $a_1, \dots, a_k$ have been defined, let a_{k+1} be the smallest integer $a > a_k$ such that $\{a_1, \dots, a_k\} \cup \{a\}$ contains no 3-term arithmetic progression. This sequence is called *the Stanley sequence* generated by A . Stanley sequence were considered, for instance, in [1]–[5].

Recently MOY [3] proved that for any $\varepsilon > 0$ and $x \geq x_0(\varepsilon, A)$,

$$S(x) \geq (\sqrt{2} - \varepsilon)\sqrt{x}.$$

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This solved a problem posed by Erdős *et al.* [1]. That means

$$\liminf_{x \rightarrow \infty} \frac{S(x)}{\sqrt{x}} \geq \sqrt{2}.$$

In this note we study

$$\limsup_{x \rightarrow \infty} \frac{S(x)}{\sqrt{x}}.$$

We have the following results:

Theorem 1. *For a given finite set $A \subset \mathbb{N}_0$ containing no 3-term arithmetic progression, let S be the Stanley sequence generated by A and $S(x)$ be its counting function. Then*

$$\limsup_{x \rightarrow \infty} \frac{S(x)}{\sqrt{x}} \geq 1/\sqrt{\tau} > 1.77,$$

where τ is the maximum value of

$$\frac{t}{2\sqrt{2}} \log(\sqrt{2t^2 - 1} + \sqrt{2t^2}) + \frac{1}{2}t^2(1 - \sqrt{2t^2 - 1})$$

on $[1/\sqrt{2}, 1]$.

We also pose the following problems.

Problem 1. *Is there any finite set $A \subset \mathbb{N}_0$ containing no 3-term arithmetic progression such that*

$$\limsup_{x \rightarrow \infty} \frac{S(x)}{\sqrt{x}} < +\infty?$$

Problem 2. *Is there any finite set $A \subset \mathbb{N}_0$ containing no 3-term arithmetic progression such that*

$$\liminf_{x \rightarrow \infty} \frac{S(x)}{\sqrt{x}} < +\infty?$$

2. Proof of the theorem

As in [3], define

$$H(S, n) = |\{(a_i, a_j) : i < j, n = 2a_j - a_i\}|.$$

Lemma 1 ([3, Lemma 2.3]). *We have*

$$\sum_{0 \leq n \leq x} H(S, n) \geq x - S(x) - \max A.$$

PROOF OF THEOREM 1. Let

$$\alpha = \liminf_{x \rightarrow \infty} \frac{S(x)}{\sqrt{x}}, \quad \beta = \limsup_{x \rightarrow \infty} \frac{S(x)}{\sqrt{x}}, \quad t = \frac{\alpha}{\beta}.$$

We assume that $\beta < 2$ as otherwise the assertion is immediate, and notice that then, in view of Moy's result, $t > \sqrt{2}/2$, whence $0 < 2t^2 - 1 \leq 1$ which will be used later.

Fix $\varepsilon \in (0, 1)$ and find a positive integer x_0 such that for any $x \geq x_0$ we have

$$(\alpha - \varepsilon)\sqrt{x} < S(x) < (\beta + \varepsilon)\sqrt{x}.$$

For the rest of the proof we assume that x runs over a strictly increasing sequence of positive integers such that

$$S(x) = (\alpha + o(1))\sqrt{x}.$$

The crucial observation (originating from Moy's paper) is that every sufficiently large integer is either in S or the largest term of a three-term arithmetic progression having its two smallest terms in S . Hence, the number of such progressions with the largest term not exceeding x is at least $x + o(x)$. Noticing that the smallest term of such a progression s and its second smallest term t satisfy $2t - s \leq x$, we conclude that

$$\begin{aligned} x + o(x) &\leq \sum_{\substack{s, t \in S \\ s < t \leq (x+s)/2}} 1 = \sum_{\substack{s \in S \\ s \leq x}} \left(S\left(\frac{x+s}{2}\right) - S(s) \right) \\ &= \sum_{\substack{s \in S \\ s \leq x}} S\left(\frac{x+s}{2}\right) - \frac{1}{2}S(x)(S(x) - 1) \\ &= \sum_{\substack{s \in S \\ s \leq x}} S\left(\frac{x+s}{2}\right) - \frac{1}{2}\alpha^2 x + o(x). \end{aligned} \tag{1}$$

To estimate the sum in the right-hand side we use partial summation:

$$\begin{aligned} \sum_{\substack{s \in S \\ s \leq x}} S\left(\frac{x+s}{2}\right) &= \sum_{s \leq x} S\left(\frac{x+s}{2}\right) (S(s) - S(s-1)) \\ &= S(x)^2 - \sum_{s \leq x-1} \left(S\left(\frac{x+s+1}{2}\right) - S\left(\frac{x+s}{2}\right) \right) S(s) \end{aligned}$$

$$\begin{aligned}
&\leq S(x)^2 - (\alpha - \varepsilon) \sum_{x_0 \leq s \leq x} \left(S\left(\frac{x+s+1}{2}\right) - S\left(\frac{x+s}{2}\right) \right) \sqrt{s} \\
&\leq S(x)^2 - (\alpha - \varepsilon) S(x) \sqrt{x} + (\alpha - \varepsilon) S((x+x_0)/2) \sqrt{x_0 - 1} \\
&\quad + (\alpha - \varepsilon) \sum_{x_0 \leq s \leq x} S\left(\frac{x+s}{2}\right) (\sqrt{s} - \sqrt{s-1}) \\
&= (\alpha - \varepsilon) \sum_{x_0 \leq s \leq x} S\left(\frac{x+s}{2}\right) (\sqrt{s} - \sqrt{s-1}) + O(\varepsilon x). \tag{2}
\end{aligned}$$

Now we split the last sum into two parts $s > \gamma x$ and $x_0 \leq s \leq \gamma x$, where γ will be chosen later to obtain the optimal splitting point. The part of the last sum corresponding to $s > \gamma x$ can be estimated by

$$\sum_{\gamma x < s \leq x} S(x) (\sqrt{s} - \sqrt{s-1}) = S(x) (\sqrt{x} - \sqrt{[\gamma x]}) = \alpha(1 - \sqrt{\gamma})x + o(x). \tag{3}$$

For the remaining part of the sum, we have

$$\begin{aligned}
\sum_{x_0 \leq s \leq \gamma x} S\left(\frac{x+s}{2}\right) (\sqrt{s} - \sqrt{s-1}) &\leq \frac{\beta + \varepsilon}{\sqrt{2}} \sum_{x_0 \leq s \leq \gamma x} \sqrt{x+s} \left(\frac{1}{2\sqrt{s}} + O\left(\frac{1}{s}\right) \right) \\
&\leq \frac{\beta + \varepsilon}{2\sqrt{2}} \sum_{1 \leq s \leq \gamma x} \sqrt{1 + \frac{x}{s}} + o(x) \leq \frac{\beta + \varepsilon}{2\sqrt{2}} \int_1^{\gamma x} \sqrt{1 + \frac{x}{t}} dt + o(x) \\
&\leq \frac{\beta + \varepsilon}{2\sqrt{2}} x \int_0^\gamma \sqrt{1 + \frac{1}{t}} dt + o(x) \\
&\leq \frac{\beta + \varepsilon}{2\sqrt{2}} \left(\sqrt{\gamma(\gamma+1)} + \log(\sqrt{\gamma} + \sqrt{\gamma+1}) \right) x + o(x). \tag{4}
\end{aligned}$$

From (1)–(4) we get

$$(\alpha - \varepsilon) \left(\alpha(1 - \sqrt{\gamma}) + \frac{\beta + \varepsilon}{2\sqrt{2}} \left(\sqrt{\gamma(\gamma+1)} + \log(\sqrt{\gamma} + \sqrt{\gamma+1}) \right) \right) - \frac{1}{2} \alpha^2 + O(\varepsilon) \geq 1,$$

and furthermore, since ε can be chosen arbitrarily small,

$$\left(\frac{1}{2} - \sqrt{\gamma} \right) \alpha^2 + \frac{\alpha\beta}{2\sqrt{2}} \left(\sqrt{\gamma(\gamma+1)} + \log(\sqrt{\gamma} + \sqrt{\gamma+1}) \right) \geq 1.$$

Dividing through by β^2 , we derive that

$$\beta^{-2} \leq \left(\frac{1}{2} - \sqrt{\gamma} \right) t^2 + \frac{t}{2\sqrt{2}} \left(\sqrt{\gamma(\gamma+1)} + \log(\sqrt{\gamma} + \sqrt{\gamma+1}) \right).$$

For a fixed $t \in (\sqrt{2}/2, 1]$, the right-hand side takes its minimal value at $\gamma = 2t^2 - 1$. So we choose $\gamma = 2t^2 - 1$. Then

$$\beta^{-2} \leq \frac{t}{2\sqrt{2}} \log(\sqrt{2t^2 - 1} + \sqrt{2t^2}) + \frac{1}{2}t^2(1 - \sqrt{2t^2 - 1}).$$

This establishes the estimate $\beta \geq 1/\sqrt{\tau}$, and numerical investigation shows that $\tau < 0.318214$, implying $1/\sqrt{\tau} > 1.77$.

This completes the proof of Theorem 1. $\square$

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