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Metric properties and exceptional sets of beta-continued fractions of Laurent series

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Abstract. In 2010, M. Jellali et al. [10] introduced a new kind of continued fractions algorithm of Laurent series, called β-continued fractions. In this paper, we discuss metric properties of the partial quotients $\{e_n(x), n \geq 1\}$ occurring in β-continued fractions, and for the β-continued fractions with sequences of partial quotients and the β-continued fractions whose sum of degrees of partial quotients tends to infinity with generally functional growth rate, we give their Hausdorff dimensions.

1. Introduction

Let $\mathbb{F}_q$ be the finite field of q elements and $\mathbb{F}_q((z^{-1}))$ denotes the field of all formal Laurent series with coefficients in $\mathbb{F}_q$. Recall that $\mathbb{F}_q[z]$ denotes the ring of polynomials in z with coefficients in $\mathbb{F}_q$.

For each $x = \sum_{n=\nu}^{\infty} c_n z^{-n} \in \mathbb{F}_q((z^{-1}))$, call $[x] = \sum_{\nu \leq n \leq 0} c_n z^{-n} \in \mathbb{F}_q[z]$ the integral part of x and $\deg x = -\inf\{n \in \mathbb{Z} : c_n \neq 0\}$ the degree of x, with the convention that $\deg 0 = -\infty$. Define the absolute value on $\mathbb{F}_q((z^{-1}))$ as $|x| = q^{\deg x}$ which is a non-Archimedean absolute value, the field $\mathbb{F}_q((z^{-1}))$ is locally compact and complete under the metric $\rho(x, y) = |x - y|$.

The regular continued fraction over the field of formal Laurent series is introduced by E. Artin [1], and the metric and ergodic properties of this dynamical system have been studied in [2], [5], [6], [13], [14], [15], [16], [17].

In 2010, M. Jellali et al. [10] introduced the β-continued fractions of Laurent series, they studied the metric and ergodic properties of this new continued

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fraction of Laurent series and gave the Hausdorff dimensions of bounded-type set and the set of series having a given rate of convergence. In this paper, we discuss metric properties of the partial quotients $\{e_n(x) : n \geq 1\}$ occurring in β-continued fraction expansions, obtain the so-called '0-1' law and limits results on the partial quotients. Furthermore, for the β-continued fractions with sequences of partial quotients and the β-continued fractions whose sum of degrees of partial quotients tends to infinity with generally functional growth rate, we give their Hausdorff dimensions. Our results are inspired by works [3], [8], [11], [12], [18].

2. Preliminary

Let $\beta = \{\beta_i\}_{i\in\mathbb{Z}}$ with $\beta_i \in \mathbb{F}_q((z^{-1}))\backslash\{0\}$ such that $\{\deg \beta_i\}_{i\in\mathbb{Z}}$ is a strictly increasing sequence of integers, and

$$\mathcal{S} = \left\{\{s_i\}_{-\infty<i\leq k} : k \in \mathbb{Z},\ s_i \in \mathbb{F}_q[z],\ \deg s_i < \deg \beta_{i+1} - \deg \beta_i\right\}.$$

Then each $x \in \mathbb{F}_q((z^{-1}))$ admits a unique representation $x = \sum_{-\infty<i\leq k} s_i\beta_i$, $\{s_i\}_{-\infty<i\leq k} \in \mathcal{S}$. Conversely, for any given string $\{s_i\}_{-\infty<i\leq k} \in \mathcal{S}$, there exists a unique $x \in \mathbb{F}_q((z^{-1}))$ such that $x = \sum_{-\infty<i\leq k} s_i\beta_i$. For any $x = \sum_{-\infty<i\leq k} s_i\beta_i \in \mathbb{F}_q((z^{-1}))$, we define the β-integer part by $[x]_\beta = \sum_{0\leq i\leq k} s_i\beta_i$. Let

$$H_0(\beta) := \{d\beta_0 : d \in \mathbb{F}_q[z],\ 0 < \deg d < \deg \beta_1 - \deg \beta_0\},$$

$$H_n(\beta) := \{d_0\beta_0 + \cdots + d_n\beta_n : d_i \in \mathbb{F}_q[z],\ \deg d_i < \deg \beta_{i+1} - \deg \beta_i, d_n \neq 0\}$$

and

$$H(\beta) := H_0(\beta) \cup \bigcup_{n\geq 1} H_n(\beta).$$

Let $\mathcal{I} := \left\{x = \sum_{-\infty<i\leq -1} s_i\beta_i, \{s_i\}_{-\infty<i\leq -1} \in \mathcal{S}\right\}$. For $r > 0$, denote by $D(x, r) = \{y \in \mathbb{F}_q((z^{-1})) : |y - x| < r\}$ the disc with center x and radius r. It is easy to see that $\mathcal{I} = D(0, |\beta_0|)$ and isomorphic to $\mathbb{F}_q^\infty$. A natural measure on $\mathcal{I}$ is the normalized Haar measure μ given by $\mu(D(x, q^{-n})) = \frac{q^{-n}}{|\beta_0|}$.

Now we recall the β-continued fraction of Laurent series. Consider the transformation $T_\beta : \mathcal{I} \to \mathcal{I}$ defined by

$$T_\beta x := \frac{\beta_0^2}{x} - \left[\frac{\beta_0^2}{x}\right]_\beta, \quad T_\beta 0 := 0.$$

Then each $x \in \mathcal{I}$ have the following unique continued fraction expansion:

$$x = \cfrac{\beta_0^2}{e_1(x) + \cfrac{\cdots}{\cdots + \cfrac{\beta_0^2}{e_n(x) + \cdots}}} = [0; e_1(x), e_2(x), \dots]_\beta,$$

where $e_n(x) \in H(\beta)$ and $e_n(x) = \big[\frac{\beta_0^2}{T_\beta^{n-1}(x)}\big]_\beta$ is called the β-partial quotient. Similarly, denote $\{p_n(x)/q_n(x),\ n \geq 0\}$ the (β, n)-th convergent of x, i.e.

$$\frac{p_n(x)}{q_n(x)} = \cfrac{\beta_0^2}{e_1(x) + \cfrac{\cdots}{\cdots + \cfrac{\beta_0^2}{e_n(x)}}} = [0; e_1(x), e_2(x), \dots, e_n(x)]_\beta.$$

Next we collect some known results which we will use frequently.

Proposition 1 ([10]). *For any $x \in \mathcal{I}$, let $p_n(x)/q_n(x)$ denote the (β, n)-th convergent of x. Then we have*

(1) $|q_n(x)| \geq |\beta_0||q_{n-1}(x)|$;

(2) $|q_n(x)| = |e_1(x) \cdots e_n(x)|$;

(3) $\deg q_n(x) = \sum_{k=1}^{n} \deg e_n(x)$;

(4) $\big|x - \frac{p_n(x)}{q_n(x)}\big| = \frac{|\beta_0|^{2n+2}}{|q_n(x) \cdot q_{n+1}(x)|} < \frac{|\beta_0|^{2n+1}}{|q_n(x)|^2}$.

Proposition 2 ([10]). *Let $e_1, \dots, e_n \in H(\beta)$, $p_n/q_n = [0; e_1, \dots, e_n]_\beta$ and put*

$$\Delta(e_1, \dots, e_n) := \big\{[0; e_1, \dots, e_n + \theta]_\beta,\ \theta \in \mathcal{I}\big\}.$$

Then

$$\Delta(e_1, \dots, e_n) = D\left(\frac{p_n}{q_n}, \frac{|\beta_0|^{2n+1}}{|q_n|^2}\right)$$

with

$$|\Delta(e_1, \dots, e_n)| = \frac{|\beta_0|^{2n+1}}{q|e_1|^2 \cdots |e_n|^2} = \frac{|\beta_0|}{q} q^{-2\sum_{j=1}^{n}(\deg e_j - \deg \beta_0)},$$

where $|\Delta(e_1, \dots, e_n)|$ denotes the diameter of $\Delta(e_1, \dots, e_n)$.

Remark 2.1. We call $\Delta(e_1, \dots, e_n)$ a (β, n)-th order cylinder. Since the valuation $|\cdot|$ is non-Archimedean, it follows that if two cylinders intersect, then one contains the other.

Theorem 2.1 ([10]). *The transformation T_β is invariant and ergodic with respect to the Haar measure μ.*

3. Metric properties of the partial quotients

In this section, we give the metric properties of the partial quotients $\{e_n(\cdot)\}$ occurring in the β-continued fraction expansions.

Theorem 3.1. *The random variable sequence* $\{e_n(\cdot)\}$ *is independent and identically distributed with respect to the Haar measure* μ.

Proof. For any $n \geq 1, b \in H(\beta)$, since the transformation T_β converse the Haar measure and $e_n(x) = e_1(T_\beta^{(n-1)}(x))$, then

$$\mu\{x \in \mathcal{I} : e_n(x) = b\} = \mu\{T_\beta^{-(n-1)}\{x \in \mathcal{I} : e_1(x) = b\}\} = \mu\{\Delta(b)\} = \frac{|\beta_0|^2}{|b|^2},$$

So, the random variable sequence $\{e_n(\cdot)\}$ is identically distributed. Next we prove that the sequence $\{e_n(\cdot)\}$ is independent, In fact

$$\mu\{x \in \mathcal{I} : e_1(x) = b_1, \ldots, e_n(x) = b_n\} = \mu\{\Delta(b_1, \ldots, b_n)\} = \frac{|\beta_0|^{2n}}{|b_1 \cdots b_n|^2}.$$

We know, for all $1 \leq j \leq n$, $\mu\{x \in \mathcal{I} : e_j(x) = b_j\} = \frac{|\beta_0|^2}{|b_j|^2}$, so

$$\mu\{x \in \mathcal{I} : e_1(x) = b_1, \ldots, e_n(x) = b_n\} \\ = \mu\{x \in \mathcal{I} : e_1(x) = b_1\} \cdots \mu\{x \in \mathcal{I} : e_n(x) = b_n\}.$$

Thus, the random variable sequence $\{e_n(\cdot)\}$ is independent. □

Corollary 3.2. *The random variable sequence* $\{\deg e_n(\cdot)\}$ *is independent and identically distributed, moreover*

$$\mu\{x \in \mathcal{I} : \deg e_1(x) = k\} = \frac{(q-1)|\beta_0|}{q^k}, \quad \text{for all } k > \deg \beta_0.$$

Proof. We only need prove the second part, we first recall the equation $\sum_{\substack{\deg b = k \\ b \in H(\beta)}} \frac{1}{|b|^2} = \frac{(q-1)}{q^k |\beta_0|}$, see [10]. Therefore

$$\mu\{x \in \mathcal{I} : \deg e_1(x) = k\} = \sum_{\substack{\deg b = k \\ b \in H(\beta)}} \mu\{x \in \mathcal{I} : e_1(x) = b\} \\ = \sum_{\substack{\deg b = k \\ b \in H(\beta)}} \frac{|\beta_0|^2}{|b|^2} = \frac{(q-1)|\beta_0|}{q^k}.$$

□

Now we consider the random variables sequence $\{|e_n(\cdot)|\}$, we have

Corollary 3.3. *The random variable sequence* $\{|e_n(\cdot)|\}$ *is independent and identically distributed.*

Theorem 3.4. *Let* $\varphi : \mathbb{N} \to \mathbb{R}^+$*, then*

(1) *If* $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} < \infty$*, then* $\mu\{x \in \mathcal{I} : |e_n(x)| \geq \varphi(n),\ \text{i.o. } n\} = 0$*;*

(2) *If* $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} = \infty$*, then* $\mu\{x \in \mathcal{I} : |e_n(x)| \geq \varphi(n),\ \text{i.o. } n\} = 1$*.*

Proof. Notice that

$$\begin{aligned}\mu\{x \in \mathcal{I} : |e_n(x)| \geq \varphi(n)\} &= \sum_{e_1,\dots,e_{n-1},|e_n|\geq\varphi(n)} \frac{|\beta_0|^{2n}}{|e_1|^2 \cdots |e_n|^2} \\ &= \sum_{e_1,\dots,e_{n-1}} \mu\Big\{\Delta(e_1,\dots,e_{n-1})\Big\} \sum_{|e_n|\geq\varphi(n)} \frac{|\beta_0|^2}{|e_n|^2} \\ &= \sum_{|e_n|\geq\varphi(n)} \frac{|\beta_0|^2}{|e_n|^2}.\end{aligned}$$

Moreover, we have

$$\sum_{|e_n|\geq\varphi(n)} \frac{|\beta_0|^2}{|e_n|^2} \geq \sum_{k=\lfloor\log_q \varphi(n)\rfloor+1}^{\infty} \sum_{\substack{\deg e_n = k \\ e_n \in H(\beta)}} \frac{|\beta_0|^2}{|e_n|^2} \geq \frac{|\beta_0|}{\varphi(n)},$$

and

$$\sum_{|e_n|\geq\varphi(n)} \frac{|\beta_0|^2}{|e_n|^2} \leq \sum_{k=\lfloor\log_q \varphi(n)\rfloor}^{\infty} \sum_{\substack{\deg e_n = k \\ e_n \in H(\beta)}} \frac{|\beta_0|^2}{|e_n|^2} \leq \frac{q^2|\beta_0|}{\varphi(n)}.$$

So,

$$\frac{|\beta_0|}{\varphi(n)} \leq \mu\{x \in \mathcal{I} : |e_n(x)| \geq \varphi(n)\} \leq \frac{q^2|\beta_0|}{\varphi(n)}.$$

By Borel–Cantelli Lemma and Corollary 3.3, we have that

$$\mu\{x \in \mathcal{I} : |e_n(x)| \geq \varphi(n),\ \text{i.o. } n\}$$

equal to 0 or 1 according as $\sum_{n=1}^{\infty} \frac{1}{\varphi(n)}$ converges or diverges. This complete the proof. □

Theorem 3.5. *For almost all* $x \in \mathcal{I}$*,*

$$\limsup_{n\to\infty} \frac{\log q \deg e_n(x) - \log n}{\log\log n} = 1, \quad \liminf_{n\to\infty} \frac{\log q \deg e_n(x) - \log n}{\log\log n} = -\infty.$$

PROOF. Let $\varphi(n) = n \log n$ and $\varphi(n) = n(\log n)^{1+\varepsilon}(\varepsilon > 0)$. By Theorem 3.4, we get the first equality, we only need prove the second one.

For all $\alpha > 0$, let $A_\alpha = \{x \in \mathcal{I} : |e_n(x)| < n(\log n)^{-\alpha},\ \text{i.o. } n\}$, we have that

$$\begin{aligned}
\mu(A_\alpha) &= \mu\{\limsup_{n\to\infty}\{x \in \mathcal{I} : |e_n(x)| < n(\log n)^{-\alpha}\}\} \\
&\geq \limsup_{n\to\infty} \mu\{x \in \mathcal{I} : |e_n(x)| < n(\log n)^{-\alpha}\} \\
&= 1 - \liminf_{n\to\infty} \mu\{x \in \mathcal{I} : |e_n(x)| \geq n(\log n)^{-\alpha}\} \\
&\geq 1 - \liminf_{n\to\infty} \frac{q^2|\beta_0|}{n(\log n)^{-\alpha}} = 1.
\end{aligned}$$

This implies $\mu(A_\alpha) = 1$. Hence for almost all $x \in \mathcal{I}$, $|e_n(x)| < n(\log n)^{-\alpha}$ holds for infinitely times. Notice $\alpha > 0$ is arbitrary, we proves the second equality. □

Now we consider

$$|L_n(x)| = \max_{1\leq j\leq n} |e_j(x)|,$$

therefore

$$\deg L_n(x) = \max_{1\leq j\leq n} \deg e_j(x),$$

we have the following two similar results.

Theorem 3.6. *For almost all* $x \in \mathcal{I}$,

$$\limsup_{n\to\infty} \frac{\log q \deg L_n(x) - \log n}{\log\log n} = 1, \quad \liminf_{n\to\infty} \frac{\log q \deg L_n(x) - \log n}{\log\log n} = 0.$$

PROOF. We only establish the second equality, the first one is similar to the above theorem.

I. First, we consider the set $\{x \in \mathcal{I} : |L_n(x)| < n(\log n)^\alpha\}, (\alpha > 0)$. Notice that $\{|e_n(x)|\}$ is independent and identically distributed, following the same line as in the proof of Theorem 3.4, we have

$$\begin{aligned}
\mu\{x \in \mathcal{I} : |L_n(x)| < n(\log n)^\alpha\} &= \mu\{x \in \mathcal{I} : |e_j(x)| < n(\log n)^\alpha,\ 1 \leq j \leq n\} \\
&= \mu\{x \in \mathcal{I} : |e_1(x)| < n(\log n)^\alpha\} \cdots \mu\{x \in \mathcal{I} : |e_n(x)| < n(\log n)^\alpha\} \\
&\geq \left(1 - \frac{q^2|\beta_0|}{n(\log n)^\alpha}\right)^n.
\end{aligned}$$

By Fatou Lemma, we have

$$\begin{aligned}
\mu\{\limsup_{n\to\infty}\{x \in \mathcal{I} : |L_n(x)| < n(\log n)^\alpha\}\} &\geq \limsup_{n\to\infty} \mu\{x \in \mathcal{I} : |L_n(x)| < n(\log n)^\alpha\} \\
&\geq \limsup_{n\to\infty} \left(1 - \frac{q^2|\beta_0|}{n(\log n)^\alpha}\right)^n = 1.
\end{aligned}$$

Hence for almost all $x \in \mathcal{I}$, $|L_n(x)| < n(\log n)^\alpha$ holds for infinitely times, because $\alpha > 0$ is arbitrary, we have

$$\liminf_{n\to\infty} \frac{\log q \deg L_n(x) - \log n}{\log\log n} \le 0.$$

II: Conversely, it suffices to show that for any $0 < \alpha < 1$, for almost all $x \in \mathcal{I}$,

$$|L_n(x)| < n(\log n)^{-\alpha} := \varphi(n)$$

holds for only finite times. But by Theorem 3.4, for almost all $x \in \mathcal{I}$, there are infinitely many n such that $|L_n(x)| \ge \varphi(n)$, it follows that "for almost all $x \in \mathcal{I}$, $|L_n(x)| < \varphi(n)$ holds for only finite times" is equivalent to "for almost all $x \in \mathcal{I}$, $|L_n(x)| < \varphi(n)$ but $|L_{n+1}(x)| \ge \varphi(n+1)$ holds for only finite times".

In fact

$$\begin{aligned}
&\mu\{x \in \mathcal{I} : |L_n(x)| < n(\log n)^{-\alpha}, |L_{n+1}(x)| \ge (n+1)(\log(n+1))^{-\alpha}\} \\
&\quad = \mu\{x \in \mathcal{I} : |L_n(x)| < n(\log n)^{-\alpha}, |e_{n+1}(x)| \ge (n+1)(\log(n+1))^{-\alpha}\} \\
&\quad \le \left(1 - \frac{|\beta_0|}{n(\log n)^{-\alpha}}\right)^n \frac{q^2|\beta_0|}{(n+1)(\log(n+1))^{-\alpha}} \ll \frac{1}{n(\log n)^{1+\alpha}}.
\end{aligned}$$

The last Vinogradov symbol ($\ll$) holds, because

$$\frac{q^2|\beta_0|}{(n+1)(\log(n+1))^{-\alpha}} \ll \frac{(\log(n+1))^\alpha}{n+1} \ll \frac{(2\log n)^\alpha}{n}. \tag{3.1}$$

and notice that $(1 - \frac{1}{x})^x$ is decreasing and for any $x > 0, e^x > x^k/k!$, then

$$\begin{aligned}
\left(1 - \frac{|\beta_0|}{n(\log n)^{-\alpha}}\right)^n &= \left(1 - \frac{|\beta_0|}{n(\log n)^{-\alpha}}\right)^{(-\frac{n(\log n)^{-\alpha}}{|\beta_0|})(-\frac{|\beta_0|}{(\log n)^{-\alpha}})} \\
&\le e^{-|\beta_0|(\log n)^\alpha} < \frac{k!}{|\beta_0|^k(\log n)^{\alpha k}}.
\end{aligned} \tag{3.2}$$

Together (3.1) and (3.2), and take k satisfying $\alpha k > 1 + 2\alpha$, we have

$$\begin{aligned}
\left(1 - \frac{|\beta_0|}{n(\log n)^{-\alpha}}\right)^n \frac{q^2|\beta_0|}{(n+1)(\log(n+1))^{-\alpha}} &\le \frac{2^\alpha(\log n)^\alpha k!}{n|\beta_0|^k(\log n)^{\alpha k}} \\
&\ll \frac{1}{n(\log n)^{\alpha k - \alpha}} \le \frac{1}{n(\log n)^{1+\alpha}}.
\end{aligned}$$

Then

$$\sum_{n=1}^{\infty} \mu\{x \in \mathcal{I} : |L_n(x)| < n(\log n)^{-\alpha}, |L_{n+1}(x)| \geq (n+1)(\log(n+1))^{-\alpha}\}$$

$$\ll \sum_{n=1}^{\infty} \frac{1}{n(\log n)^{1+\alpha}} < +\infty,$$

by Borel–Cantelli Lemma, we have

$$\liminf_{n\to\infty} \frac{\log q \deg L_n(x) - \log n}{\log\log n} \geq -\alpha.$$

Because $1 > \alpha > 0$ is arbitrary, therefore

$$\liminf_{n\to\infty} \frac{\log q \deg L_n(x) - \log n}{\log\log n} \geq 0. \qquad \square$$

4. β−continued fractions with sequences of partial quotients

Let $\mathfrak{B} = \{b_1, b_2, \dots, b_n, \dots\}$ be a non-empty infinite set of elements of $H(\beta)$, and

$$F_{\mathfrak{B}} = \{x \in \mathcal{I} : e_n(x) \in \mathfrak{B}, \text{ for } n \geq 1\},$$

$$F(\mathfrak{B}) = \{x \in \mathcal{I} : e_n(x) \in \mathfrak{B}, \text{ for all } n \geq 1 \text{ and } \deg e_n(x) \to \infty\}.$$

We calculate the Hausdorff dimension of these two sets:

Theorem 4.1. $\dim_H F_{\mathfrak{B}} = t$, *where* $t = \inf\left\{s : \sum_{b\in\mathfrak{B}} \left|\frac{\beta_0}{b}\right|^{2s} \leq 1\right\}$.

Theorem 4.2. $\dim_H F(\mathfrak{B}) = \alpha$, *where* $\alpha = \inf\left\{s : \sum_{b\in\mathfrak{B}} \left|\frac{\beta_0}{b}\right|^{2s} < \infty\right\}$.

Now we prove Theorem 4.1. First we give the following lemma.

Lemma 4.3 ([10])**.** *Let S be a non-empty finite set of elements of $H(\beta)$, say $S = \{e_1, e_2, \dots e_m\}$. Write*

$$F_S = \{x \in \mathcal{I} : e_n(x) \in S, \text{ for } n \geq 1\}.$$

Then $\dim_H F_S = t$, *where t is given by* $\sum_{k=1}^{m} \left|\frac{\beta_0}{e_k}\right|^{2t} = 1$.

Recall that $\mathfrak{B} = \{b_1, b_2, \dots, b_n, \dots\}$ and write $\mathfrak{B}_n = \{b_1, b_2, \dots, b_n\}$ for any $n \geq 1$. Let

$$F_{\mathfrak{B}_n} = \{x \in \mathcal{I} : e_i(x) \in \mathfrak{B}_n, \text{ for } i \geq 1\} \text{ and } t_n = \dim_H F_{\mathfrak{B}_n}.$$

Lemma 4.4. $\lim_{n\to\infty} t_n = t$, *where t is given in Theorem 4.1*

Proof. It is clear that t_n is increasing and $t_n \leq t$ for any $n \geq 1$. Suppose $\lim_{n\to\infty} t_n = s$, then $s \leq t$. For any $n \geq 1$, we have $\sum_{k=1}^{n} \left|\frac{\beta_0}{b_k}\right|^{2s} \leq \sum_{k=1}^{n} \left|\frac{\beta_0}{b_k}\right|^{2t_n} = 1$, thus $\sum_{k=1}^{\infty} \left|\frac{\beta_0}{b_k}\right|^{2s} \leq 1$, by definition of t, we have $s \geq t$. $\square$

4.1. upper bound. It is clear that

$$F_{\mathfrak{B}} = \bigcap_{n=1}^{\infty} \bigcup_{(e_1,\dots,e_n)\in\mathfrak{B}^n} \Delta(e_1,\dots,e_n).$$

For any $s > t$, write $s = (s+t)/2 + (s-t)/2$, we have

$$\sum_{k=1}^{\infty} \left|\frac{\beta_0}{b_k}\right|^{2s} = \sum_{k=1}^{\infty} \left|\frac{\beta_0}{b_k}\right|^{(s+t)+(s-t)} \le q^{-(s-t)} \sum_{k=1}^{\infty} \left|\frac{\beta_0}{b_k}\right|^{(s+t)} \le q^{-(s-t)}.$$

Therefore

$$\begin{aligned}\mathcal{H}^s(F_{\mathfrak{B}}) &\le \liminf_{n\to\infty} \sum_{(e_1,\dots,e_n)\in\mathfrak{B}^n} |\Delta(e_1,\dots,e_n)|^s \\ &\le \liminf_{n\to\infty} q^{-(s-t)} \sum_{(e_1,\dots,e_{n-1})\in\mathfrak{B}^{n-1}} |\Delta(e_1,\dots,e_{n-1})|^s \\ &\ll \liminf_{n\to\infty} q^{-n(s-t)} = 0\end{aligned}$$

Thus $\dim_H F_{\mathfrak{B}} \le s$. Since $s > t$ is arbitrary, we have $\dim_H F_{\mathfrak{B}} \le t$.

4.2. Lower bound. It is clear that $F_{\mathfrak{B}_n} \subset F_{\mathfrak{B}}$ for any $n \ge 1$. So for any $n \ge 1$, $\dim_H F_{\mathfrak{B}} \ge t_n$. By Lemma 4.4, we have $\dim_H F_{\mathfrak{B}} \ge \lim_{n\to\infty} t_n = t$. □

In the following, we are devoted to prove Theorem 4.2. For any $n \ge 1$, let $\mathfrak{C}_n$ be a set of elements of $H(\beta)$ and denote:

$$\mathcal{C} = \{x \in \mathcal{I} : e_n(x) \in \mathfrak{C}_n, \text{ for } n \ge 1\},$$

$$\mathcal{C}_N = \{x \in \mathcal{I} : e_n(x) \in \mathfrak{C}_n, \text{ for all } n \ge N\}, \quad N \in \mathbb{N}.$$

We give the following lemma which is essentially due to I. J. Good [7], or see [9].

Lemma 4.5. *For any $N \in \mathbb{N}$, $\dim_H \mathcal{C} = \dim_H \mathcal{C}_N$.*

Lemma 4.6. *For any $n \ge 1$, let $D_n = \sharp\{b \in \mathfrak{B} : \deg b = n + \deg \beta_0\}$. Then*

$$\alpha = \limsup_{n\to\infty} \frac{\log D_n}{2n \log q},$$

where α is given in Theorem 4.2.

Proof. For any $\eta < \alpha$, since $\sum_{b\in\mathfrak{B}} \left|\frac{\beta_0}{b}\right|^{2\eta} = \sum_{n=1}^{\infty} D_n q^{-2n\eta}$ diverges, there exists infinitely many n, say $\{n_k : k \ge 1\}$, such that $D_{n_k} q^{-2n_k\eta} \ge \frac{1}{n_k^2}$, which implies $\limsup_{n\to\infty} \frac{\log D_n}{2n\log q} \ge \eta$.

On the other hand, for any $\zeta > \alpha$, since $\sum_{b\in\mathfrak{B}} \left|\frac{\beta_0}{b}\right|^{2\zeta} = \sum_{n=1}^{\infty} D_n q^{-2n\zeta} < \infty$, we have $D_n q^{-2n\zeta} \le 1$ when n is large enough. Thus $\limsup_{n\to\infty} \frac{\log D_n}{2n\log q} \le \zeta$. □

4.3. upper bound. For any $\eta > \alpha$, since $\sum_{b\in\mathfrak{B}} \left|\frac{\beta_0}{b}\right|^{2\eta} < \infty$, there exists $M_\eta \in \mathbb{N}$ such that $\sum_{\substack{b\in\mathfrak{B}\\ \deg b\geq M_\eta}} \left|\frac{\beta_0}{b}\right|^{2\eta} < 1$. Denote

$$F(\mathfrak{B})_{N,m} = \{x \in \mathcal{I} : e_n(x) \in \mathfrak{B} \text{ for all } n \geq 1 \text{ and } \deg e_n(x) \geq m \text{ for all } n \geq N\},$$

then $F(\mathfrak{B}) = \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} F(\mathfrak{B})_{N,m}$.

From Lemma 4.5, we have

$$\dim_H F(\mathfrak{B}) \leq \inf_{m\geq 1} \dim_H F(\mathfrak{B})_{1,m} \leq \dim_H F(\mathfrak{B})_{1,M_\eta}.$$

Since

$$\begin{aligned} F(\mathfrak{B})_{1,M_\eta} &= \{x \in \mathcal{I} : e_n(x) \in \mathfrak{B} \text{ and } \deg e_n(x) \geq M_\eta \text{ for all } n \geq 1\} \\ &= \bigcap_{n=1}^{\infty} \bigcup_{\substack{(e_1,\ldots,e_n)\in\mathfrak{B}^n\\ \deg e_i\geq M_\eta, i=1,\ldots,n}} \Delta(e_1,\ldots,e_n), \end{aligned}$$

we have

$$\begin{aligned} \mathcal{H}^\eta(F(\mathfrak{B})_{1,M_\eta}) &\leq \liminf_{n\to\infty} \sum_{\substack{(e_1,\ldots,e_n)\in\mathfrak{B}^n\\ \deg e_i\geq M_\eta, i=1,\ldots,n}} \left|\Delta(e_1,\ldots,e_n)\right|^\eta \\ &\leq \liminf_{n\to\infty} \left(\sum_{\substack{b\in\mathfrak{B}\\ \deg b\geq M_\eta}} \left|\frac{\beta_0}{b}\right|^{2\eta} \right) \sum_{\substack{(e_1,\ldots,e_{n-1})\in\mathfrak{B}^{n-1}\\ \deg e_i\geq M_\eta, i=1,\ldots,n-1}} \left|\Delta(e_1,\ldots,e_{n-1})\right|^\eta \\ &\ll \liminf_{n\to\infty} \left(\sum_{\substack{b\in\mathfrak{B}\\ \deg b\geq M_\eta}} \left|\frac{\beta_0}{b}\right|^{2\eta} \right)^n = 0. \end{aligned}$$

Thus

$$\dim_H F(\mathfrak{B}) \leq \dim_H F(\mathfrak{B})_{1,M_\eta} \leq \eta.$$

Since $\eta > \alpha$ is arbitrary, we have $\dim_H F(\mathfrak{B}) \leq \alpha$.

4.4. Lower bound. Now we prove $\dim_H F(\mathfrak{B}) \geq \alpha$. If $\alpha = 0$ we have the desired result, we assume $\alpha > 0$. Let $n_0 = \min\{\deg b : b \in \mathfrak{B}\}$.

For any $\varepsilon > 0$ satisfying $\alpha - \varepsilon > 0$, from Lemma 4.6, there exist $n_0 < n_1 < n_2 < \cdots$ such that for any $k \geq 1$,

$$D_{n_k} \geq q^{2n_k(\alpha-\varepsilon)}. \tag{4.1}$$

Choose an integer sequence $\{t_k : k \geq 0\}$ satisfying

$$t_0 = 0, \text{ and } t_k = n_{k+1}^2 \quad \text{for any } k \geq 1. \tag{4.2}$$

Let

$$F^*(\mathfrak{B}) = \Bigg\{ x \in \mathcal{I} : e_n(x) \in \mathfrak{B} \text{ for all } n \geq 1 \text{ and } n_k \leq \deg e_n(x) - \deg \beta_0 < n_{k+1} \\ \text{if } \sum_{i=0}^{k} t_i \leq n < \sum_{i=0}^{k+1} t_i \text{ for some } k \geq 0 \Bigg\}.$$

Then $F^*(\mathfrak{B})$ is compact and $F^*(\mathfrak{B}) \subset F(\mathfrak{B})$, we only need to prove that

$$\dim_H F^*(\mathfrak{B}) \geq \alpha.$$

For any $\delta > 0$ satisfying $\alpha - \varepsilon - \delta > 0$, by (4.2), there exists $K_0 \in \mathbb{N}$ such that for any $k \geq K_0$,

$$\frac{(\alpha - \varepsilon) n_{k+1}}{\sum_{i=1}^{k} t_i n_{i-1}} < \delta. \tag{4.3}$$

Let $U = \{u_1, u_2, \dots\}$ be any disc covering system of $F^*(\mathfrak{B})$ satisfying

$$|u_i| < \frac{|\beta_0|}{q} q^{-2\sum_{j=1}^{K_0} t_j n_j} \quad \text{for any } i \geq 1. \tag{4.4}$$

For any $s > 0$, write $\Lambda_s(U) = \sum_{i=1}^{\infty} |u_i|^s$. Since $F^*(\mathfrak{B})$ is compact, we can choose finite subsystem $V = \{v_1, v_2, \dots\}$ which also covers $F^*(\mathfrak{B})$. It is clear that

$$\Lambda_{\alpha-\varepsilon-\delta}(U) \geq \Lambda_{\alpha-\varepsilon-\delta}(V).$$

For any $J \in V$, choose $x \in F^*(\mathfrak{B}) \cap J$. Suppose $x = [e_1, e_2, \dots]_\beta$, there exists unique $n = n(J)$ such that

$$\Delta(e_1, \dots, e_n) \subseteq J \subseteq \Delta(e_1, \dots, e_{n-1}).$$

Suppose $\sum_{i=0}^{k} t_i \leq n < \sum_{i=0}^{k+1} t_i$ for some $k \geq 0$. By the definition of $F^*(\mathfrak{B})$ and Proposition 2, we have

$$|J| \geq |\Delta(e_1, \dots, e_n)| = \frac{|\beta_0|}{q} q^{-2\sum_{i=1}^{n}(\deg e_i - \deg \beta_0)} \geq \frac{|\beta_0|}{q} q^{-2\sum_{i=1}^{k+1} t_i n_i}.$$

From (4.4), we have

$$k \geq K_0. \tag{4.5}$$

Write $I_J = \Delta(e_1, \ldots, e_{n-1})$. Since

$$\begin{aligned}|J|^{\alpha-\varepsilon} &\geq |\Delta(e_1, \ldots, e_n)|^{\alpha-\varepsilon} \\ &= \left|\frac{\beta_0}{e_n}\right|^{2(\alpha-\varepsilon)} |\Delta(e_1, \ldots, e_{n-1})|^{\alpha-\varepsilon} = \left|\frac{\beta_0}{e_n}\right|^{2(\alpha-\varepsilon)} |I_J|^{\alpha-\varepsilon},\end{aligned}$$

by (4.3) and (4.5), we have

$$\begin{aligned}|I_J|^{\alpha-\varepsilon} &\leq \left|\frac{e_n}{\beta_0}\right|^{2(\alpha-\varepsilon)} |J|^{\alpha-\varepsilon} \\ &\leq \left(\frac{|\beta_0|}{q}\right)^{\delta} q^{(\deg e_n - \deg \beta_0)2(\alpha-\varepsilon) - 2\delta \sum_{i=1}^{n-1}(\deg e_i - \deg \beta_0)} |J|^{\alpha-\varepsilon-\delta} \\ &\leq \left(\frac{|\beta_0|}{q}\right)^{\delta} q^{2(\alpha-\varepsilon)n_{k+1} - 2\delta \sum_{i=1}^{k} t_i n_{i-1}} |J|^{\alpha-\varepsilon-\delta} \leq \left(\frac{|\beta_0|}{q}\right)^{\delta} |J|^{\alpha-\varepsilon-\delta}. \quad (4.6)\end{aligned}$$

Let $\widetilde{W} = \{I_J : J \in V\}$, we select all those discs in $\widetilde{W}$ which are maximal (I_J is maximal if there is no $J' \in V$ such that $I_J \subseteq I_{J'}$ and $J \neq J'$). Let W be the set consisting of all maximal discs in $\widetilde{W}$. It is obvious that W is a covering system of $F^*(\mathfrak{B})$ by fundamental cylinders. By (4.6), we have

$$\Lambda_{\alpha-\varepsilon-\delta}(U) \geq \Lambda_{\alpha-\varepsilon-\delta}(V) \geq \left(\frac{q}{|\beta_0|}\right)^{\delta} \Lambda_{\alpha-\varepsilon}(W).$$

Suppose the largest order of the fundamental cylinders in W is ι. Then there exists $\Delta(e_1, \ldots, e_\iota) \in W$. Suppose $\sum_{i=0}^{k} t_i \leq \iota < \sum_{i=0}^{k+1} t_i$ for some $k \geq 0$. If $\iota > 0$, since each fundamental cylinder $\Delta(e_1, \ldots, e_{\iota-1}, b)$, where $b \in \mathfrak{B}$ and $n_k \leq \deg b - \deg \beta_0 < n_{k+1}$, contains infinitely many points in $F^*(\mathfrak{B})$, the fundamental cylinders $\Delta(e_1, \ldots, e_{\iota-1}, b)$ with $b \in \mathfrak{B}$ and $n_k \leq \deg b - \deg \beta_0 < n_{k+1}$ must all be elements of W. By (4.1) we have

$$\begin{aligned}&\sum_{b \in \mathfrak{B}, n_k \leq \deg b - \deg \beta_0 < n_{k+1}} |\Delta(e_1, \ldots, e_{\iota-1}, b)|^{\alpha-\varepsilon} \\ &= \sum_{b \in \mathfrak{B}, n_k \leq \deg b - \deg \beta_0 < n_{k+1}} \left|\frac{\beta_0}{b}\right|^{2(\alpha-\varepsilon)} |\Delta(e_1, \ldots, e_{\iota-1})|^{\alpha-\varepsilon} \\ &\geq D_{n_k} q^{-2n_k(\alpha-\varepsilon)} |\Delta(e_1, \ldots, e_{\iota-1})|^{\alpha-\varepsilon} \geq |\Delta(e_1, \ldots, e_{\iota-1})|^{\alpha-\varepsilon}\end{aligned}$$

Denote by R the new covering system of $F^*(\mathfrak{B})$ obtained by just replacing all fundamental cylinders $\Delta(e_1, \ldots, e_{\iota-1}, e_\iota) \in W$ by the fundamental cylinders $\Delta(e_1, \ldots, e_{\iota-1})$. Then

$$\Lambda_{\alpha-\varepsilon}(W) \geq \Lambda_{\alpha-\varepsilon}(R).$$

Proceeding in this manner, after a finite number of steps we reach a system whose largest order is zero, thus

$$\Lambda_{\alpha-\varepsilon}(W) \geq \Lambda_{\alpha-\varepsilon}(R) \geq \cdots \geq |\mathcal{I}|^{\alpha-\varepsilon} = |\beta_0|^{\alpha-\varepsilon}.$$

Therefore

$$\Lambda_{\alpha-\varepsilon-\delta}(U) \geq q^{\delta}|\beta_0|^{\alpha-\varepsilon-\delta}$$

By the definition of Hausdorff dimension, we have $\dim_H F^*(\mathfrak{B}) \geq \alpha-\varepsilon-\delta$. Since $\varepsilon > 0$ and $\delta > 0$ are arbitrary, we have $\dim_H F^*(\mathfrak{B}) \geq \alpha$. □

5. On the sums of degrees of partial quotients occurring in β-continued fractions

Let $\phi : \mathbb{N} \to \mathbb{R}^+$ be a function satisfying $\phi(n)/n \to \infty$ as $n \to \infty$. and

$$E(\phi) := \left\{ x \in \mathcal{I} : \lim_{n\to\infty} \frac{\deg e_1(x) + \cdots + \deg e_n(x)}{\phi(n)} = 1 \right\}.$$

We determine the Hausdorff dimension of $E(\phi)$.

Theorem 5.1. *If $E(\phi) \neq \emptyset$, then*

$$\dim_H E(\phi) = \frac{1}{1+b}, \qquad \textit{where } b = \limsup_{n\to\infty} \frac{\phi(n+1)}{\phi(n)}.$$

We first give a remark on ϕ. For any given positive function $\phi(n)$, if $E(\phi) \neq \emptyset$, then there exists an $x_0 \in E(\phi)$, define $\bar{\phi}(n) = \sum_{k=1}^{n} \deg e_k(x_0) - n \deg \beta_0$ for all $n \geq 1$. Obviously, we have $\phi(n)/\bar{\phi}(n) \to 1$ as $n \to \infty$, so $E(\phi) = E(\bar{\phi})$. Hence, in what follows, we can always assume that $\phi : \mathbb{N} \to \mathbb{N}$ and $\phi(n+1) - \phi(n) \geq 1$ once $E(\phi)$ is non-empty.

5.1. Lower bound. The lower bound is obtained by estimating the Hausdorff dimension of a homogeneous Moran subset of $E(\phi)$. We recall the definition and a basic dimensional result of the homogeneous Moran set at first, see [4], [9] for details.

Let $\{n_k\}_{k\geq1}$ be a sequence of positive integers and $\{c_k\}_{k\geq1}$ be a sequence of positive numbers satisfying $n_k \geq 2, 0 < c_k < 1, n_1c_1 \leq \delta$ and $n_kc_k \leq 1 (k \geq 2)$, where δ is some positive number.

Let

$$D_0 = \{\emptyset\}, \quad D_k = \{(i_1, \ldots, i_k) : 1 \leq i_j \leq n_j,\ 1 \leq j \leq k\}.$$

and $D=\bigcup_{k\geq 0} D_k$. If $\sigma=(\sigma_1,\ldots,\sigma_k)\in D_k, \tau=(\tau_1,\ldots,\tau_m)\in D_m$, we define the concatenation of σ and τ as $\sigma*\tau=(\sigma_1,\ldots,\sigma_k,\tau_1,\ldots,\tau_m)$. Let (X,d) be a metric space. Suppose that $J\subset X$ is a closed subset with positive diameter. A collection $\mathfrak{F}=\{J_\sigma:\sigma\in D\}$ of closed subsets of J is said to have a homogeneous Moran structure if it satisfies:

(1) $J_\emptyset=J$;

(2) For any $k\geq 1$ and $\sigma\in D_{k-1}, J_{\sigma*1}, J_{\sigma*2},\ldots,J_{\sigma*n_k}$ are subsets of J_σ and satisfying $\mathrm{int}(J_{\sigma*i})\bigcap \mathrm{int}(J_{\sigma*j})=\emptyset$ $(i\neq j)$. where $\mathrm{int}(A)$ denotes the interior of A;

(3) For any $k\geq 1$ and $\sigma\in D_{k-1}, 1\leq j\leq n_k$, we have $\frac{|J_{\sigma*j}|}{|J_\sigma|}=c_k$.

If $\mathfrak{F}$ is such a collection, $E:=\bigcap_{k\geq 1}\bigcup_{\sigma\in D_k} J_\sigma$ is called a homogeneous Moran set determined by $\mathfrak{F}$.

Lemma 5.2 ([4], [9])**.** *For the above defined homogeneous Moran set, we have*

$$\dim_H E\geq \liminf_{k\to\infty}\frac{\log n_1n_2\cdots n_k}{-\log c_1c_2\cdots c_{k+1}n_{k+1}}.$$

Lemma 5.3. *Assume that $\{s_n\}_{n=1}^\infty$ is a sequence of positive integers satisfying $\frac{1}{n}\sum_{k=1}^n s_k\to\infty$ as $n\to\infty$. Let $E=\{x\in\mathcal{I}:\deg e_n(x)-\deg\beta_0=s_n\}$. Then we have*

$$\dim_H E\geq \liminf_{n\to\infty}\frac{s_1+s_2+\cdots+s_n}{2(s_1+s_2+\cdots+s_n)+s_{n+1}}.$$

Proof. Let

$$D_n=\{\sigma:\sigma=(e_1,e_2,\ldots,e_n)\in H(\beta)^n,\ \deg e_k-\deg\beta_0=s_k,\ 1\leq k\leq n\}.$$

Put

$$E_0=\mathcal{I},\quad E_n=\bigcup_{(e_1,e_2,\ldots,e_n)\in D_n}\Delta(e_1,e_2,\ldots,e_n),\ \forall\, n\geq 1.$$

Then $E=\bigcap_{n=1}^{+\infty}E_n$.

Take $n_k=(q-1)q^{s_k}, c_k=q^{-2s_k}$. From the above structure, it follows that each component $\Delta(e_1,e_2,\ldots,e_{k-1})$ in E_{k-1} contains n_k many elements $\Delta(e_1,e_2,\ldots,e_k)$ in E_k with the same ratio c_k. Thus E is a standard homogeneous Moran set. By Lemma 5.2, we have

$$\begin{aligned}\dim_H E&\geq \liminf_{n\to\infty}\frac{n\log(q-1)+(s_1+\cdots+s_n)\log q}{-\log(q-1)+(2(s_1+\cdots+s_n)+s_{n+1})\log q}\\ &\geq \liminf_{n\to\infty}\frac{s_1+s_2+\cdots+s_n}{2(s_1+s_2+\cdots+s_n)+s_{n+1}}.\end{aligned}$$

□

Take $s_1 = \phi(1)$, $s_n = \phi(n) - \phi(n-1)$ for $n \geq 2$. It is easy to see that $E \subset E(\phi)$. In fact, for any $n \geq 1, \deg e_n = s_n + \deg \beta_0$, therefore

$$\lim_{n\to\infty} \frac{\deg e_1(x) + \cdots + \deg e_n(x)}{\phi(n)} = \lim_{n\to\infty} \frac{\phi(n) + n \deg \beta_0}{\phi(n)} = 1,$$

since $\phi(n)/n \to \infty$ as $n \to \infty$. By lemma 5.3, we have

$$\dim_H E(\phi) \geq \liminf_{n\to\infty} \frac{\phi(n)}{\phi(n) + \phi(n+1)} = \frac{1}{1+b}.$$

5.2. Upper bound. Set

$$E^*(\phi) := \left\{ x \in \mathcal{I} : \lim_{n\to\infty} \frac{\deg e_1(x) + \cdots + \deg e_n(x) - n \deg \beta_0}{\phi(n)} = 1 \right\}.$$

It is easy to see that $E^*(\phi) = E(\phi)$, so we only need estimate the upper bound of $\dim_H E^*(\phi)$. Since ϕ is monotonic increasing, we have $b \geq 1$. The proof is distinguished into three cases according to $1 < b < \infty$, $b = 1$ and $b = \infty$.

Case I: $1 < b < \infty$.

We begin with the construction of a family of measures $\{\mu_t, t > 1\}$. For any $t > 1$ and $\{b_1, b_2, \ldots, b_n\} \subset H(\beta)$ with $\deg b_j \geq \deg \beta_0 + 1 (1 \leq j \leq n)$, set

$$\mu_t(\Delta(b_1, b_2, \ldots, b_n)) = q^{-t \sum_{j=1}^{n} (\deg b_j - \deg \beta_0) - nP(t)},$$

where

$$P(t) = \log_q(q(q-1)) - \log_q(q^t - q).$$

By using the equalities

$$\begin{aligned} \sum_{b:\deg b \geq \deg \beta_0 + 1} q^{-t(\deg b - \deg \beta_0)} &= \sum_{n=1}^{\infty} \sum_{\deg b = \deg \beta_0 + n} q^{-tn} \\ &= \sum_{n=1}^{\infty} q^{-tn}(q-1)q^n = \frac{q(q-1)}{q^t - q} = q^{P(t)}, \end{aligned}$$

it is easy to check that

$$\sum_{b_{n+1}:\deg b_{n+1} \geq \deg \beta_0 + 1} \mu_t(\Delta(b_1, b_2, \ldots, b_{n+1})) = \mu_t(\Delta(b_1, b_2, \ldots, b_n)),$$

$$\sum_{b_1, b_2, \ldots, b_n} \mu_t(\Delta(b_1, b_2, \ldots, b_n)) = 1,$$

where the sum is taken over all b_j and $\deg b_j \geq \deg \beta_0 + 1$. So the measure μ_t is well defined.

Fix $t > 1$ and $\varepsilon > 0$. By the given condition $\phi(n)/n \to \infty$ and the definition of b, we have

- There exists $N(t,\varepsilon) \in \mathbb{N}$ such that for all $n \geq N(t,\varepsilon)$,

$$nP(t) \leq \varepsilon(1-\varepsilon)\phi(n). \tag{5.1}$$

- We can choose a subsequence $\{n_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ with $n_k \geq N(t,\varepsilon)$ for all $k \geq 1$ and

$$\phi(n_k+1) \geq \phi(n_k)b(1-\varepsilon). \tag{5.2}$$

Now we give a cover of the set $E^*(\phi)$. For any $n \geq 1$, let

$$\Delta_n(\varepsilon) = \Big\{(e_1,\ldots,e_n) \in H(\beta)^n : (1-\varepsilon) < \frac{1}{\phi(n)}\sum_{j=1}^{n}(\deg e_j - \deg\beta_0) < (1+\varepsilon)\Big\} \tag{5.3}$$

For any $(e_1,\ldots,e_n) \in \Delta_n(\varepsilon)$, let

$$D_{n+1}(\varepsilon;(e_1,\ldots,e_n)) = \{e_{n+1} \in H(\beta) : (e_1,\ldots,e_n,e_{n+1}) \in \Delta_{n+1}(\varepsilon)\}$$

and

$$J(e_1,\ldots,e_n) = \bigcup_{e_{n+1}\in D_{n+1}(\varepsilon;(e_1,\ldots,e_n))} \Delta(e_1,\ldots,e_n,e_{n+1}).$$

Then

$$E^*(\phi) \subset \bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}\bigcup_{(e_1,\ldots,e_n)\in\Delta_n(\varepsilon)} J(e_1,\ldots,e_n). \tag{5.4}$$

For each $N \geq 1$, $(e_1,\ldots,e_{n_k}) \in \Delta_{n_k}(\varepsilon)$ with $n_k \geq N$, we will estimate the length of $J(e_1,\ldots,e_{n_k})$.

For any $e_{n_k+1} \in D_{n_k+1}(\varepsilon;(e_1,\ldots,e_{n_k}))$, by the definition of $\Delta_n(\varepsilon)$ and $D_{n+1}(\varepsilon;(e_1,\ldots,e_n))$, together with (5.2), we have

$$\sum_{j=1}^{n_k+1}(\deg e_j - \deg\beta_0) \geq \phi(n_k+1)(1-\varepsilon) \geq \phi(n_k)b(1-\varepsilon)^2 \geq \frac{b(1-\varepsilon)^2}{1+\varepsilon}\sum_{j=1}^{n_k}(\deg e_j - \deg\beta_0).$$

Thus

$$\deg e_{n_k+1} - \deg\beta_0 \geq \left(\frac{b(1-\varepsilon)^2}{1+\varepsilon} - 1\right)\sum_{j=1}^{n_k}(\deg e_j - \deg\beta_0). \tag{5.5}$$

Write $\gamma = \left(\frac{b(1-\varepsilon)^2}{1+\varepsilon} - 1\right)$, by Proposition 2 we have

$$\begin{aligned}
|J(e_1,\ldots,e_{n_k})| &\le \sum_{e_{n_k+1}:\deg e_{n_k+1}-\deg\beta_0\ge\gamma\sum_{j=1}^{n_k}(\deg e_j-\deg\beta_0)} |\Delta(e_1,\ldots,e_{n_k},e_{n_k+1})| \\
&\le \sum_{k=\gamma\sum_{j=1}^{n_k}(\deg e_j-\deg\beta_0)}^{\infty} \sum_{e_{n_k+1}:\deg e_{n_k+1}-\deg\beta_0=k} |\Delta(e_1,\ldots,e_{n_k},e_{n_k+1})| \\
&\ll q^{-2\sum_{j=1}^{n_k}(\deg e_j-\deg\beta_0)} \sum_{k=\gamma\sum_{j=1}^{n_k}(\deg e_j-\deg\beta_0)}^{\infty} q^{-k} \\
&\ll |\Delta(e_1,\ldots,e_{n_k})|^{\frac{2+\gamma}{2}}.
\end{aligned}$$

For each $(e_1,\ldots,e_{n_k}) \in \Delta_{n_k}(\varepsilon)$, by (5.1), we have

$$\begin{aligned}
|\Delta(e_1,\ldots,e_{n_k})|^{\frac{t+\varepsilon}{2}} &\ll q^{-(t+\varepsilon)\sum_{j=1}^{n_k}(\deg e_j-\deg\beta_0)} \\
&\ll q^{-t\sum_{j=1}^{n_k}(\deg e_j-\deg\beta_0)-n_kP(t)} = \mu_t(\Delta(e_1,\ldots,e_{n_k})).
\end{aligned}$$

After these preliminaries, we estimate the $\frac{t+\varepsilon}{2+\gamma}$-dimensional Hausdorff measure of $E^*(\phi)$.

$$\begin{aligned}
\mathcal{H}^{\frac{t+\varepsilon}{2+\gamma}}(E^*(\phi)) &\le \liminf_{k\to\infty} \sum_{(e_1,\ldots,e_{n_k})\in\Delta_{n_k}(\varepsilon)} |J(e_1,\ldots,e_{n_k})|^{\frac{t+\varepsilon}{2+\gamma}} \\
&\ll \liminf_{k\to\infty} \sum_{(e_1,\ldots,e_{n_k})\in\Delta_{n_k}(\varepsilon)} \left(|\Delta(e_1,\ldots,e_{n_k})|^{\frac{2+\gamma}{2}}\right)^{\frac{t+\varepsilon}{2+\gamma}} \\
&\ll \liminf_{k\to\infty} \sum_{(e_1,\ldots,e_{n_k})\in\Delta_{n_k}(\varepsilon)} \mu_t(\Delta(e_1,\ldots,e_{n_k})) \ll 1.
\end{aligned}$$

So, we have $\dim_H E^*(\phi) \le \frac{t+\varepsilon}{2+\gamma}$. Letting $\varepsilon \to 0$ and $t \to 1$, we obtain

$$\dim_H E^*(\phi) \le \frac{1}{1+b}.$$

Case II: $b = 1$.

In this case, it can be proved by just applying to the natural covering system

$$E^*(\phi) \subset \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcup_{(e_1,\ldots,e_n)\in\Delta_n(\varepsilon)} \Delta(e_1,\ldots,e_n). \tag{5.6}$$

Then, following from formula (5.1), we have

$$\begin{aligned}\mathcal{H}^{\frac{t+\varepsilon}{2}}(E^*(\phi)) &\leq \liminf_{k\to\infty} \sum_{(e_1,\ldots,e_{n_k})\in\Delta_{n_k}(\varepsilon)} |\Delta(e_1,\ldots,e_{n_k})|^{\frac{t+\varepsilon}{2}} \\ &\ll \liminf_{k\to\infty} \sum_{(e_1,\ldots,e_{n_k})\in\Delta_{n_k}(\varepsilon)} q^{-(t+\varepsilon)\sum_{j=1}^{n_k}(\deg e_j-\deg\beta_0)} \\ &\ll \liminf_{k\to\infty} \sum_{(e_1,\ldots,e_{n_k})\in\Delta_{n_k}(\varepsilon)} q^{-t\sum_{j=1}^{n_k}(\deg e_j-\deg\beta_0)-n_kP(t)} \\ &\ll \liminf_{k\to\infty} \sum_{(e_1,\ldots,e_{n_k})\in\Delta_{n_k}(\varepsilon)} \mu_t(\Delta(e_1,\ldots,e_{n_k})) \ll 1.\end{aligned}$$

It follows that $\dim_H E^*(\phi) \leq \frac{t+\varepsilon}{2}$. Letting $\varepsilon \to 0$ and $t \to 1$, we have

$$\dim_H E^*(\phi) \leq \frac{1}{2}.$$

Case III: $b = \infty$.

In this case, we replace b in case I by arbitrary large number n. Then, we have $\dim_H E^*(\phi) \leq \frac{1}{1+n}$. Letting $n \to \infty$, we have $\dim_H E^*(\phi) = 0$. □

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