

Rational points in geometric progressions on certain hyperelliptic curves

By ANDREW BREMNER (Tempe) and MACIEJ ULAS (Kraków)

Abstract. We pose a simple Diophantine problem which may be expressed in the language of geometry. Let C be a hyperelliptic curve given by the equation $y^2 = f(x)$, where $f \in \mathbb{Z}[x]$ is without multiple roots. We say that points $P_i = (x_i, y_i) \in C(\mathbb{Q})$ for $i = 1, 2, \dots, k$, are in geometric progression if the numbers x_i for $i = 1, 2, \dots, k$, are in geometric progression.

Let $n \geq 3$ be a given integer. In this paper we show that there exist polynomials $a, b \in \mathbb{Z}[t]$ such that on the curve $y^2 = a(t)x^n + b(t)$ (defined over the field $\mathbb{Q}(t)$) we can find four points in geometric progression. In particular this result generalizes earlier results of Bérczes and Ziegler concerning the existence of geometric progressions on Pell type quadrics $y^2 = ax^2 + b$. We also investigate for fixed $b \in \mathbb{Z}$, when there can exist rationals y_i , $i = 1, \dots, 4$, with $\{y_i^2 - b\}$ forming a geometric progression, with particular attention to the case $b = 1$. Finally, we show that there exist infinitely many parabolas $y^2 = ax + b$ which contain five points in geometric progression.

1. Introduction

Let $f \in \mathbb{Z}[x]$ be without multiple roots, and consider the hyperelliptic curve $C : y^2 = f(x)$. We say that the rational points $P_i = (x_i, y_i)$, $i = 1, 2, \dots, k$, lying on the curve C , are in geometric progression if the numbers $x_1, x_2, \dots, x_k$ are in geometric progression, i.e. there exist $p, t \in \mathbb{Q}$ such that $x_i = pt^i$ for $i = 1, \dots, k$. In a recent paper, BÉRCZES and ZIEGLER [1, Theorem 4] proved that for any four term geometric progression $0 < x_1 < x_2 < x_3 < x_4$ there exist infinitely many $a, b \in \mathbb{Z}$ such that there exist $y_i \in \mathbb{Q}$ with the property that the points $P_i = (x_i, y_i)$

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for $i = 1, 2, 3, 4$ lie on the curve $y^2 = ax^2 + b$. Moreover one can choose a, b in such a way that a is not a square, $b \neq 0$, and $\gcd(a, b)$ is squarefree. In view of this result it may be asked what can be proved in the case of a more general curve such as $y^2 = ax^n + b$, where $n \in \mathbb{N}_+$ is a fixed integer. More precisely: can one obtain a straight generalization of the cited result for the curve $y^2 = ax^n + b$, for all n ? This problem is most interesting in the case $n = 1$. In order to see this, note that we may concentrate on geometric progressions of the form $x_i = t^i$ for $i = 0, 1, \dots, k$; for if the points $P'_i = (x'_i, y_i)$ are in geometric progression on $C' : y^2 = ax^n + b$, with $x'_i = pt^i$, then the points $P_i = (t^i, y_i)$ lie in geometric progression on the curve $C : y^2 = ap^n x^n + b$, which is of the same type. We say that the geometric progression of the form t^i for $i = 0, 1, \dots, k-1$, is *the geometric progression generated by t of length k* . Next, note that we can indeed reduce the investigation to the case $n = 1$. For if the points $P_i = (t^i, y_i)$ lie in geometric progression on $C : y^2 = ax^n + b$, then the points $Q_i = (t^{in}, y_i)$ lie in geometric progression on the curve $y^2 = ax + b$. Thus, the problem for a given positive integer n is equivalent to the investigation of geometric progressions of the form $x_i = t^{in}$ on the parabola $y^2 = ax + b$. In other words, if we denote by $S(t, n)$ the problem of existence of four term geometric progressions generated by t on curves of type $y^2 = ax^n + b$, then we have the equivalence $S(t, n) \Leftrightarrow S(t^n, 1)$. We thus see that the cited result from [1] immediately implies that if n is even then the problem $S(t, n)$ has an affirmative answer. Henceforth, in this paper we shall consider only the case n odd. Moreover, we should note that the problem $S(t, 1)$ can also be rephrased as a problem of the existence of four values of the polynomial $(y^2 - b)/a$ which are in geometric progression. This is clearly equivalent to the investigation of the Diophantine system

$$\frac{Y^2 - b}{X^2 - b} = \frac{Z^2 - b}{Y^2 - b} = \frac{W^2 - b}{Z^2 - b}. \quad (1)$$

This system was investigated in [8], where it is shown that there exists a homogeneous polynomial $b \in \mathbb{Z}[u, v]$ of degree 18 such that there is a solution $X, Y, Z, W \in \mathbb{Z}[u, v]$ of (1). However, the discussion presented there is far from exhaustive. Indeed, in [8] we were interested in *integer* solutions of (1) only. This assumption is very restrictive, and rational solutions of the system are missed. For example when $b = -6$ there is a solution $(X, Y, Z, W) = (3/11, 3, 39/7, 453/49)$. A natural question arises as to whether there are infinitely many parametric solutions of the system (1) (where b is treated as a variable for this question).

The strategy we adopt is the following. The variety which parameterizes the instances of a, b such that there is a four term geometric progression, say

$1, t, t^2, t^3$, on the curve $y^2 = ax + b$ results in the study of a certain elliptic curve $\mathcal{C}$ defined over the field $\mathbb{Q}(t)$. Now $\mathcal{C}$ may also be viewed as an elliptic surface, and using the Shioda theory of elliptic surfaces, we compute the rank of $\mathcal{C}$ over $\mathbb{C}(t)$ (it is equal to one) and find on $\mathcal{C}$ a point of infinite order. This allow us to find infinitely many $b \in \mathbb{Q}(t)$ such that the system (1) has a non-trivial solution in polynomials $X, Y, Z, W \in \mathbb{Q}(t)$. Moreover, using the Silverman specialization theorem and a theorem of Hurwitz we prove that the set of all rational points on $\mathcal{C}$, that is, the set $\mathcal{C}(\mathbb{Q})$, is dense in the real topology in the set of all real points on $\mathcal{C}$.

Secondly, we investigate when there can exist solutions of the system (1) for a fixed value of b , and show that there exist infinitely many solutions for b of certain type, including $b = 1$. Note that the “three-term geometric progression” corresponding to the system

$$\frac{Y^2 - 1}{X^2 - 1} = \frac{Z^2 - 1}{Y^2 - 1}$$

has been much studied in the past; see, for example, GUY [3], Section D23, and the references given there; and ULAS [10].

Thirdly, we prove that there exist infinitely many distinct parabolas $y^2 = ax + b$ which contain *five* points in geometric progression. Finally, some computational remarks are made.

2. A parameterizing curve

In this section we construct a curve $\mathcal{E}$ defined over the field $\mathbb{Q}(t)$ which parameterizes pairs of rational functions a, b with the property that on the parabola $y^2 = f(x) = ax + b$ there lie four points in geometric progression, say the geometric progression $1, t, t^2, t^3$ generated by t of length 4. Demanding $a + b = U^2$, $at + b = V^2$, gives

$$a = \frac{U^2 - V^2}{1 - t}, \quad b = \frac{V^2 - tU^2}{1 - t}. \quad (2)$$

It remains to satisfy $f(t^2), f(t^3)$ both squares. Thus we investigate the curve $\mathcal{C}$ given by the intersection of the following two quadrics:

$$\mathcal{C} : -tU^2 + (1 + t)V^2 = R^2, \quad -t(1 + t)U^2 + (1 + t + t^2)V^2 = S^2. \quad (3)$$

We prove the following result.

Theorem 2.1. *Consider the curve $\mathcal{C}$ in $\mathbb{P}^3$ over $\mathbb{Q}(t)$ defined by (3). Then $\mathcal{C}$ is birationally equivalent over $\mathbb{Q}(t)$ to an elliptic curve $\mathcal{E}$ with $\text{rank } \mathcal{E}(\mathbb{Q}(t)) = 1$. Moreover, regarding $\mathcal{E}$ as a surface in $\mathbb{R}^3$ then the set $\mathcal{E}(\mathbb{Q}) \subset \mathbb{R}^3$ of all rational points is dense in the set $\mathcal{E}(\mathbb{R})$ of all real points lying on $\mathcal{E}$.*

PROOF. Taking $(1, 1, 1, 1)$ as the zero point, then a cubic model for the elliptic curve $\mathcal{C}$ is given by

$$\mathcal{E} : Y^2 = X(X + t^2)(X + t(1 + t)^2).$$

The discriminant $\Delta(\mathcal{E})$ of $\mathcal{E}$ is

$$\Delta(\mathcal{E}) = 16t^8(1 + t)^4(1 + t + t^2)^2,$$

so the specialization of $\mathcal{E}$ at $t \in \mathbb{C}$ is singular for the values $t \in \mathcal{A}$, where

$$\mathcal{A} = \left\{ \infty, -1, 0, -\frac{1 + \sqrt{-3}}{2}, -\frac{1 - \sqrt{-3}}{2} \right\}.$$

Now $\mathcal{E}$ represents a K3-surface, and the Néron–Severi group over $\mathbb{C}$, denoted by $\text{NS}(\mathcal{E}) = \text{NS}(\mathcal{E}, \mathbb{C})$, is a finitely generated $\mathbb{Z}$ -module. From SHIMODA [5], we have

$$\text{rank NS}(\mathcal{E}, \mathbb{C}) = \text{rank } \mathcal{E}(\mathbb{C}(t)) + 2 + \sum_{\nu} (m_{\nu} - 1),$$

where the sum ranges over all fibers of the pencil $\mathcal{E}_t$, with m_{ν} the number of irreducible components of the fiber. Recall that if the fiber in the pencil $\mathcal{E}_t$ is smooth then $m_{\nu} - 1 = 0$, thus the series on the right hand side is finite. For $t \in \mathcal{A}$, the decomposition is of Kodaira classification as follows. For $t = 0$ and $t = \infty$ we have type I_2^* , each with $m_{\nu} = 7$. For $t = -\frac{1 \pm \sqrt{-3}}{2}$ we have type I_2 and then $m_{\nu} = 2$. For $t = -1$ we have type I_4 and then $m_{\nu} = 4$. Summing up gives

$$\text{rank NS}(\mathcal{E}, \mathbb{C}) = \text{rank } \mathcal{E}(\mathbb{C}(t)) + 2 + 2(7 - 1) + (4 - 1) + 2(2 - 1).$$

Since the rank of the Néron–Severi group of a K3-surface cannot exceed 20, then $\text{rank } \mathcal{E}(\mathbb{C}(t)) \leq 1$. The curve $\mathcal{E}$ has three two-torsion points

$$T_1 = (0, 0), \quad T_2 = (-t^2, 0), \quad T_3 = (-t(1 + t)^2, 0)$$

and the point

$$P = (t^3(1 + t), t^3(1 + t)(1 + t + t^2)).$$

The height of P equals $3/4$, so that P is of infinite order; and hence $\mathcal{E}(\mathbb{Q}(t))$ (and so $\mathcal{E}(\mathbb{C}(t))$) has rank 1.

We will prove that the set of rational points on the surface $\mathcal{E}$ is dense in the Euclidean topology. However, we first prove Zariski density of the set of rational points. Because the curve $\mathcal{E}$ is of positive rank over $\mathbb{Q}(t)$, the set of multiples of the point P , i.e. $mP = (X_m(t), Y_m(t))$ for $m = 1, 2, \dots$, gives infinitely many $\mathbb{Q}(t)$ -rational points on the curve $\mathcal{E}$. Now, regarding the curve $\mathcal{E}$ as an elliptic surface in the space with coordinates (X, Y, t) we see that each rational curve (X_m, Y_m, t) is included in the Zariski closure, say $\mathcal{R}$, of the set of rational points on $\mathcal{E}$. Because this closure consists of only finitely many components, it has dimension two, and as the surface $\mathcal{E}$ is irreducible, $\mathcal{R}$ is the whole surface. Thus the set of rational points on $\mathcal{E}$ is dense in the Zariski topology.

To obtain the density of the set $\mathcal{E}(\mathbb{Q})$ in the Euclidean topology, we use two beautiful results: a theorem of HURWITZ [4] (see also [7, p. 78]) and a theorem of SILVERMAN [6, p. 368]. The theorem of Hurwitz states that if an elliptic curve E defined over $\mathbb{Q}$ has positive rank and one torsion point of order two (defined over $\mathbb{Q}$) then the set $E(\mathbb{Q})$ is dense in $E(\mathbb{R})$. The same result holds if E has three torsion points of order two under the assumption that we have a rational point of infinite order on the bounded branch of the set $E(\mathbb{R})$.

Silverman's theorem states that if $\mathcal{E}$ is an elliptic curve defined over $\mathbb{Q}(t)$ with positive rank, then for all but finitely many $t_0 \in \mathbb{Q}$, the curve $\mathcal{E}_{t_0}$ obtained from the curve $\mathcal{E}$ by specialization at $t = t_0$ has positive rank. From this result we see that for all but finitely many $t \in \mathbb{Q}$ the elliptic curve $\mathcal{E}_t$ is of positive rank. Denote by $\mathcal{G}$ the set of $t \in \mathbb{Q}$ such that the specialization P_t of the point P at $t \in \mathbb{Q}$ is of infinite order on the curve $\mathcal{E}_t$. From Mazur's theorem we know that the order of a torsion point on an elliptic curve defined over $\mathbb{Q}$ is at most 12. Thus, in order to find $\mathcal{G}$ it is enough to find all $t \in \mathbb{Q}$ such that P_t has finite order. This is straightforward: compute the expression $mP = (X(m), Y(m))$ for $m \in \mathbb{N}$ and $m \leq 12$, and determine for any given m those $t \in \mathbb{Q}$ such that the denominator of $X(m)$ has a zero at t . The only $t \in \mathbb{Q}$ with such a property for which $\mathcal{E}_t$ is nonsingular is $t = 1$. In this case P_1 is of order four on the curve $\mathcal{E}_1$. Moreover the rank of $\mathcal{E}_1(\mathbb{Q})$ is equal to zero. We thus get that $\mathcal{G} = \mathbb{Q} \setminus \{-1, 0, 1\}$. Note that the values $t = -1, 0, 1$ are without interest because they lead to trivial geometric progressions.

Now define the polynomial $X_i(t)$ to be the X -coordinate of the torsion point T_i for $i = 1, 2, 3$. We have the following equalities

$$\begin{aligned} P + T_1 &= (t + 1, -(1 + t)(1 + t + t^2)), \\ P + T_2 &= (-t(t + 1), t^2(t + 1)), \end{aligned}$$

$$P + T_3 = (-t^2(t+1), -t^3(t+1)),$$

and it is straightforward to verify the following inequalities:

$$\begin{aligned} X_2(t) < X_{P+T_1}(t) < X_1(t) < X_3(t) & \text{for } t \in (-\infty, -1), \\ X_2(t) < X_{P+T_3}(t) < X_1(t) < X_2(t) & \text{for } t \in (-1, 0), \\ X_3(t) < X_{P+T_3}(t) < X_2(t) < X_1(t) & \text{for } t \in (0, \infty). \end{aligned}$$

For each $i = 1, 2, 3$, the point $P + T_i$ is of infinite order on the curve $\mathcal{E}$. Moreover, for all $t \in \mathcal{G}$ and $i = 1, 2, 3$, the specialization of the point $P + T_i$ is of infinite order on the curve $\mathcal{E}_t$. From the above inequalities we deduce that for all $t \in \mathcal{G}$ there is a point of infinite order lying on the bounded branch of the real curve $\mathcal{E}_t$. Using the Hurwitz theorem, it follows that for all $t \in \mathcal{G}$ the set $\mathcal{E}_t(\mathbb{Q})$ is dense in the set $\mathcal{E}_t(\mathbb{R})$. This proves that the set $\mathcal{E}(\mathbb{Q})$ is dense in the set $\mathcal{E}(\mathbb{R})$ in the Euclidean topology. Note, it follows from the birational equivalence that $\mathcal{C}(\mathbb{Q})$ is dense in $\mathcal{C}(\mathbb{R})$. $\square$

Remark 2.2. Consider the system (1). If X, Y, Z, W is a solution of (1) for some b , and the common value of the equalities is t , it follows immediately that there exists a such that

$$X^2 - b = a, \quad Y^2 - b = at, \quad Z^2 - b = at^2, \quad W^2 - b = at^3.$$

Solving the first three equations with respect to a, b, t gives

$$a = \frac{(X^2 - Y^2)^2}{X^2 - 2Y^2 + Z^2}, \quad b = \frac{-Y^4 + X^2 Z^2}{X^2 - 2Y^2 + Z^2}, \quad t = \frac{Y^2 - Z^2}{X^2 - Y^2}.$$

Substituting into the fourth equation,

$$t = \frac{Y^2 - Z^2}{X^2 - Y^2} = \frac{Z^2 - W^2}{Y^2 - Z^2}. \quad (4)$$

If we are not interested in the value of t , we need to investigate the projective surface $(Y^2 - Z^2)^2 = (X^2 - Y^2)(Z^2 - W^2)$, and essentially this approach was used in [8] in order to find one polynomial solution of the system. If we are interested in the solutions (4) with given t , this leads to the intersection of the two quadratic surfaces

$$Z^2 = -tX^2 + (1+t)Y^2, \quad W^2 = -t(t+1)X^2 + (1+t+t^2)Y^2,$$

which, on renaming the variables, is exactly the same intersection defining the curve $\mathcal{C}$ from (3). Theorem 2.1 now implies that the set of rational points on the surface $(Y^2 - Z^2)^2 = (X^2 - Y^2)(Z^2 - W^2)$ is dense in the Euclidean topology.

Example 2.3. Using the pullbacks on $\mathcal{C}$ of the points P and $mP + T_i$ for $m \in \mathbb{Z}$ and $i = 1, 2, 3$, one can compute $a(t)$, $b(t)$, given by (2); and without loss of generality, a , b may be taken as polynomials in $\mathbb{Z}[t]$. For such b , provided that $a \neq 0$, we get a solution of the system (1). The pullbacks when $m = 1$ lead only to trivial solutions; the pullback of $2P$ however leads to

$$a = 8(1+t)(1+t^2), \quad b = (-1-t-3t^2+t^3)(-1+3t+t^2+t^3),$$

with solution of system (1) given by

$$(X, Y, Z, W) = (t^3 - t^2 - t - 3, t^3 - t^2 + 3t + 1, t^3 + 3t^2 - t + 1, -3t^3 - t^2 - t + 1).$$

Note that the degree of b is equal to six. This improves upon the degree 18 polynomial obtained in [8]. The expressions $X^2 - b$, $Y^2 - b$, $Z^2 - b$, $W^2 - b$ are in geometric progression with quotient t . This progression is clearly non-trivial for $t \neq -1, 0, 1$.

Returning to the initial question about the existence of geometric progressions on hyperelliptic curves of the form $y^2 = ax^n + b$, we note the following.

Corollary 2.4. *Let n be a given positive (odd) integer. For any nontrivial four term geometric progression x_i , $i = 1, 2, 3, 4$, there exist infinitely many pairwise non-isomorphic hyperelliptic curves $C : y^2 = ax^n + b$ such that x_i is the x -coordinate of a rational point on C .*

PROOF. It is clear that we may assume $x_i = u^i$, say, for $i = 0, 1, 2, 3$. In the previous example it was shown that there exist infinitely many rational functions $a_m(t)$, $b_m(t)$ (corresponding to the point mP) such that $a_m(t)t^i + b_m(t)$ is the square of a rational function, say $r_m(t)$, for $i = 0, 1, 2, 3$, and $m = 1, 2, \dots$. Putting $t = u^n$ we immediately obtain $a_m(u^n)(u^i)^n + b_m(u^n) = r_m(u^n)^2$. This implies that for $i = 0, 1, 2, 3$, the point $(x_i, r_m(t))$ lies on the hyperelliptic curve $C : y^2 = a_m(t)x^n + b_m(t)$ for $m = 1, 2, \dots$. That there are infinitely many distinct such curves is a simple consequence of the following reasoning. Let $C_m : y^2 = a_m(t)x^n + b_m(t)$, where m is a positive integer. The coefficients a_m and b_m are given by (2) and are calculated from the $\mathbb{Q}(t)$ -rational point mP on the curve $\mathcal{C}$, where P is the point of infinite order on $\mathcal{E}$ given in the proof of Theorem 2.1. Note that the curves C_p and C_q are isomorphic if and only if

$$a_p(t)^{n-1}b_p(t) = a_q(t)^{n-1}b_q(t)W^{2n}$$

for some $W \in \mathbb{Q}(t)$. Suppose we have constructed the integers $k_1, k_2, \dots, k_m$ such that the curves C_{k_i} are pairwise non-isomorphic over $\mathbb{Q}(t)$. Consider the m curves

$$C^i : a(U, V)^{n-1}b(U, V) = a_{k_i}(t)^{n-1}b_{k_i}(t)W^{2n}$$

for $i = 1, 2, \dots, m$, where $a(U, V)$, $b(U, V)$ are given by (2). The polynomial defining the curve C^i is homogenous of degree $2n$ in the coordinates $(U : V : W)$; and it is clear from (2) that C^i is defined over $\mathbb{Q}(t)$. The curve C^i for $i = 1, 2, \dots, m$ is of genus ≥ 2 , so that the set $C^1(\mathbb{Q}(t)) \cup \dots \cup C^m(\mathbb{Q}(t))$ is finite (this is just the function field analogue of Faltings Theorem [2]). Because the elliptic curve $\mathcal{C}$ has infinitely many rational points we can find an integer $k_{m+1} > k_m$ such that the curve $C_{k_{m+1}}$ is not isomorphic over $\mathbb{Q}(t)$ to any of the curves C_{k_i} for $i = 1, 2, \dots, m$. By induction we can construct an infinite set $\mathcal{A}$ with the required property. $\square$

Corollary 2.5. *There exists $k \in \mathbb{Z}[t]$ such that on the elliptic curve $\mathcal{C} : y^2 = x^3 + k$ there are four independent rational points in geometric progression.*

PROOF. In order to prove the result it is enough to take

$$k(t) = (1 + t^3)^2(1 + t^6)(-1 - t^3 - 3t^6 + t^9)(-1 + 3t^3 + t^6 + t^9).$$

This corresponds to the values of a, b presented in Example 2.3, and is equal to $\frac{1}{64}a(t^3)^2b(t^3)$. Then on the curve $\mathcal{C} : y^2 = x^3 + k(t)$ we have the four points in geometric progression:

$$\begin{aligned} P_1 &= (2(1 + t^3)(1 + t^6), (1 + t^3)(1 + t^6)(-3 - t^3 - t^6 + t^9)), \\ P_2 &= (2t(1 + t^3)(1 + t^6), (1 + t^3)(1 + t^6)(1 + 3t^3 - t^6 + t^9)), \\ P_3 &= (2t^2(1 + t^3)(1 + t^6), (1 + t^3)(1 + t^6)(1 - t^3 + 3t^6 + t^9)), \\ P_4 &= (2t^3(1 + t^3)(1 + t^6), (1 + t^3)(1 + t^6)(-1 + t^3 + t^6 + 3t^9)). \end{aligned}$$

The above points are seen to be independent in the group $\mathcal{C}(\mathbb{Q}(t))$ by means of a simple specialization argument. Specialize the curve $\mathcal{C}$ at $t = 2$ to get the elliptic curve

$$\mathcal{C}_2 : y^2 = x^3 + 63752753025.$$

The points P_i , $i = 1, 2, 3, 4$, specialize respectively to

$$\begin{aligned} R_1 &= (1170, 255645), & R_2 &= (2340, 276705), \\ R_3 &= (4680, 407745), & R_4 &= (9360, 940095), \end{aligned}$$

and the determinant of the height pairing matrix of the four points is equal to 326.8430126208496567501056976. This proves independence of the points R_i on the curve $\mathcal{C}_2$, and thus the independence of the points P_i on the curve $\mathcal{C}$. $\square$

Remark 2.6. The result above is very similar to the one obtained in ULAS [9], where it is proved that there exists $k \in \mathbb{Z}[t]$ such that on the curve $\mathcal{E} : y^2 = x^3 + k(t)$ there are four points in arithmetic progression which are independent in the group $\mathcal{E}(\mathbb{Q}(t))$.

Further, it is possible to prove that for each odd n which is divisible by 3 there exists $k \in \mathbb{Z}[t]$ such that the rank of the Jacobian of the curve $A : Y^2 = X^n + K(t)$ defined over $\mathbb{Q}(t)$ is greater than or equal to 4. Indeed, it is enough to take $K(t) = k(t^{n/3})$, where $k \in \mathbb{Z}[t]$ is given above. Then there is a map from A to the elliptic curve $C' : y^2 = x^3 + K(t)$ given by $(X, Y) \mapsto (X^{n/3}, Y)$. Thus C' is a factor of the Jacobian $\mathcal{J}(A)$ (up to isogeny), which implies that $\text{rank } \mathcal{J}(A(\mathbb{Q}(t))) \geq \text{rank } C'(\mathbb{Q}(t))$. From the reasoning presented in Corollary 2.5 it follows that the rank of C' over $\mathbb{Q}(t)$ is greater than or equal to 4, and thus the same is true for $\mathcal{J}(A(\mathbb{Q}(t)))$.

3. The case of fixed b

It is interesting to ask whether there exist solutions in rationals of the system (1) for a given squarefree integer value of b . We do not know how to answer this question, but can make some inroads.

Consider the intersection of (1) with

$$X^2 Z^2 = (4X^2 - 3Y^2)Y^2.$$

It follows from the first equation at (1) that

$$Y^2(4X^2 - 3b) = bX^2. \quad (5)$$

We suppose that b is of the form $b = c^2 + 3d^2$, so that this latter curve (5) of genus 0 may be parametrized by

$$X = b(m^2 + 3n^2)/(2(dm^2 - 2cmn - 3dn^2)), \quad (6)$$

$$Y = b(m^2 + 3n^2)/(2(cm^2 + 6dmn - 3cn^2)); \quad (7)$$

and the requirement that $4X^2 - 3Y^2 = (XZ/Y)^2$ gives

$$Z = b(m^2 + 3n^2)k/(2(cm^2 + 6dmn - 3cn^2)^2), \quad (8)$$

where

$$(4c^2 - 3d^2)m^4 + 60cdm^3n - 18(2c^2 - 9d^2)m^2n^2 - 180cdmn^3 + 9(4c^2 - 3d^2)n^4 = k^2. \quad (9)$$

The second equation at (1) gives, on eliminating Z , then Y :

$$W^2 = bX^2(8X^2 - 9b)^2/(4X^2 - 3b)^3 = (8X^2 - 9b)^2Y^6/(b^2X^4),$$

so that

$$\begin{aligned} W &= (8X^2 - 9b)Y^3/(bX^2) \\ &= \frac{b(m^2 + 3n^2)((2c^2 - 3d^2)m^4 + 36cdm^3n}{2(cm^2 + 6dmn - 3cn^2)^3} \\ &\quad + \frac{-6(4c^2 - 15d^2)m^2n^2 - 108cdmn^3 + 9(2c^2 - 3d^2)n^4}{2(cm^2 + 6dmn - 3cn^2)^3}. \end{aligned} \quad (10)$$

Then (X, Y, Z, W) at (6), (7), (8), (10), give a solution of (1) with $b = c^2 + 3d^2$, and common ratio equal to

$$-3(dm^2 - 2cmn - 3dn^2)^2/(cm^2 + 6dmn - 3cn^2)^2.$$

It follows that when the quartic (9) represents an elliptic curve of positive rank, then there will be infinitely many distinct rational solutions of the system (1) when $b = c^2 + 3d^2$. In the range $1 \leq b < 100$, the only such values of b occur for $b = 1, 19, 31, 61, 79$.

Perhaps the most interesting case is $(c, d) = (1, 0)$, with $b = 1$. The curve (9) is now

$$k^2 = 4(m^4 - 9m^2n^2 + 9n^4),$$

with cubic model

$$y^2 = x^3 - 63x + 162,$$

of rational rank 1, and generator $(x, y) = (1, 10)$. Modulo torsion, the multiples of the generator pull back to $(m, n) = (0, 1), (1, 1), (4, 1), (-45, 7), \dots$ The first two result in trivial solutions, and the next two give the following solutions of (1), in the case $b = 1$:

$$\begin{aligned} (X, Y, Z, W) &= (-19/16, 19/26, 209/169, 1387/2197), \\ &\quad (181/105, 181/313, 108419/97969, 29478203/30664297), \dots \end{aligned}$$

More generally, the rank of (9) can be forced to be positive by setting $4c^2 - 3d^2 = e^2$, say. Take

$$(c, d, e) = (g^2 + 3h^2, \quad g^2 + 2gh - 3h^2, \quad g^2 - 6gh - 3h^2),$$

so that

$$b = 4(g^4 + 3g^3h - 9gh^3 + 9h^4),$$

and (9) becomes

$$\begin{aligned} k^2 = & (g^2 - 6gh - 3h^2)^2 m^4 + 60(g - h)(g + 3h)(g^2 + 3h^2)m^3n \\ & + 18(7g^4 + 36g^3h - 30g^2h^2 - 108gh^3 + 63h^4)m^2n^2 \\ & - 180(g - h)(g + 3h)(g^2 + 3h^2)mn^3 + 9(g^2 - 6gh - 3h^2)^2n^4. \end{aligned}$$

With $(m, n, k) = (0, 1, 3(g^2 - 6gh - 3h^2))$ as zero of the group, then the point $(m, n, k) = ((0, 1, -3(g^2 - 6gh - 3h^2)))$ has height 4, and is of infinite order. Thus the system (1) has infinitely many solutions in the case that

$$b \equiv (g^4 + 3g^3h - 9gh^3 + 9h^4) \pmod{\mathbb{Q}^{*2}}.$$

4. Some remarks on five points in geometric progression on $y^2 = ax + b$

The problem of finding five points in geometric progression on the parabola $y^2 = ax + b$ reduces to considering the system

$$x^2 - b = A/q^2, \quad y^2 - b = A/q, \quad z^2 - b = A, \quad t^2 - b = Aq, \quad u^2 - b = Aq^2, \quad (11)$$

where, by absorbing squares into x, y, z, t, u , and A , we may assume without loss of generality that b is a squarefree integer.

We show that there are infinitely many essentially distinct solutions of this system.

Solving for A, b, q ,

$$A = \frac{(z^2 - t^2)(y^2 - z^2)}{t^2 + y^2 - 2z^2}, \quad b = \frac{t^2y^2 - z^4}{t^2 + y^2 - 2z^2}, \quad q = \frac{z^2 - t^2}{y^2 - z^2},$$

and substituting into the remaining two equations,

$$x^2(t^2 - z^2) = y^2(t^2 - y^2) + y^2z^2 - z^4, \quad u^2(y^2 - z^2) = t^2(y^2 - t^2) + t^2z^2 - z^4.$$

Equivalently,

$$(t^2 - z^2)(y^2(t^2 - y^2) + y^2z^2 - z^4) = \square, \quad (y^2 - z^2)(t^2(y^2 - t^2) + t^2z^2 - z^4) = \square.$$

If we set

$$t^2(y^2 - 4z^2) = -3z^4,$$

then

$$(t^2 - z^2)(y^2(t^2 - y^2) + y^2z^2 - z^4) = \left(\frac{z(y^2 - z^2)(y^2 - 2z^2)}{y^2 - 4z^2} \right)^2,$$

and

$$(y^2 - z^2)(t^2(y^2 - t^2) + t^2z^2 - z^4) = (-4y^2 + 13z^2) \left(\frac{z^2(y^2 - z^2)}{y^2 - 4z^2} \right)^2.$$

Accordingly, we demand

$$-4y^2 + 13z^2 = \square, \quad y^2 - 4z^2 = -3\square, \quad (12)$$

the equation of an elliptic curve on taking $(1, 1, 3, 1)$ as zero. The curve has rational rank 1 with $P = (-1, -1, -3, 1)$ as generator. It follows that we can construct an infinite chain of solutions to the system (11) by pulling back multiples of the generator. Note that $b = (t^2y^2 - z^4)/(t^2 + y^2 - 2z^2)$ and $t^2(y^2 - 4z^2) = -3z^4$ imply that $b(y^2 - 5z^2) = -4z^4$. Thus, with (12), we have for fixed b that

$$b(y^2 - 5z^2) = -\square, \quad -4y^2 + 13z^2 = \square, \quad y^2 - 4z^2 = -3\square,$$

the equation of a curve of genus 5, with only finitely many rational points. Accordingly, infinitely many b arise from this construction. In particular, this implies that there are infinitely many distinct quadratic polynomials $f(x) = x^2 - b$ such that the set $f(\mathbb{Q})$ contains a non-constant geometric progression of length 5. This shows that the conjecture of ULAS [8, Conjecture 3.3] is false.

As example, the points

$$2P = (-23, 13, 9, 7), \quad 3P = (-1873, -1117, 1479, 703), \quad \dots$$

give rise to $(y, z) = (-23, 13), (-1873, -1117), \dots$ giving the following solutions to the system (11):

$$(b, A, q) = \left(79, -\frac{7110}{169}, -\frac{169}{147} \right), \quad (x, y, z, t, u) = \left(\frac{15089}{2197}, \frac{1817}{169}, \frac{79}{13}, \frac{79}{7}, \frac{237}{49} \right),$$

and

$$(b, A, q) = \left(682579, -\frac{385732218690}{1247689}, -\frac{1247689}{1482627} \right),$$

$$(x, y, z, t, u) = \left(\frac{691282564829}{1393668613}, \frac{1278470467}{1247689}, \frac{682579}{1117}, \frac{682579}{703}, \frac{336511447}{494209} \right),$$

etc.

5. Computational remarks

Finally, a search was undertaken for small solutions of (1) in the range $-100 < b < 100$, and the results are presented in the Table in the Appendix. The parameterization at (6)–(10) above was discovered by focussing on solutions in which the common ratio was of type $-3\Box$. It seems highly plausible that there should be other parameterizable solutions corresponding (say) to the common ratio being a square.

Two solutions found exhibit $Z = 0$. It is straightforward to analyze those solutions in which $XYZW = 0$. By symmetry, we may suppose $W = 0$ or $Z = 0$. In the former case, (1) reduces to

$$-tX^2 + (1+t)Y^2 = Z^2, \quad -t(1+t)X^2 + (1+t+t^2)Y^2 = 0.$$

But then $t(1+t)(1+t+t^2) = \Box$, representing an elliptic curve of rational rank 0. The finite rational points occur for $t = 0, -1$, affording no solution to the original problem.

In the latter case, (1) reduces to

$$-tX^2 + (1+t)Y^2 = 0, \quad -t(1+t)X^2 + (1+t+t^2)Y^2 = W^2.$$

Thus $t(1+t) = \Box$; set $t = 1/(u^2 - 1)$. Then $X^2 = u^2Y^2$, $Y^2 = (1 - u^2)W^2$, so put $u = 2v/(v^2 + 1)$, giving $X = 2v/(v^2 + 1)Y$, $W = (v^2 + 1)/(v^2 - 1)Y$, $b = (v^2 + 1)^2/(2(v^4 + 1))Y^2$. Accordingly, we have the infinite family

$$(X, Y, Z, W) = \left(\frac{4v(1+v^4)}{(1+v^2)^2}, \quad \frac{2(1+v^4)}{1+v^2}, \quad 0, \quad \frac{2(1+v^4)}{v^2-1} \right), \quad b = 2(1+v^4),$$

with common ratio $t = -(1+v^2)^2/(1-v^2)^2$. It is worth remarking that there will be infinitely many distinct solutions of the system (1) for b of the form $2(1+v_0^4)$. For a solution with such b , we demand $v \in \mathbb{Q}$ such that $b = 2(1+v_0^4) = 2(1+v^4)y^2$, equivalently, $(1+v_0^4)(1+v^4) = w^2$, say. This latter equation is that of an elliptic curve, with points given by $(\pm v, \pm w) = (v_0, 1+v_0^4), (1/v_0, (1+v_0^4)/v_0^2)$. Taking $(v_0, 1+v_0^4)$ as zero of the group, then the point $(v_0, -1-v_0^4)$ has height 4, and so is non-torsion. Its multiples (v, w) correspond to the solution

$$(X, Y, Z, W) = \left(\frac{4vw}{(1+v^2)^2}, \quad \frac{2w}{1+v^2}, \quad 0, \quad \frac{2w}{1-v^2} \right), \quad b = 2(1+v_0^4),$$

with common ratio $-(1+v^2)^2/(1-v^2)^2$.

6. Appendix

b	X	Y	Z	W	ratio
-95	16	29	49	81	8/3
-87	145/64	29/4	145/13	2581/169	256/169
-79	79/135	7979/765	82871/4335	766221/24565	684/289
-79	419/15	119/3	167/3	233/3	25/13
-74	3086/529	578/23	202/3	4678/27	529/81
-59	3599/361	1711/19	649	4661	361/7
-39	3	21	69	219	10
-39	5	11	19	31	5/2
-29	29/64	551/16	899/4	1450	208/5
-23	5251/1681	181/41	17/3	188/27	1681/1296
-11	3	7	13	23	3
-11	19/3	37	193	1003	27
-11	2085/529	3485/644	11105/1568	794405/87808	4761/3136
-7	23/27	29/9	17/3	9	9/4
-6	3/11	3	39/7	453/49	121/49
1	299/289	23/17	23/7	529/49	578/49
1	19/16	19/26	209/169	1387/2197	-192/169
1	9951/7168	771/448	699/308	2649/847	256/121
1	2201/1849	155/43	93/5	7471/75	12943/450
1	475799/243049	5357/493	487/7	21915/49	243049/5880
1	181/105	181/313	108419/97969	29478203/30664297	-33075/97969
2	557/368	13/8	229/124	4343/1922	2116/961
11	83/25	17/5	5	19	25
14	19/5	5	17	83	25
15	453/121	39/11	3	3/7	121/49
19	1349/343	247/49	19/7	19/3	-49/27
22	9/2	17/4	29/8	23/16	9/4
22	230/49	34/7	10	62	49
23	187933/52822	3335/1078	2231/946	24265/40678	2401/1849
29	6887/1331	733/121	17/11	11	-121/35
29	37/7	41/7	11/7	89/7	-5
31	31/5	31/7	341/49	713/343	-75/49
31	188/41	17/4	181/48	5251/1728	1681/1296
34	136/25	34/5	0	34/3	-25/9
34	10	32	122	472	-25/9
41	123/25	41/5	0	41/4	-25/16
41	4343/529	229/23	13	557/31	2116/961
43	6	31/4	19/16	769/64	-39/16
51	9771/961	579/31	543/13	16629/169	961/169
61	262/21	138/7	242/7	442/7	45/13
69	933/125	249/25	3/5	15	-25/11
69	7187/729	1267/81	307/9	83	81/13
78	62/7	10	34	230	49
79	97/11	103/11	47/11	247/11	-7
79	15089/2197	1817/169	79/13	79/7	-169/147
79	1817/169	79/13	79/7	237/49	-169/147

89	13	43	197	923	22
91	9	11	1	19	-3
93	2445/361	111/19	75/17	309/289	361/289
93	5815/529	337/23	25	541/11	529/121

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ANDREW BREMNER
SCHOOL OF MATHEMATICAL
AND STATISTICAL SCIENCES
ARIZONA STATE UNIVERSITY
TEMPE AZ 85287-1804
USA

E-mail: bremner@asu.edu

MACIEJ ULAS
JAGIELLONIAN UNIVERSITY
FACULTY OF MATHEMATICS
AND COMPUTER SCIENCE
INSTITUTE OF MATHEMATICS
ŁOJASIEWICZA 6
30-348 KRAKÓW
POLAND

E-mail: maciej.ulas@uj.edu.pl

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