

Generalized near-derivations and their applications in Lie algebras

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Abstract. Let L be a Lie algebra. The aim of this paper is to investigate generalized near-derivations of L , which is a generalization of near-derivation initiated by BREŠAR in 2008. As an application we determine all linear maps $f : L \rightarrow L$ with the property that $[\dots[[f, \delta_1], \delta_2], \dots, \delta_n]$ is a derivation whenever $\delta_1, \delta_2, \dots, \delta_n$ are derivations of L , where n is a fixed positive integer.

1. Introduction

Let n be a fixed positive integer. Let R be a not necessarily associative algebra with multiplication $\cdot$. Recall that a linear map $\delta : R \rightarrow R$ is said to be a derivation if $\delta(x \cdot y) = \delta(x) \cdot y + x \cdot \delta(y)$ for all $x, y \in R$. A linear map $f : R \rightarrow R$ is said to be a *generalized derivation* if there exist linear maps $g, h : R \rightarrow R$ such that

$$f(x) \cdot y = g(x \cdot y) + x \cdot h(y) \quad \text{for all } x, y \in R.$$

LEGER and LUKS [4] investigated generalized derivations of Lie algebras. As it is well known, $[\delta', \delta] = \delta'\delta - \delta\delta'$ is a derivation whenever δ and δ' are derivations. Is it possible to determine all linear maps $f : R \rightarrow R$ with the property that $[f, \delta]$ is a derivation whenever δ is a derivation? BREŠAR [2] discussed this question in Lie algebras. He answered this question by introducing an interesting concept of near-derivations in Lie algebras.

Mathematics Subject Classification: 16R60, 16W25, 17B60 17B65.

Key words and phrases: functional identities, generalized derivations, near-derivations, generalized near-derivations, Lie algebras.

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Let L be a Lie algebra over a field $\mathbb{F}$. By adx we denote the inner derivation induced by $x \in L$, i.e. $(\text{ad } x)(y) = [x, y]$ for all $y \in L$. A linear map $f : L \rightarrow L$ is said to be a *near-derivation* of L if there exists a linear map $g : L \rightarrow L$ such that

$$(\text{ad } x)f - g(\text{ad } x)$$

is a derivation for every $x \in L$ (see [2, Section 1]). Note that every generalized derivation is a near-derivation. BREŠAR gave a description of f in certain Lie algebras that arise from associative ones. The typical result in [2] states that a near-derivation f of L is of the form $f = \delta + \gamma I + \tau$, where δ is a derivation, γ is an element in the center C of a certain associative algebra containing L , and τ is a linear map from L into C .

It is natural for us to consider the following general question: How to determine all linear maps $f : L \rightarrow L$ with the property that $[\dots, [f, \delta_1], \delta_2], \dots, \delta_n]$ is a derivation whenever $\delta_1, \delta_2, \dots, \delta_n$ are derivations of L ?

We shall solve this problem on Lie algebras. For this purpose we shall extend the definition of near-derivations. Let Q be a unital associative algebra, containing L as its Lie subalgebra (the Q always exists). First, we give a slight generalization of near-derivations as follows: A linear map $f : L \rightarrow L$ is said to be a *near-derivation* if there exists a linear map $g : L \rightarrow Q$ such that

$$(\text{ad } x)f - g(\text{ad } x)$$

is a derivation for every $x \in L$. It is easy to see that the typical result $\delta + \gamma I + \tau$ in [2] is a near derivation in the above sense. Now, we say that a linear map $f : L \rightarrow L$ is a *generalized near-derivation* if there exists a linear map $g : L \rightarrow Q$ such that

$$(\text{ad } x)f - g(\text{ad } x)$$

is a near-derivation for every $x \in L$. It is easy to see that every near-derivation is a generalized near-derivation.

We shall apply the powerful theory of functional identities [3] to the descriptions of generalized near-derivations in certain Lie algebras that arise from associative ones.

2. Functional identities preliminaries

Let Q be a unital algebra and let S be a d -free subset of Q for some positive integer d . Denote by P as the set of all quasi-polynomials. We refer the reader to

the recent book [3] for the basic properties of d -free subsets and quasi-polynomials, upon which the present paper heavily rests.

The following result is a slight generalization of [2, Lemma 2].

Lemma 2.1. *Let Q be a unital algebra and let S be a subset of Q . Let $B : S \times S \rightarrow Q$ be a skew-symmetric map. Suppose that*

$$[B(x, y), z] + [B(z, x), y] + [B(y, z), x] \in P$$

for all $x, y, z \in S$. If S is a 4-free subset of Q , then there exist $\lambda \in C$, the center of Q , a linear map $\mu : S \rightarrow C$, and a skew-symmetric map $\nu : S^2 \rightarrow C$ such that

$$B(x, y) = \lambda[x, y] + \mu(x)y - \mu(y)x + \nu(x, y) \quad \text{for all } x, y \in S.$$

PROOF. Using [3, Theorem 4.13] it follows that B is a quasi-polynomial. This means that there exist $\lambda_1, \lambda_2 \in C$ and maps $\mu_1, \mu_2 : S \rightarrow C$, $\nu : S^2 \rightarrow C$ such that

$$B(x, y) = \lambda_1 xy + \lambda_2 yx + \mu_1(x)y + \mu_2(y)x + \nu(x, y).$$

Since $B(x, y) = -B(y, x)$ it follows that

$$(\lambda_1 + \lambda_2)(xy + yx) + (\mu_1 + \mu_2)(x)y + (\mu_1 + \mu_2)(y)x + \nu(x, y) + \mu(y, x) = 0.$$

But then $\lambda_1 = -\lambda_2$, $\mu_1 = -\mu_2$ and ν is skew-symmetric [3, Lemma 4.4]. Setting $\lambda = \lambda_1$ and $\mu = \mu_1$ we thus have

$$B(x, y) = \lambda[x, y] + \mu(x)y - \mu(y)x + \nu(x, y) \quad \text{for all } x, y \in S. \quad \square$$

Let A be a prime associative algebra. By $Q_{ml}(A)$ we denote the maximal left ring of quotients of A (see [1, Chapter 2] or [3, Appendix A]). The center C of $Q_{ml}(A)$ is a field called the extended centroid of A . By $\deg(x)$ we denote the degree of the algebraicity of $x \in A$ over C . If x is not algebraic, then we write $\deg(x) = \infty$. Further, we set

$$\deg(A) = \sup\{\deg(x) \mid x \in A\}.$$

The condition that $\deg(A) = \infty$ is equivalent to the condition that A is not a PI-algebra, while the condition that $\deg(A) = n < \infty$ is equivalent to the condition that A is a PI-algebra satisfying the standard polynomial identity of degree $2n$, but not satisfying a polynomial identity of degree $< 2n$. If A is a central simple algebra, then $\deg(A) = \infty$ is the same as saying that A is infinite-dimensional over $\mathbb{F}$, while $\deg(A) = n < \infty$ is equivalent to $\dim_{\mathbb{F}} A = n^2$ [3, Appendix C]. If $\deg(A) \geq d + 1$, then every noncommutative Lie ideal L of A is a d -free subset of $Q_{ml}(A)$ [3, Corollary 5.16]. If $\deg(A) \geq 2d + 3$, A has an involution and K is the set of skew elements in A , then every noncentral Lie ideal L of K is a d -free subset of $Q_{ml}(A)$ [3, Corollary 5.19].

3. Generalized near-derivations on Lie algebras

We begin with the following crucial result.

Lemma 3.1. *Let L be a Lie algebra with $\text{char}(\mathbb{F}) \neq 2, 3$ and let Q be a unital associative algebra, containing L as its Lie subalgebra. Let f be a generalized near-derivation of L . Suppose L is a 4-free subset of Q . Then there exist $\gamma \in C$, the center of Q , and a skew-symmetric bilinear map $\beta : L^2 \rightarrow C$ such that*

$$(f + \gamma I)([x, y]) = [f(x), y] + [x, f(y)] + \beta(x, y) \quad \text{for all } x, y \in L.$$

PROOF. Our assumption is that there exists $g : L \rightarrow Q$ such that $(\text{ad } x)f - g(\text{ad } x)$ is a near-derivation for every $x \in L$. In view of [2, Lemma 3.1] we have that there exists $\lambda_x \in C$ (depending on x) such that

$$\begin{aligned} ((\text{ad } x)f - g(\text{ad } x) + \lambda_x I)([y, z]) - [((\text{ad } x)f - g(\text{ad } x))(y), z] \\ - [y, ((\text{ad } x)f - g(\text{ad } x))(z)] \in C. \end{aligned}$$

That is,

$$\begin{aligned} [x, f([y, z])] - g([x, [y, z]]) + \lambda_x [y, z] - [[x, f(y)] - g([x, y]), z] \\ - [y, [x, f(z)] - g([x, z])] \in C \quad (1) \end{aligned}$$

for all $x, y, z \in L$. Now define $\theta : L \rightarrow C$ by the rule

$$\theta(x) = \lambda_x \quad \text{for all } x \in L.$$

We claim that θ is well-defined. It is enough to show that $x = 0$ implies $\lambda_x = 0$. Picking $x = 0$ in (1) we obtain $\lambda_0 [y, z] \in C$ for all $y, z \in L$. Applying [3, Lemma 4.4] we get $\lambda_0 = 0$ as desired. Hence, the identity (1) can be rewritten as

$$\begin{aligned} [x, f([y, z])] - g([x, [y, z]]) + \theta(x)[y, z] - [[x, f(y)] - g([x, y]), z] \\ - [y, [x, f(z)] - g([x, z])] \in C \quad (2) \end{aligned}$$

for all $x, y, z \in L$. In view of the Jacobi identity we have

$$g([x, [y, z]]) + g([z, [x, y]]) + g([y, [z, x]]) = 0;$$

according to (2) this can be rewritten as

$$\begin{aligned} [x, f([y, z])] - [[x, f(y)], z] + [g([x, y]), z] - [y, [x, f(z)]] + [y, g([x, z])] \\ + [z, f([x, y])] - [[z, f(x)], y] + [g([z, x]), y] - [x, [z, f(y)]] \\ + [x, g([z, y])] + [y, f([z, x])] - [[y, f(z)], x] + [g([y, z]), x] \\ - [z, [y, f(x)]] + [z, g([y, x])] + \theta(x)[y, z] \\ + \theta(z)[x, y] + \theta(y)[z, x] \in C. \end{aligned}$$

Rearranging the terms we get

$$\begin{aligned} & [(2g - f)([y, z]) - [f(y), z] - [y, f(z)], x] \\ & + [(2g - f)([x, y]) - [f(x), y] - [x, f(y)], z] \\ & + [(2g - f)([z, x]) - [f(z), x] - [z, f(x)], y] \\ & + \theta(x)[y, z] + \theta(z)[x, y] + \theta(y)[z, x] \in C \end{aligned} \quad (3)$$

for all $x, y, z \in L$. That is,

$$\begin{aligned} & [(2g - f)([y, z]) - [f(y), z] - [y, f(z)], x] \\ & + [(2g - f)([x, y]) - [f(x), y] - [x, f(y)], z] \\ & + [(2g - f)([z, x]) - [f(z), x] - [z, f(x)], y] \in P \end{aligned}$$

for all $x, y, z \in L$. We are now in a position to apply Lemma 2.1. Thus there exist $\lambda \in C$, $\mu_1 : L \rightarrow C$, and a skew-symmetric map $\nu : L^2 \rightarrow C$ such that

$$(2g - f)([x, y]) - [f(x), y] - [x, f(y)] = \lambda[x, y] + \mu_1(x)y - \mu_1(y)x + \nu(x, y). \quad (4)$$

Substituting (4) into (3) we obtain

$$(2\mu_1(x) + \theta(x))[y, z] + (2\mu_1(z) + \theta(z))[x, y] + (2\mu_1(y) + \theta(y))[z, x] \in C.$$

Applying [3, Lemma 4.4] it follows that $2\mu_1(x) + \theta(x) = 0$ for all $x \in L$ and hence $\mu_1 = -\frac{1}{2}\theta$. Thus, the expression (4) becomes

$$(2g - f)([x, y]) - [f(x), y] - [x, f(y)] = \lambda[x, y] - \frac{1}{2}\theta(x)y + \frac{1}{2}\theta(y)x + \nu(x, y). \quad (5)$$

Set $h = 2g - f - \lambda I$. Then

$$h([x, y]) - [f(x), y] - [x, f(y)] = -\frac{1}{2}\theta(x)y + \frac{1}{2}\theta(y)x + \nu(x, y). \quad (6)$$

Since

$$h([x, [y, z]]) + h([z, [x, y]]) + h([y, [z, x]]) = 0$$

for all $x, y, z \in L$, we get from (6) that

$$\begin{aligned} & [f(x), [y, z]] + [x, f([y, z])] + [f(z), [x, y]] + [z, f([x, y])] + [f(y), [z, x]] \\ & + [y, f([z, x])] - \frac{1}{2}\theta(x)[y, z] + \frac{1}{2}\theta([y, z])x - \frac{1}{2}\theta(z)[x, y] \\ & + \frac{1}{2}\theta([x, y])z - \frac{1}{2}\theta(y)[z, x] + \frac{1}{2}\theta([z, x])y \\ & + \nu(x, [y, z]) + \nu(z, [x, y]) + \nu(y, [z, x]) = 0. \end{aligned}$$

We rewrite this as

$$\begin{aligned}
& [f([y, z]) - [f(y), z] - [y, f(z)], x] + [f([x, y]) - [f(x), y] - [x, f(y)], z] \\
& + [f([z, x]) - [f(z), x] - [z, f(x)], y] - \frac{1}{2}\theta(x)[y, z] + \frac{1}{2}\theta([y, z])x \\
& - \frac{1}{2}\theta(z)[x, y] + \frac{1}{2}\theta([x, y])z - \frac{1}{2}\theta(y)[z, x] + \frac{1}{2}\theta([z, x])y \\
& + \nu(x, [y, z]) + \nu(z, [x, y]) + \nu(y, [z, x]) = 0.
\end{aligned} \tag{7}$$

That is,

$$\begin{aligned}
& [f([y, z]) - [f(y), z] - [y, f(z)], x] + [f([x, y]) - [f(x), y] - [x, f(y)], z] \\
& + [f([z, x]) - [f(z), x] - [z, f(x)], y] \in P.
\end{aligned}$$

Applying Lemma 2.1 again we get

$$f([x, y]) - [f(x), y] - [x, f(y)] = \alpha[x, y] + \mu_2(x)y - \mu_2(y)x + \beta(x, y) \tag{8}$$

for some $\alpha \in C$, $\mu_2 : L \rightarrow C$, and skew-symmetric map $\beta : L^2 \rightarrow C$. It is clear that the linearity of f implies the bilinearity of β .

Substituting (8) into (7) we obtain

$$\begin{aligned}
& (2\mu_2(x) - \frac{1}{2}\theta(x))[y, z] + (2\mu_2(z) - \frac{1}{2}\theta(z))[x, y] + (2\mu_2(y) - \frac{1}{2}\theta(y))[z, x] \\
& + \frac{1}{2}\theta([y, z])x + \frac{1}{2}\theta([x, y])z + \frac{1}{2}\theta([z, x])y \in C.
\end{aligned}$$

Applying [3, Lemma 4.4] it follows that $2\mu_2(x) - \frac{1}{2}\theta(x) = 0$ for all $x \in L$ and hence $\mu_2 = \frac{1}{4}\theta$. Thus, the expression (8) becomes

$$f([x, y]) - [f(x), y] - [x, f(y)] = \alpha[x, y] + \frac{1}{4}\theta(x)y - \frac{1}{4}\theta(y)x + \beta(x, y) \tag{9}$$

We claim that $\theta = 0$. Indeed, subtracting (9) from (5) we get

$$2(g - f)([x, y]) = (\lambda - \alpha)[x, y] - \frac{3}{4}\theta(x)y + \frac{3}{4}\theta(y)x + \nu(x, y) - \beta(x, y).$$

Set $k = 2g - 2f - (\lambda - \alpha)I$. Then

$$k([x, y]) = -\frac{3}{4}\theta(x)y + \frac{3}{4}\theta(y)x + \nu(x, y) - \beta(x, y). \tag{10}$$

Since

$$k([x, [y, z]]) + k([z, [x, y]]) + k([y, [z, x]]) = 0$$

by the Jacobi identity, we get from (10) that

$$-\theta(x)[y, z] + \theta([y, z])x - \theta(z)[x, y] + \theta([x, y])z - \theta(y)[z, x] + \theta([z, x])y \in C.$$

Applying [3, Lemma 4.4] it follows that $\theta(x) = 0$ for all $x \in L$.

Setting $\gamma = -\alpha$ we can get from (9) that

$$(f + \gamma I)([x, y]) = [f(x), y] + [x, f(y)] + \beta(x, y) \quad \text{for all } x, y \in L.$$

This proves the lemma. $\square$

We denote by $H_2(L, \mathbb{F})$ the second cohomology group of L . Applying Lemma 3.1 we have the following:

Theorem 3.1. *Let L be a Lie algebra with $\text{char}(\mathbb{F}) \neq 2, 3$ and $H_2(L, \mathbb{F}) = 0$. Let Q be a unital associative algebra, containing L as its Lie subalgebra. Let f be a generalized near-derivation of L . Suppose L is a 4-free subset of Q . Then there exist $\gamma \in C$, the center of Q , a derivation $\delta : L \rightarrow Q$ and a linear map $\tau : L \rightarrow C$ such that $f = \delta + \gamma I + \tau$.*

PROOF. By Lemma 3.1 the map $d = f - \gamma I : L \rightarrow CL \subseteq Q$ satisfies

$$d([x, y]) - [d(x), y] - [x, d(y)] = \beta(x, y) \in C$$

for all $x, y \in L$. Consequently,

$$\beta(x, [y, z]) = d([x, [y, z]]) - [d(x), [y, z]] - [x, [d(y), z]] - [x, [y, d(z)]],$$

since $[x, \beta(y, z)] = 0$. Using the Jacobi identity it readily follows that

$$\beta(x, [y, z]) + \beta(z, [x, y]) + \beta(y, [z, x]) = 0.$$

Since $H^2(L, \mathbb{F}) = 0$ then exists a linear map $\tau : L \rightarrow C$ such that $\beta(x, y) = \tau([x, y])$ for all $x, y \in L$ (see [2, P. 3769]). That is,

$$d([x, y]) - [d(x), y] - [x, d(y)] = \tau([x, y]).$$

It follows that $\delta = d - \tau$ is a derivation from L into Q . $\square$

Let L be a Lie algebra over $\mathbb{F}$. We denote $\text{End}(L)$ by the $\mathbb{F}$ -linear space of all $\mathbb{F}$ -linear transformations on L . Set

$$\text{Cent}(L) = \{\chi \in \text{End}(L) \mid \chi([x, y]) = [\chi(x), y] = [x, \chi(y)] \text{ for all } x, y \in L\}.$$

We call $\text{Cent}(L)$ the *centroid* of L . Applying Lemma 3.1 we also have the following:

Theorem 3.2. *Let L be a Lie algebra with $\text{char}(\mathbb{F}) \neq 2, 3$. Let Q be a unital associative algebra, containing L as its Lie subalgebra. Assume further that L has trivial center and $[L, L] = L$. Let f be a generalized near-derivation of L . Suppose that L is a 4-free subset of Q . Then there exist a derivation $\delta : L \rightarrow L$ and $\zeta \in \text{Cent}(L)$, the centroid of L , such that $f = \delta + \zeta$.*

PROOF. Lemma 3.1 implies that

$$\gamma[[x, y], z] = [[f(x), y], z] + [[x, f(y)], z] - [f([x, y]), z] \in L$$

for all $x, y, z \in L$. Since $[L, L] = L$, and hence also $[[L, L], L] = L$, it follows that $\gamma L \subseteq L$. That is, the map $\zeta : x \mapsto \gamma x$ maps L into L and so $\zeta \in \text{Cent}(L)$. Further,

$$\beta(x, y) = f([x, y]) + \gamma[x, y] - [f(x), y] - [x, f(y)]$$

then lies in $L \cap C$ which is zero since L has trivial center. But then $\delta = f - \zeta$ is a derivation of L . $\square$

Applying Lemma 3.1 and using the same arguments as in the corresponding results in [2, Section 3], we can obtain the following results. We omit their proofs for brevity.

Corollary 3.1. *Let A be a central simple algebra with $\text{char}(\mathbb{F}) \neq 2, 3$. Suppose that one of the following conditions is satisfied:*

- (i) $\dim_{\mathbb{F}} A \geq 25$ and $1 \in L = [A, A]$;
- (ii) Suppose that A has an involution of the first kind and $\dim_{\mathbb{F}} A \geq 121$. Let K be the set of skew elements in A , and set $L = [K, K]$.

Then every generalized near-derivation f of L is of the form $f = \delta + \gamma I$ where δ is a derivation of L and $\gamma \in \mathbb{F}$;

Corollary 3.2. *Let A be a prime algebra with $\text{char}(\mathbb{F}) \neq 2, 3$, let C be the extended centroid of A . Suppose that one of the following conditions is satisfied:*

- (i) Let L be a noncommutative Lie ideal of A . Suppose that $\deg(A) \geq 5$;

(ii) Suppose that A has an involution and $\deg(A) \geq 11$. Let K be the skew elements of A , and let L be a noncentral Lie ideal of K .

If f is a generalized near-derivation of L , then there exist an (associative) derivation $\delta : \langle L \rangle \rightarrow \langle L \rangle C + C$, $\gamma \in C$, and a linear map $\tau : L \rightarrow C$ such that $f = \delta + \gamma I + \tau$.

4. An application of generalized near-derivations

As an application of generalized near-derivations we shall answer the question posed in Section 1. More precisely, we have the following result.

Theorem 4.1. *Let L be a Lie algebra with $\text{char}(\mathbb{F}) \neq 2, 3$ and $H_2(L, \mathbb{F}) = 0$. Let f be a linear maps of L with the property that*

$$[\dots [[f, \delta_1], \delta_2], \dots, \delta_n]$$

is a derivation whenever $\delta_1, \delta_2, \dots, \delta_n$ are derivations of L . Suppose there exists a unital associative algebra Q , containing L as its Lie subalgebra, such that L is a 4-free subset of Q . Then there exist $\gamma \in C$, the center of Q , a derivation $\delta : L \rightarrow Q$ and a linear map $\tau : L \rightarrow C$ such that $f = \delta + \gamma I + \tau$.

PROOF. For every $x_1, x_2, \dots, x_n \in L$ we get from our assumption that

$$[\dots [[f, \text{ad } x_1], \text{ad } x_2], \dots, \text{ad } x_n]$$

is a derivation and so a near derivation. That is

$$[\dots [[f, \text{ad } x_1], \text{ad } x_2], \dots, \text{ad } x_{n-1}]$$

is a generalized near-derivation. By Theorem 3.1 we get that

$$[\dots [[f, \text{ad } x_1], \text{ad } x_2], \dots, \text{ad } x_{n-1}]$$

is also a near derivation. Following the same process we finally obtain that f is a near-derivation. Then the result follows from Theorem 3.1. $\square$

Similarly, applying the corresponding results in the above section we can obtain the following results. We omit their proofs for brevity.

Theorem 4.2. *Let L be a Lie algebra with $\text{char}(\mathbb{F}) \neq 2, 3$ such that L has trivial center and $[L, L] = L$. Let f be a linear maps of L with the property that*

$$[\dots [[f, \delta_1], \delta_2], \dots, \delta_n]$$

is a derivation whenever $\delta_1, \delta_2, \dots, \delta_n$ are derivations of L . Suppose that there

exists a unital associative algebra Q , containing L as its Lie subalgebra, such that L is a 4-free subset of Q . Then there exist a derivation $\delta : L \rightarrow L$ and $\zeta \in \text{Cent}(L)$, the centroid of L , such that $f = \delta + \zeta$.

Corollary 4.1. *Let A be a central simple algebra with $\text{char}(\mathbb{F}) \neq 2, 3$. Suppose that one of the following conditions is satisfied:*

- (i) $\dim_{\mathbb{F}} A \geq 25$ and $1 \in L = [A, A]$;
- (ii) Suppose that A has an involution of the first kind and $\dim_{\mathbb{F}} A \geq 121$. Let K be the set of skew elements in A , and $L = [K, K]$.

If f is a linear map of L with the property that

$$[\dots [[f, \delta_1], \delta_2], \dots, \delta_n]$$

is a derivation whenever $\delta_1, \delta_2, \dots, \delta_n$ are derivations of L , then $f = \delta + \gamma I$ where δ is a derivation of L and $\gamma \in \mathbb{F}$.

Corollary 4.2. *Let A be a prime algebra with $\text{char}(\mathbb{F}) \neq 2, 3$, let C be the extended centroid of A . Suppose that one of the following conditions is satisfied:*

- (i) Suppose that L is a noncommutative Lie ideal of A and $\deg(A) \geq 5$;
- (ii) Suppose that A has an involution with $\deg(A) \geq 11$, K is the skew elements of A and let L be a noncentral Lie ideal of K .

If f is a linear map of L with the property that

$$[\dots [[f, \delta_1], \delta_2], \dots, \delta_n]$$

is a derivation whenever $\delta_1, \delta_2, \dots, \delta_n$ are derivations of L , then there exist an (associative) derivation $\delta : \langle L \rangle \rightarrow \langle L \rangle C + C$, $\gamma \in C$, and a linear map $\tau : L \rightarrow C$ such that $f = \delta + \gamma I + \tau$.

Let us show that there exists a generalized near-derivation that is not a near-derivation.

Let L be the usual Lie algebra of all 4×4 strict upper triangular matrices over $\mathbb{F}$, i.e.,

$$L = \left\{ \sum_{1 \leq i < j \leq 4} a_{ij} e_{ij} \mid a_{ij} \in \mathbb{F} \right\}.$$

Let φ be a linear functional on L such that $\varphi(e_{14}) = 1$. Now define $f : L \rightarrow L$ by $f(y) = \varphi(y)e_{24}$. We claim that f is not a near-derivation of L . Indeed, for every linear map $g : L \rightarrow L$ we have

$$((\text{ad } e_{12})f - g(\text{ad } e_{12}))([e_{13}, e_{34}]) = e_{14}.$$

On the other hand, we have

$$[((\operatorname{ad} e_{12})f - g(\operatorname{ad} e_{12}))(e_{13}), e_{34}] + [e_{13}, ((\operatorname{ad} e_{12}) - g(\operatorname{ad} e_{12}))(e_{34})] = 0.$$

This implies that $(\operatorname{ad} e_{12})f - g(\operatorname{ad} e_{12})$ is not a derivation of L . However, since $(\operatorname{ad} x)(\operatorname{ad} y)f = 0$ for all $x, y \in L$, we get that $(\operatorname{ad} y)f$ is a near-derivation for every $y \in L$ and so f is a generalized near-derivation.

ACKNOWLEDGMENT. The authors would like to express their sincere thanks to the referee for his/her careful reading of the manuscript. The valuable suggestions have clarified the paper greatly. The first author is supported in part by the Natural Science Foundation Grants of Jilin Province (20125220) and the second author is supported in part by the innovation program of Shanghai Municipal Education Commission (11ZZ119).

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(Received February 20, 2012; revised June 18, 2012)