

On group algebras with unit groups of derived length three in characteristic three

By HARISH CHANDRA (Lucknow) and MEENA SAHAI (Lucknow)

Abstract. Let K be a field of characteristic 3 and let G be a finite 3-group of class 2. Necessary and sufficient conditions are obtained for the group of units $U(KG)$ to be solvable of derived length 3.

1. Introduction

Let KG be the group algebra of a finite group G over a field K of characteristic p . Let $U = U(KG)$ be the group of units of the group algebra KG . First description of the solvability of the unit group U is given in [16, Chapter VI]. This problem has been discussed by many authors as can be seen in [3], [4], [5], [6], [10], [11]. Computation of the derived length of U and its connection with the order and nature of the commutator subgroup G' of G is an interesting problem. SHALEV [17] has found necessary and sufficient conditions for U to be metabelian when $p \geq 3$. This work was completed by COLEMAN and SANDLING [7] and independently by KURDICS [9] for $p = 2$. For $p \neq 2$, a complete description of group algebras KG with centrally metabelian unit groups is given in [13]. The group algebras with $\gamma_3(\delta^1(U)) = 1$ have been listed in [15]. BAGINSKI [2] and BALOGH and LI [1] have computed the derived length of U for finite p -groups and arbitrary groups with cyclic commutator subgroup of order p^n ($p > 2$), respectively. Recently we have obtained the necessary and sufficient conditions for U to have derived length 3 when $p \neq 2, 3$, see [8]. As in [8], for $p = 3$, if $U''' = 1$, then

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$G = P \rtimes H$ where P is a normal Sylow 3-subgroup of G and H is an abelian 3'-subgroup of G . So in this paper, we continue our work on this problem when $p = 3$ and G is a finite 3-group. In addition, we assume that $\gamma_3(G) = 1$. We will use $(x, y) = x^{-1}y^{-1}xy$ for the group commutator of elements x and y of a group G and if $o(x) = n$ then $\hat{x} = 1 + x + x^2 + \cdots + x^{n-1}$. The Lie commutators are denoted by $[x, y] = xy - yx$, $x, y \in KG$.

Our main result is as follows:

Theorem 1.1. *Let K be a field such that $\text{Char } K = 3$ and let G be a finite 3-group of class 2. Then the following conditions are equivalent:*

- (i) $U''' = 1$;
- (ii) G' is elementary abelian 3-subgroup such that $|G'| \leq 3^3$.

2. Proof of the Theorem

Throughout this section, K is a field of characteristic 3.

Lemma 2.1. *Let G be a finite 3-group of class 2 such that $U''' = 1$. Then G' has exponent 3.*

PROOF. Let $x, y \in G$, $z = (x, y)$, $o(z) = 3^n$, where $n \geq 2$. If $u = 1 + x - y$, then

$$\begin{aligned} u_1 &= (u, y) = 1 + u^{-1}x(z - 1), \\ u_2 &= (u, x) = 1 + u^{-1}y(z - 1)z^{-1} \\ \text{and } u_3 &= (u, y^{-1}) = 1 - u^{-1}x(z - 1)z^{-1}. \end{aligned}$$

Clearly $(u_1, u_3) = 1$. Now

$$\begin{aligned} v &= (u_1, u_2) = 1 + u_1^{-1}u_2^{-1}u^{-2}yxu^{-1}(z - 1)^3z^{-1} \\ \text{and } w &= (u_3, u_2) = 1 - u_3^{-1}u_2^{-1}u^{-2}yxu^{-1}(z - 1)^3z^{-2}. \end{aligned}$$

Since $U''' = 1$, so $[v, w] = [u_1^{-1}\beta, u_3^{-1}\beta](z - 1)^6z^{-3} = 0$ where $\beta = u_2^{-1}u^{-2}yxu^{-1}$. The annihilator A of $(z - 1)^6$ in KG is a two sided ideal. The above equation implies that $\bar{\beta}^{-1}\bar{u}_1$ and $\bar{\beta}^{-1}\bar{u}_3$ commute in KG/A where $\bar{w}$ is the image of $w \in KG$ in KG/A . Hence

$$[\beta^{-1}u_1, \beta^{-1}u_3](z - 1)^6 = 0.$$

On simplifying we get

$$\begin{aligned} & \{2uu_2 - x^{-1}yu^{-1}xy^{-1}u^2u_2 - ux^{-1}u^{-1}y^{-1}xu^{-1}xu^2u_2z^{-1} \\ & + ux^{-1}u^{-1}y^{-1}xu^2u_2z^{-1} + ux^{-1}y^{-1}u^{-1}yxuu_2 - (z-1)z^{-1}\}(z-1)^8 = 0. \end{aligned}$$

As $u_2 \in 1 + (z-1)KG$, hence on multiplying this equation by $(z-1)^{3^n-9}$, we get $(z-1)^{3^n-1} = 0$, which is a contradiction. Hence $n = 1$ and exponent of G' is 3. $\square$

To prove the theorem we only need to show that $|G'| \leq 3^3$. Suppose that $G' \neq C_3$. Then, there exist $x, y, z \in G$ such that $(x, y) \neq 1$ and $(x, z) \notin \langle (x, y) \rangle$. If for all such triplets $x, y, z \in G$, $(y, z), (x, g) \in \langle (x, y), (x, z) \rangle$ for all $g \in G$, then $G' = C_3 \times C_3$, [14, Theorem 14, Step III]. So if $G' \neq C_3 \times C_3$, then there exists a triplet $x, y, z \in G$ such that either $(y, z) \notin \langle (x, y), (x, z) \rangle$ or $(x, g) \notin \langle (x, y), (x, z) \rangle$ for some $g \in G$. We first prove some preliminary results based on these two cases.

Lemma 2.2. *Let G be a finite 3-group of class 2 such that $U''' = 1$. Let $u, v, x, y, z \in G$ such that $a = (x, y) \neq 1$, $b = (x, z) \notin \langle a \rangle$ and $c = (y, z) \notin \langle a, b \rangle$. If $(u, x), (u, y), (u, z) \in \langle a, b, c \rangle$ and $(u, v) \notin \langle a, b, c \rangle$, then $(u, x) = (u, y) = (u, z) = 1$.*

PROOF. If $t \in G'$, then $1 + g(t-1)$ is a unit for all $g \in G$. Let $\alpha = (u, v) - 1$ and $(u, x) = a^l b^m c^n$. Now

$$r_1 = (1 + x\alpha, y) = 1 + x\alpha(a-1) - x^2(a-1)\alpha^2$$

$$\text{and } r_2 = (1 + u^{-1}, v^{-1}) = 1 + (1 + u)^{-1}\alpha.$$

$$\begin{aligned} \text{Then } u_1 &= (r_1, r_2) = 1 + r_1^{-1}r_2^{-1}[r_1 - 1, r_2 - 1] = 1 + [x, (1 + u)^{-1}](a-1)\alpha^2 \\ &= 1 - (1 + u)^{-1}ux(1 + u)^{-1}(a-1)((x, u) - 1)\alpha^2 \in U''. \end{aligned}$$

Also if

$$r_3 = (1 + y^{-1}, x^{-1}) = 1 + (1 + y)^{-1}(a^{-1} - 1)$$

$$\text{and } r_4 = (1 + z^{-1}, y^{-1}) = 1 + (1 + z)^{-1}(c^{-1} - 1).$$

$$\text{Then } u_2 = (r_3, r_4) = 1 - r_3^{-1}(1 + y)^{-2}(1 + z)^{-2}zy(a^{-1} - 1)\widehat{c} \in U''.$$

As $[u_1, u_2] = 0$, so we have

$$[(1 + u)^{-1}ux(1 + u)^{-1}, (1 + y)^{-2}(1 + z)^{-2}zy]\widehat{a}\widehat{c}((x, u) - 1)\alpha^2 = 0.$$

If $M = \langle a, b, c \rangle$, then $\Delta^7(M) = 0$. Since $(u, x), (u, y), (u, z) \in M$, hence we have

$$[(1 + u)^{-2}ux, (1 + y)^{-2}(1 + z)^{-2}zy]\widehat{a}\widehat{c}((x, u) - 1)\alpha^2 = 0.$$

Thus

$$[(1+u)^{-2}ux, y^{-1}z^{-1}(1+z)^2(1+y)^2]\widehat{ac}((x, u) - 1)\alpha^2 = 0.$$

Equivalently

$$\begin{aligned} & \{(1+u)^{-2}u[x, y^{-1}z^{-1}(1+z)^2(1+y)^2]x^{-1} \\ & + [(1+u)^{-2}u, y^{-1}z^{-1}(1+z)^2(1+y)^2]\}\widehat{ac}((x, u) - 1)\alpha^2 = 0. \end{aligned} \quad (2.1)$$

On replacing x by x^{-1} in r_1 we get

$$r'_1 = 1 + x^{-1}\alpha(a^{-1} - 1) - x^{-2}\alpha^2(a^{-1} - 1).$$

Then

$$u'_1 = (r'_1, r_2) = 1 - (1+u)^{-1}ux^{-1}(1+u)^{-1}((x, u)^{-1} - 1)(a^{-1} - 1)\alpha^2.$$

As $[u'_1, u_2] = 0$, so we have

$$\begin{aligned} & \{(1+u)^{-2}u[x^{-1}, y^{-1}z^{-1}(1+z)^2(1+y)^2]x \\ & + [(1+u)^{-2}u, y^{-1}z^{-1}(1+z)^2(1+y)^2]\}\widehat{ac}((x, u) - 1)\alpha^2 = 0. \end{aligned} \quad (2.2)$$

Subtracting (2.2) from (2.1) we get

$$0 = \{[x, z^{-1} + z]x^{-1} - [x^{-1}, z^{-1} + z]x\}\widehat{ac}((x, u) - 1)\alpha^2 = 2m(z - z^{-1})\widehat{ab\hat{c}}\alpha^2.$$

Thus $m = 0$.

Now take $r_5 = (1 + x\alpha, z)$, $r'_5 = (1 + x^{-1}\alpha, z)$, $r_6 = (1 + y^{-1}, z^{-1})$, $r_7 = (1 + z^{-1}, x^{-1})$ and the commutators $u_3 = (r_5, r_2)$, $u'_3 = (r'_5, r_2)$, $u_4 = (r_6, r_7)$ we have

$$0 = [u_3, u_4] = [(1+u)^{-1}ux(1+u)^{-1}, (1+z)^{-2}(1+y)^{-2}yz]\widehat{bc}((x, u) - 1)\alpha^2.$$

Equivalently

$$\begin{aligned} & \{(1+u)^{-2}u[x, z^{-1}y^{-1}(1+y)^2(1+z)^2]x^{-1} \\ & + [(1+u)^{-2}u, z^{-1}y^{-1}(1+y)^2(1+z)^2]\}\widehat{bc}((x, u) - 1)\alpha^2 = 0. \end{aligned} \quad (2.3)$$

Also $[u'_3, u_4] = 0$ implies

$$\begin{aligned} & \{(1+u)^{-2}u[x^{-1}, z^{-1}y^{-1}(1+y)^2(1+z)^2]x \\ & + [(1+u)^{-2}u, z^{-1}y^{-1}(1+y)^2(1+z)^2]\}\widehat{bc}((x, u) - 1)\alpha^2 = 0. \end{aligned} \quad (2.4)$$

Subtracting (2.4) from (2.3) we get

$$0 = \{[x, y^{-1} + y]x^{-1} - [x^{-1}, y^{-1} + y]x\}\widehat{b}\widehat{c}((x, u) - 1)\alpha^2 = 2l(y - y^{-1})\widehat{a}\widehat{b}\widehat{c}\alpha^2.$$

Thus $l = 0$. Now

$$\begin{aligned} u_5 &= (r_3, r_7) \\ &= 1 + r_3^{-1}r_7^{-1}(1+z)^{-1}(1+y)^{-1}zy(1+y)^{-1}(1+z)^{-1}(a^{-1}-1)(b^{-1}-1)(c-1). \end{aligned}$$

Hence $[u_1, u_5] = 0$ yields

$$\begin{aligned} 0 &= [(1+u)^{-1}ux(1+u)^{-1}, r_7^{-1}(1+z)^{-1}(1+y)^{-1} \\ &\quad \times zy(1+y)^{-1}(1+z)^{-1}]\widehat{a}(b-1)(c-1)((x, u) - 1)\alpha^2. \end{aligned}$$

Equivalently

$$\begin{aligned} &\{(1+u)^{-2}u[x, y^{-1}z^{-1}(1+z)^2(1+y)^2]x^{-1} \\ &\quad + [(1+u)^{-2}u, y^{-1}z^{-1}(1+z)^2(1+y)^2]\} \\ &\quad \times \widehat{a}(b-1)(c-1)((x, u) - 1)\alpha^2 = 0. \end{aligned} \quad (2.5)$$

Also $[u'_1, u_5] = 0$ yields

$$\begin{aligned} &\{(1+u)^{-2}u[x^{-1}, y^{-1}z^{-1}(1+z)^2(1+y)^2]x \\ &\quad + [(1+u)^{-2}u, y^{-1}z^{-1}(1+z)^2(1+y)^2]\} \\ &\quad \times \widehat{a}(b-1)(c-1)((x, u) - 1)\alpha^2 = 0. \end{aligned} \quad (2.6)$$

Subtracting (2.6) from (2.5) we get

$$\begin{aligned} 0 &= \{[x, z^{-1} + z]x^{-1} - [x^{-1}, z^{-1} + z]x\}\widehat{a}(b-1)(c-1)((x, u) - 1)\alpha^2 \\ &= 2n(z - z^{-1})\widehat{a}\widehat{b}\widehat{c}\alpha^2. \end{aligned}$$

Thus $n = 0$ and $(u, x) = 1$.

On interchanging x and y in the above proof we get $(u, y) = 1$. Similarly, if we interchange x and z we get $(u, z) = 1$. $\square$

If for all triplets $x, y, z \in G$ such that $a = (x, y) \neq 1$, $b = (x, z) \notin \langle a \rangle$, we have $c = a^i b^j$, then on replacing z by $x^{i-1}z$ and y by $x^{1-j}y$, we can assume that $c = ab$.

Lemma 2.3. *Let G be a finite 3-group of class 2 such that $U''' = 1$. Let for all triplets x, y, z in G such that $a = (x, y) \neq 1$, $b = (x, z) \notin \langle a \rangle$, $c = (y, z) \in \langle a, b \rangle$. If $u, v, g \in G$ such that $d = (x, g) \notin \langle a, b \rangle$, $(u, v) \notin \langle a, b, d \rangle$ and $(u, x), (u, y), (u, z) \in \langle a, b, d \rangle$. Then $(u, x) = (u, y) = (u, z) = 1$.*

PROOF. Let $\alpha = (u, v) - 1$ and $(u, x) = a^l b^m d^n$. Now

$$\begin{aligned} r_8 &= (1 + x\alpha, g) = 1 + x\alpha(d - 1) - x^2(d - 1)\alpha^2, \\ r'_8 &= (1 + x^{-1}\alpha, g) = 1 + x^{-1}\alpha(d^{-1} - 1) - x^{-2}\alpha^2(d^{-1} - 1) \end{aligned}$$

and

$$r_9 = (1 + x^{-1}, g^{-1}) = 1 + (1 + x)^{-1}(d - 1).$$

Then

$$\begin{aligned} u_6 &= (r_8, r_2) = 1 - (1 + u)^{-1}ux(1 + u)^{-1}(d - 1)((x, u) - 1)\alpha^2, \\ u'_6 &= (r'_8, r_2) = 1 - (1 + u)^{-1}ux^{-1}(1 + u)^{-1}(d^{-1} - 1)((x, u)^{-1} - 1)\alpha^2 \end{aligned}$$

and

$$\begin{aligned} u_7 &= (r_9, r_6) \\ &= 1 + r_9^{-1}r_6^{-1}(1 + x)^{-1}(1 + y)^{-1}yx(1 + y)^{-1}(1 + x)^{-1}(a - 1)(c - 1)(d - 1). \end{aligned}$$

As $[u_6, u_7] = 0$, so we have

$$\begin{aligned} &[(1 + u)^{-1}ux(1 + u)^{-1}, r_6^{-1}(1 + x)^{-1}(1 + y)^{-1}yx(1 + y)^{-1}(1 + x)^{-1}] \\ &\quad (a - 1)(c - 1)\widehat{d}((x, u) - 1)\alpha^2 = 0. \end{aligned}$$

Let $N = \langle a, b, d \rangle$, then $\Delta^7(N) = 0$. Since $(u, x), (u, y) \in N$, hence we have

$$[(1 + u)^{-2}ux, (1 + x)^{-2}(1 + y)^{-2}yx](a - 1)(c - 1)\widehat{d}((x, u) - 1)\alpha^2 = 0.$$

Thus

$$[(1 + u)^{-2}ux, x^{-1}y^{-1}(1 + y)^2(1 + x)^2](a - 1)(c - 1)\widehat{d}((x, u) - 1)\alpha^2 = 0.$$

Equivalently

$$\begin{aligned} &\{(1 + u)^{-2}u[x, x^{-1}y^{-1}(1 + y)^2(1 + x)^2]x^{-1} \\ &\quad + [(1 + u)^{-2}u, x^{-1}y^{-1}(1 + y)^2(1 + x)^2]\} \\ &\quad \times (a - 1)(c - 1)\widehat{d}((x, u) - 1)\alpha^2 = 0. \end{aligned} \tag{2.7}$$

Also $[u'_6, u_7] = 0$, so we have

$$\begin{aligned} & \{(1+u)^{-2}u[x^{-1}, x^{-1}y^{-1}(1+y)^2(1+x)^2]x \\ & \quad + [(1+u)^{-2}u, x^{-1}y^{-1}(1+y)^2(1+x)^2]\} \\ & \quad \times (a-1)(c-1)\widehat{d}((x, u) - 1)\alpha^2 = 0. \end{aligned} \quad (2.8)$$

Subtracting (2.8) from (2.7) we get

$$\begin{aligned} 0 &= \{[x, y^{-1} + y]x^{-1} - [x^{-1}, y^{-1} + y]x\}(a-1)(c-1)\widehat{d}((x, u) - 1)\alpha^2 \\ &= 2(y - y^{-1})\widehat{a}(c-1)\widehat{d}((x, u) - 1)\alpha^2 = 2m(y - y^{-1})\widehat{a}\widehat{b}\widehat{d}\alpha^2. \end{aligned}$$

Thus $m = 0$.

Now take $r_{10} = (1 + x^{-1}, z^{-1})$, $r_{11} = (1 + y^{-1}, g^{-1})$ and $u_8 = (r_{10}, r_{11})$. We have

$$\begin{aligned} 0 &= [u_3, u_8] = [(1+u)^{-1}ux(1+u)^{-1}, r_{11}^{-1}(1+x)^{-1}(1+y)^{-1} \\ & \quad \times yx(1+y)^{-1}(1+x)^{-1}](a-1)\widehat{b}((y, g) - 1)((x, u) - 1)\alpha^2. \end{aligned}$$

Equivalently

$$\begin{aligned} & (1+u)^{-2}u[x, x^{-1}y^{-1}(1+y)^2(1+x)^2]x^{-1} \\ & \quad + [(1+u)^{-2}u, x^{-1}y^{-1}(1+y)^2(1+x)^2]\} \\ & \quad \times (a-1)\widehat{b}((y, g) - 1)((x, u) - 1)\alpha^2 = 0. \end{aligned} \quad (2.9)$$

Also $[u'_3, u_8] = 0$ implies

$$\begin{aligned} & \{(1+u)^{-2}u[x^{-1}, x^{-1}y^{-1}(1+y)^2(1+x)^2]x \\ & \quad + [(1+u)^{-2}u, x^{-1}y^{-1}(1+y)^2(1+x)^2]\} \\ & \quad \times (a-1)\widehat{b}((y, g) - 1)((x, u) - 1)\alpha^2 = 0. \end{aligned} \quad (2.10)$$

Subtracting (2.10) from (2.9) we get

$$\begin{aligned} 0 &= \{[x, y^{-1} + y]x^{-1} - [x^{-1}, y^{-1} + y]x\}(a-1)\widehat{b}((y, g) - 1)((x, u) - 1)\alpha^2 \\ &= 2(y - y^{-1})\widehat{a}\widehat{b}((y, g) - 1)((x, u) - 1)\alpha^2. \end{aligned}$$

Let $(y, g) = a^r d^s$. On replacing y by xy , if needed, we can assume that $s \neq 0$. Hence

$$2n(y - y^{-1})\widehat{a}\widehat{b}\widehat{d}\alpha^2 = 0.$$

Thus $n = 0$. Now let $u_9 = (r_9, r_4)$. Then

$$0 = [u_6, u_9] = [(1+u)^{-1}ux(1+u)^{-1}, r_4^{-1}(1+x)^{-1}(1+z)^{-1} \\ \times zx(1+z)^{-1}(1+x)^{-1}](b-1)(c-1)\widehat{d}((x, u) - 1)\alpha^2.$$

Equivalently

$$\{(1+u)^{-2}u[x, x^{-1}z^{-1}(1+z)^2(1+x)^2]x^{-1} \\ + [(1+u)^{-2}u, x^{-1}z^{-1}(1+z)^2(1+x)^2]\} \\ \times (b-1)(c-1)\widehat{d}((x, u) - 1)\alpha^2 = 0. \quad (2.11)$$

Also $[u'_6, u_9] = 0$ implies

$$\{(1+u)^{-2}u[x^{-1}, x^{-1}z^{-1}(1+z)^2(1+x)^2]x \\ + [(1+u)^{-2}u, x^{-1}z^{-1}(1+z)^2(1+x)^2]\} \\ \times (b-1)(c-1)\widehat{d}((x, u) - 1)\alpha^2 = 0. \quad (2.12)$$

Subtracting (2.12) from (2.11) we get

$$0 = \{[x, z^{-1} + z]x^{-1} - [x^{-1}, z^{-1} + z]x\}(b-1)\widehat{d}(c-1)((x, u) - 1)\alpha^2 \\ = 2(z - z^{-1})\widehat{b}(c-1)\widehat{d}((x, u) - 1)\alpha^2 = 2l(z - z^{-1})\widehat{a}\widehat{b}\widehat{d}\alpha^2.$$

Thus $l = 0$ and $(u, x) = 1$. On interchanging x and y in the above proof we get $(u, y) = 1$. Similarly interchanging x and z leads to $(u, z) = 1$. $\square$

Theorem 2.1. *Let G be a group of class 2. Then $U''' = 1$ if and only if G' is an elementary abelian 3-subgroup of G such that $|G'| \leq 3^3$.*

PROOF. Let $U''' = 1$. Then by Lemma 2.1, G' is an elementary abelian 3-subgroup of G . Suppose that $G' \neq C_3$, then there exist $x, y, z \in G$ such that $(x, y) \neq 1$ and $(x, z) \notin \langle (x, y) \rangle$. If for all such triplets $x, y, z \in G$, $(y, z), (x, g) \in \langle (x, y), (x, z) \rangle$ for all $g \in G$, then $G' = C_3 \times C_3$ by [14]. Now we have two cases which we examine one by one:

Case (I): Let $x, y, z \in G$ such that $a = (x, y) \neq 1$, $b = (x, z) \notin \langle a \rangle$, $c = (y, z) \notin \langle a, b \rangle$. Then we show that $G' = \langle a, b, c \rangle = M$. Let, if possible, $u, v \in G$ such that $(u, v) \notin M$. For any $t \in G$, let $r_{12} = (1+x^{-1}, y^{-1})$, $r_{13} = (1+z((x, t)-1), x)$, $r_{14} = (1+x^{-1}, t^{-1})$,

$$u_{10} = (r_{12}, r_6) = 1 + r_6^{-1}(1+x)^{-2}(1+y)^{-2}yx\widehat{a}(c-1) \in U''$$

and

$$u_{11} = (r_{13}, r_{14}) = 1 - (1+x)^{-2}zx\widehat{b(x,t)} \in U''.$$

Thus $[u_{10}, u_{11}] = 0$ leads to

$$[r_6^{-1}(1+x)^{-2}(1+y)^{-2}yx, (1+x)^{-2}zx]\widehat{ab(c-1)(x,t)} = 0.$$

Equivalently

$$0 = [y^{-1}(1+y)^2, z]\widehat{ab(c-1)(x,t)} = [y+y^{-1}, z]\widehat{ab(c-1)(x,t)} = (1-y)\widehat{ab\hat{c}(x,t)}.$$

Thus $(x, t) \in M$, for all $t \in G$. On interchanging x and y in the above proof, we get $(y, t) \in M$, for all $t \in G$. Similarly, interchanging x and z leads to $(z, t) \in M$, for all $t \in G$.

Now by Lemma 2.2, $(u, x) = 1 = (u, y)$. So $(zu, x) = b^{-1}$, $(zu, y) = c^{-1}$ and $a = (x, y) \notin \langle (zu, x), (zu, y) \rangle = \langle b, c \rangle$. This yields $(zu, v) = (z, v)(u, v) \in M$. Hence $(u, v) \in M$.

Case (II): For all $x, y, z \in G$ such that $(x, y) \neq 1$ and $(x, z) \notin \langle (x, y) \rangle$, let $(y, z) \in \langle (x, y), (x, z) \rangle$. Out of all such triplets, there is a triplet $x, y, z \in G$ such that $(x, g) \notin \langle (x, y), (x, z) \rangle$ for some $g \in G$. Let $a = (x, y)$, $b = (x, z)$, $c = (y, z)$ and $d = (x, g)$. Then we shall prove that $G' = \langle a, b, d \rangle = N$. As noted earlier, we can assume that $c = ab$. For all $t \in G$, let $r_{15} = (1+g^{-1}, x^{-1})$, $r_{16} = (1+y((x, t)-1), z)$,

$$u_{12} = (r_{16}, r_{14}) = 1 - (1+x)^{-1}xy(1+x)^{-1}(a^{-1}-1)(c-1)\widehat{d(x,t)} \in U''$$

and

$$u_{13} = (r_{15}, r_{10}) = 1 + r_{10}^{-1}(1+g)^{-2}(1+x)^{-2}gx(b-1)\widehat{d} \in U''.$$

Thus

$$0 = [u_{12}, u_{13}]$$

$$= [(1+x)^{-1}xy(1+x)^{-1}, r_{10}^{-1}(1+g)^{-2}(1+x)^{-2}gx](a^{-1}-1)(b-1)(c-1)\widehat{d(x,t)}.$$

Equivalently

$$\begin{aligned} 0 &= [y, x^{-1}(1+x)^2g^{-1}(1+g)^2](a-1)(b-1)(c-1)\widehat{d(x,t)} \\ &= \{[y, x^{-1}+x]g^{-1}(1+g)^2 + x^{-1}(1+x)^2[y, g^{-1}+g]\} \\ &\quad \times (a-1)(b-1)(c-1)\widehat{d(x,t)}. \end{aligned} \tag{2.13}$$

Since $d \notin \langle a \rangle$, we can write $(y, g) = a^r d^s$. On replacing g by $x^r g$, we get

$(y, x^r g) = d^s$ and the above equation yields

$$(x-1)\widehat{abd}(x,t) = 0.$$

Thus $(x, t) \in N$. On interchanging x and y in (2.13) we get

$$\{[x, y^{-1} + y]g^{-1}(1+g)^2 + y^{-1}(1+y)^2[x, g^{-1} + g]\} \\ (a-1)(b-1)(c-1)\widehat{(y,g)}\widehat{(y,t)} = 0.$$

As before, on replacing g by $x^r g$ we get

$$s^2(y-1)\widehat{abd}(y,t) = 0.$$

So if $s \neq 0$, then $(y, t) \in N$. If $s = 0$, then replace y by xy to get

$$(xy-1)\widehat{abd}(xy,t) = 0.$$

Since $(x, t) \in N$, we conclude $(y, t) \in N$. Similarly on interchanging x and z in (2.13) we get

$$\{[y, z^{-1} + z]g^{-1}(1+g)^2 + z^{-1}(1+z)^2[y, g^{-1} + g]\}(a-1)(b-1)(c-1)\widehat{(z,g)}\widehat{(z,t)} = 0.$$

Now $d \notin \langle b \rangle$, so $(z, g) = b^l d^m$ and on replacing g by $x^l g$, we get $(z, x^l g) = d^m$ and $(y, x^l g) = a^{r-l} d^s$. Thus if $r = l$, we get

$$m^2(z-1)\widehat{abd}(z,t) = 0$$

and $(z, t) \in N$, if $m \neq 0$. For $m = 0$, replacing z by $x^2 z$ leads to the same conclusion. If $r \neq l$, then replacing z by xz for $m \neq 2$ and by $x^2 z$ for $m = 2$, yields the same result.

Let $u, v \in G$ such that $(u, v) \notin N$. Then $(u, x) = 1 = (u, y)$, by Lemma 2.3. Thus $(x, uy) = a$, $(x, uz) = b$, $(x, ug) = d$, and hence $(uy, v) = (u, v)(y, v) \in N$. So $(u, v) \in N$.

Conversely, if G' is a central and elementary abelian 3-subgroup of G of order $\leq 3^3$ then by [12, Theorem 2.3], we have $\delta^{(3)}(KG) = 0$ and hence $U''' = 1$. $\square$

Finally, we give an example of a finite group G with $G' = C_3 \times C_3 \times C_3$ but non-central and show that for this group derived length of U is more than 3.

Example 2.1. Let $G = \langle a, b, c, d \mid a^3 = b^3 = c^3 = d^2 = 1, (a, b) = (a, c) = (b, c) = 1, (a, d) = a, (b, d) = b, (c, d) = c \rangle$. Then $G' = C_3 \times C_3 \times C_3 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ is not central in G and $U''' \neq 1$.

Let K be a field with $\text{Char } K = 3$, then

$$u_1 = (1 + d(a - 1), d) = 1 + da(a - 1) + \widehat{a},$$

$$r = (u_1, b) = 1 + da(a - 1)(b^{-1} - 1) + \widehat{a}(b^{-1} - 1),$$

$$u_2 = (1 + d(c - 1), d) = 1 + dc(c - 1) + \widehat{c}$$

$$\text{and } s = (u_2, b) = 1 + dc(c - 1)(b^{-1} - 1) + \widehat{c}(b^{-1} - 1).$$

Then

$$\begin{aligned} (r, s) &= 1 + r^{-1}s^{-1}[da(a - 1)(b^{-1} - 1) + \widehat{a}(b^{-1} - 1), dc(c - 1)(b^{-1} - 1) + \widehat{c}(b^{-1} - 1)] \\ &= 1 + r^{-1}s^{-1}\widehat{db}(a - 1)\{c(a - 1) - a(c - 1)\} = 1 + \widehat{db}(a - 1)(c - 1)(a - c). \end{aligned}$$

This implies that $\delta^3(U(KG)) \neq 1$.

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HARISH CHANDRA
 DEPARTMENT OF MATHEMATICS
 AND ASTRONOMY
 UNIVERSITY OF LUCKNOW
 LUCKNOW
 U.P. (INDIA) 226007
E-mail: hcp_1985@yahoo.co.in

MEENA SAHAI
 DEPARTMENT OF MATHEMATICS
 AND ASTRONOMY
 UNIVERSITY OF LUCKNOW
 LUCKNOW
 U.P. (INDIA) 226007
E-mail: meena_sahai@hotmail.com

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