

On a theorem of Erdős and Sárközy

By YONG-GAO CHEN (Nanjing) and MIN TANG (Wuhu)

Abstract. Let $A = \{a_1, a_2, \dots\}$ ($a_1 \leq a_2 \leq \dots$) be an infinite sequence of nonnegative integers, $k \geq 2$ be a fixed integer and denote by $R_k(n)$ the number of solutions of $a_{i_1} + a_{i_2} + \dots + a_{i_k} = n$. In this paper, we prove that if $g(n)$ is a monotonically increasing arithmetic function with $g(n) \rightarrow +\infty$ and $g(n) = o(n(\log n)^{-2})$, then for any $0 < \varepsilon < 1$, $|R_k(n) - g(n)| > ([k/2]! - \varepsilon)\sqrt{g(n)}$ holds for infinitely many positive integers n . We also prove that for a positive integer d , if $R_k(n) \geq d$ for all sufficiently large integers n , then $R_k(n) \geq d + 2[k/2]!\sqrt{d} + ([k/2]!)^2$ for infinitely many positive integers n .

1. Introduction

Let $A = \{a_1, a_2, \dots\}$ ($0 \leq a_1 \leq a_2 \leq \dots$) be an infinite sequence of nonnegative integers, $k \geq 2$ be a fixed integer and denote by $R_k(n)$ the number of solutions of

$$a_{i_1} + a_{i_2} + \dots + a_{i_k} = n.$$

In 1985, P. ERDŐS and A. SÁRKÖZY [2] proved that if $g(n)$ is an arithmetic function satisfying $g(n) \rightarrow +\infty$, $g(n+1) \geq g(n)$ for $n \geq n_0$ and $g(n) = o(n(\log n)^{-2})$, then $R_2(n) - g(n) = o((g(n))^{1/2})$ cannot hold. In 2002, G. HORVÁTH [3] extend this result to all $k > 2$. In 2007, G. HORVÁTH [4] improved the above result, he proved that if $0 < \varepsilon < 1$ and $g(n)$ is a real arithmetic function such that

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$g(n) \rightarrow +\infty$, $g(n+1) \geq g(n)$ for $n \geq n_0$ and $g(n) = o(n(\log n)^{-2})$, then there does not exist n_1 such that $|R_k(n) - g(n)| \leq (1 - \varepsilon)\sqrt{g(n)}$ for all $n \geq n_1$.

In this paper, we obtain a stronger version of the above results:

Theorem 1. *If $g(n)$ is an arithmetic function such that*

$$g(n) \rightarrow +\infty, \quad g(n+1) \geq g(n) \text{ for } n \geq n_0 \quad (1)$$

and

$$g(n) = o(n(\log n)^{-2}), \quad (2)$$

then for any $0 < \varepsilon < 1$,

$$|R_k(n) - g(n)| > ([k/2]! - \varepsilon)\sqrt{g(n)}$$

holds for infinitely many positive integers n .

Remark 1. In [1], the authors proved a similar result for $g(n) = cn$, where c is a positive constant.

In [5], the authors proved that for a positive integer d , if $R_k(n) \geq d$ for all sufficiently large integers n , then $R_k(n) \geq d + 2\sqrt{d} + 1$ for infinitely many positive integers n . It happens that the method in this paper can be used to improve this result. The other results in [5] can be improved similarly.

Theorem 2. *Let d be a positive integer. If $R_k(n) \geq d$ for all sufficiently large integers n , then $R_k(n) \geq d + 2[k/2]!\sqrt{d} + ([k/2]!)^2$ for infinitely many positive integers n .*

2. Preliminary lemmas

Let $z = re(\alpha)$, where $r = 1 - 1/N$, N is a large integer and $e(\alpha) = e^{2\pi i\alpha}$, α is a real variable. We write

$$F(z) = \sum_{a \in A} z^a.$$

Suppose that $1 \leq R_k(n) \leq n$ for all sufficiently large integers n . Then, by

$$F^k(r^2) = \sum_{n=0}^{\infty} R_k(n)r^{2n},$$

the infinite series $F(z) = \sum_{a \in A} z^a$ is absolutely convergent for $|z| < 1$ and $F(r^2) \rightarrow +\infty$ as $N \rightarrow +\infty$.

Lemma 1. *Let ℓ be a positive integer. Then*

$$\int_0^1 |F^{2\ell}(z)| d\alpha \geq (\ell! + o(1)) F^\ell(r^2).$$

PROOF. We have

$$\begin{aligned} \int_0^1 |F^{2\ell}(z)| d\alpha &= \int_0^1 F^\ell(z) \overline{F^\ell(z)} d\alpha \\ &= \int_0^1 \sum_{a_{i_1}, \dots, a_{i_{2\ell}} \in A} r^{a_{i_1} + \dots + a_{i_{2\ell}}} e((a_{i_1} + \dots + a_{i_\ell} - a_{i_{\ell+1}} - \dots - a_{i_{2\ell}})\alpha) d\alpha \\ &= \sum_{a_{i_1} + \dots + a_{i_\ell} - a_{i_{\ell+1}} - \dots - a_{i_{2\ell}} = 0} r^{a_{i_1} + \dots + a_{i_{2\ell}}} \geq \ell! \sum_{i_1, \dots, i_\ell \text{ pairwise distinct}} r^{2a_{i_1} + \dots + 2a_{i_\ell}} \\ &\geq \ell! \cdot \left(\sum_{i_1, \dots, i_\ell} r^{2a_{i_1} + \dots + 2a_{i_\ell}} - \sum_{1 \leq u < v \leq \ell} \sum_{\substack{i_1, \dots, i_\ell \\ i_u = i_v}} r^{2a_{i_1} + \dots + 2a_{i_\ell}} \right) \\ &= \ell! \cdot \left(F^\ell(r^2) - \frac{1}{2} \ell(\ell-1) F(r^4) F^{\ell-2}(r^2) \right) \\ &\geq \ell! \cdot \left(F^\ell(r^2) - \frac{1}{2} \ell(\ell-1) F^{\ell-1}(r^2) \right) = (\ell! + o(1)) F^\ell(r^2). \end{aligned}$$

This completes the proof of Lemma 1. $\square$

Lemma 2. *Let k be an integer with $k \geq 2$. Then*

$$\int_0^1 |F^k(z)| d\alpha \geq ([k/2]! + o(1)) F^{k/2}(r^2).$$

PROOF. If k is even, then Lemma 2 follows from Lemma 1. If $k = 2\ell + 1$ is odd, then, by Hölder inequality and Lemma 1, we have

$$\begin{aligned} \left(\int_0^1 |F^k(z)| d\alpha \right)^{\frac{k-1}{k}} &= \left(\int_0^1 |F^k(z)| d\alpha \right)^{\frac{k-1}{k}} \cdot \left(\int_0^1 1 d\alpha \right)^{1/k} \\ &\geq \int_0^1 |F(z)|^{k-1} d\alpha = \int_0^1 |F(z)|^{2\ell} d\alpha \geq (\ell! + o(1)) F^\ell(r^2). \end{aligned}$$

Thus

$$\int_0^1 |F^k(z)| d\alpha \geq (\ell! + o(1))^{\frac{k}{k-1}} F^{k/2}(r^2) \geq ([k/2]! + o(1)) F^{k/2}(r^2).$$

This completes the proof of Lemma 2. $\square$

3. Proof of Theorem 1

Suppose that there exists an infinite sequence $A = \{a_1 \leq a_2 \leq \dots\}$ of nonnegative integers, $\varepsilon_0 > 0$ and $n_1 \in \mathbb{N}$ such that

$$|R_k(n) - g(n)| \leq ([k/2]! - \varepsilon_0) \sqrt{g(n)} \quad \text{for all } n \geq n_1. \quad (3)$$

By (1), (2) and (3) we have $1 \leq R_k(n) \leq n$ for all sufficiently large integers n . For $|z| < 1$, we have

$$F^k(z) = \sum_{n=0}^{\infty} R_k(n) z^n = \sum_{n=0}^{\infty} g(n) z^n + \sum_{n=0}^{\infty} (R_k(n) - g(n)) z^n.$$

Let

$$\begin{aligned} J &= \int_0^1 |F^k(z)| d\alpha, & J_1 &= \int_0^1 \left| \sum_{n=0}^{\infty} g(n) z^n \right| d\alpha, \\ J_2 &= \int_0^1 \left| \sum_{n=0}^{\infty} (R_k(n) - g(n)) z^n \right| d\alpha. \end{aligned}$$

Then

$$J \leq J_1 + J_2. \quad (4)$$

By Lemma 2 we have

$$J \geq ([k/2]! + o(1)) F^{k/2}(r^2). \quad (5)$$

Similar to the proof of HORVÁTH [4], we have

$$J_1 = o\left(\left(\sum_{n=0}^{\infty} R_k(n) r^{2n}\right)^{1/2}\right) = o(F^{k/2}(r^2)). \quad (6)$$

By (3), for all $n \geq n_2 (\geq n_1)$, we have

$$R_k(n) \geq g(n) - ([k/2]! - \varepsilon_0) \sqrt{g(n)} \geq g(n) (1 + \varepsilon_0 / (2[k/2]!))^{-2}. \quad (7)$$

By Cauchy's inequality, Parseval's formula, the assumption and (7) we have

$$\begin{aligned} J_2 &\leq \left(\int_0^1 \left| \sum_{n=0}^{\infty} (R_k(n) - g(n)) z^n \right|^2 d\alpha \right)^{\frac{1}{2}} = \left(\sum_{n=0}^{\infty} (R_k(n) - g(n))^2 r^{2n} \right)^{\frac{1}{2}} \\ &= \left(\sum_{n=0}^{n_2-1} (R_k(n) - g(n))^2 r^{2n} + \sum_{n=n_2}^{\infty} (R_k(n) - g(n))^2 r^{2n} \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{n=0}^{n_2-1} (R_k(n) - g(n))^2 + ([k/2]! - \varepsilon_0)^2 \sum_{n=n_2}^{\infty} g(n) r^{2n} \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq \left(\sum_{n=0}^{n_2-1} (R_k(n) - g(n))^2 + ([k/2]! - \varepsilon_0)^2 (1 + \varepsilon_0 / (2[k/2]!))^2 \sum_{n=n_2}^{\infty} R_k(n) r^{2n} \right)^{\frac{1}{2}} \\
&\leq \left([k/2]! - \frac{1}{2} \varepsilon_0 \right) \left(\sum_{n=0}^{\infty} R_k(n) r^{2n} \right)^{1/2} + O(1) \\
&= \left([k/2]! - \frac{1}{2} \varepsilon_0 \right) F^{k/2}(r^2) + O(1). \tag{8}
\end{aligned}$$

By (4), (5), (6) and (8), dividing by $F^{k/2}(r^2)$ and letting $N \rightarrow \infty$, we have $[k/2]! \leq [k/2]! - \varepsilon_0/2$, which is impossible. This completes the proof of Theorem 1.

4. Proof of Theorem 2

Suppose that $d \leq R_k(n) < d + 2[k/2]!\sqrt{d} + ([k/2]!)^2$ for all sufficiently large integers n . Then $1 \leq R_k(n) \leq n$ for all sufficiently large integers n . Let l be the largest integer with

$$l < 2[k/2]!\sqrt{d} + ([k/2]!)^2.$$

Then

$$d + l - [k/2]!\sqrt{d+l} < d + [k/2]!\sqrt{d}.$$

Let c be a real number with

$$\max\{d + l - [k/2]!\sqrt{d+l}, d - [k/2]!\sqrt{d}\} < c < d + [k/2]!\sqrt{d}.$$

Then

$$(d + j - c)^2 < ([k/2]!)^2(d + j), \quad j = 0, 1, \dots, l.$$

Choose a real number $0 < \delta < [k/2]!$ such that

$$(d + j - c)^2 < ([k/2]! - \delta)^2(d + j), \quad j = 0, 1, \dots, l.$$

Thus

$$(R_k(n) - c)^2 < ([k/2]! - \delta)^2 R_k(n) \tag{9}$$

for all integers $n \geq n_3$. For $|z| < 1$, we have

$$F^k(z) = \sum_{n=0}^{\infty} R_k(n) z^n = \sum_{n=0}^{\infty} c z^n + \sum_{n=0}^{\infty} (R_k(n) - c) z^n.$$

Let

$$J = \int_0^1 |F^k(z)| d\alpha, \quad J_1 = \int_0^1 \left| \sum_{n=0}^{\infty} cz^n \right| d\alpha,$$

$$J_2 = \int_0^1 \left| \sum_{n=0}^{\infty} (R_k(n) - c)z^n \right| d\alpha.$$

By Lemma 2 we have

$$J \geq ([k/2]! + o(1))F^{k/2}(r^2). \quad (10)$$

By [4] (see also [5]) we have

$$J_1 = |c| \int_0^1 \frac{1}{|1-z|} d\alpha \ll \log N. \quad (11)$$

By Cauchy's inequality, Parseval's formula, the assumption and (9) we have

$$\begin{aligned} J_2 &\leq \left(\int_0^1 \left| \sum_{n=0}^{\infty} (R_k(n) - c)z^n \right|^2 d\alpha \right)^{\frac{1}{2}} = \left(\sum_{n=0}^{\infty} (R_k(n) - c)^2 r^{2n} \right)^{\frac{1}{2}} \\ &= \left(\sum_{n=0}^{n_3-1} (R_k(n) - c)^2 r^{2n} + \sum_{n=n_3}^{\infty} (R_k(n) - c)^2 r^{2n} \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{n=0}^{n_3-1} (R_k(n) - c)^2 + ([k/2]! - \delta)^2 \sum_{n=n_3}^{\infty} R_k(n) r^{2n} \right)^{\frac{1}{2}} \\ &\leq ([k/2]! - \delta) F^{k/2}(r^2) + O(1). \end{aligned} \quad (12)$$

By $J \leq J_1 + J_2$, (10), (11) and (12), dividing by $F^{k/2}(r^2)$ and letting $N \rightarrow \infty$, we have $[k/2]! \leq [k/2]! - \delta/2$, which is impossible. This completes the proof of Theorem 2.

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YONG-GAO CHEN
SCHOOL OF MATHEMATICAL SCIENCES
AND INSTITUTE OF MATHEMATICS
NANJING NORMAL UNIVERSITY
NANJING 210023
P. R. CHINA

E-mail: ygchen@njnu.edu.cn

MIN TANG
SCHOOL OF MATHEMATICS
AND COMPUTER SCIENCE
ANHUI NORMAL UNIVERSITY
WUHU 241003
P. R. CHINA

E-mail: tmzzz2000@163.com

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