

Diophantine equations involving normalized binomial mid-coefficients

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Abstract. For a positive integer n , let μ_n be the normalized binomial mid-coefficients. We discuss the following Diophantine equation involving power means of n variables μ_i ,

$$M_k(\mu_{a_1}, \dots, \mu_{a_n}) = M_l(\mu_{b_1}, \dots, \mu_{b_n}), \quad k, l \in \mathbb{Z}.$$

For $n = 2, 3$ and other general cases, we get some results on this equation. Moreover, for $k = l = 0$ and for every $n \geq 3$, we obtain infinitely many solutions of equation $\mu_{a_1} \mu_{a_2} \cdots \mu_{a_n} = \mu_{b_1} \mu_{b_2} \cdots \mu_{b_n}$.

1. Introduction

Let $\mathbb{Z}, \mathbb{N}, \mathbb{Q}$ be the sets of integers, positive integers and rational numbers respectively. For any nonnegative integer n , the normalized binomial mid-coefficients is defined by

$$\mu_n = 2^{-2n} \binom{2n}{n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

This coefficient μ_n is closely connected to the Euler's gamma function $\Gamma(x)$, Gauss's hypergeometric function, etc. For more details, see [2], [7], [12]. There are many results for the lower and upper bounds of the estimates of μ_n . The proofs and other inequalities for μ_n can be found in [13], [14].

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Let t be a real number. The power mean of order t of the positive real numbers $x_1, \dots, x_n$ is defined by

$$M_t(x_1, \dots, x_n) = \left(\frac{1}{n} \sum_{j=1}^n x_j^t \right)^{\frac{1}{t}}, \quad \text{if } t \neq 0,$$

and

$$M_0(x_1, \dots, x_n) = \lim_{t \rightarrow 0} M_t(x_1, \dots, x_n) = \left(\prod_{j=1}^n x_j \right)^{\frac{1}{n}}.$$

The most interesting properties of power means are collected in the monograph [6].

It is very interesting to study the Diophantine equation involving power means of n variables μ_i ,

$$M_k(\mu_{a_1}, \dots, \mu_{a_n}) = M_l(\mu_{b_1}, \dots, \mu_{b_n}), \quad k, l \in \mathbb{Z}. \quad (1)$$

In fact, the normalized binomial mid-coefficient has been the subject of an intensive research in number theory and potential theory. See for examples [4] and [11].

In 1990, BANG and FUGLEDE [3] first studied the Diophantine equation

$$M_0(\mu_p, \mu_q) = M_0(\mu_r, \mu_s). \quad (2)$$

They proved that equation (2) only has the trivial solutions $(p, q) = (r, s), (s, r)$. In 2005, ALZER and FUGLEDE [1] studied equation (1), for $n = 2$ and $n = 3$. They solved this equation for integers $k, l \neq 0$ when $n = 2$ and for integers $k = l \neq 0, -1$ when $n = 3$. In [1], ALZER and FUGLEDE set the following problems:

Open problems: (i) Determine all solutions of equation (1) in case $n = 3$ with $k = l = 0$ and $k = l = -1$.

(ii) Study Diophantine equation (1). In particular, determine all solutions of the arithmetic mean-geometric mean equation (1) for $k = 1, l = 0$, i.e.

$$\frac{1}{n}(\mu_{a_1} + \dots + \mu_{a_n}) = (\mu_{b_1} \cdots \mu_{b_n})^{\frac{1}{n}}. \quad (3)$$

In this paper, we discuss Diophantine equation (1) for $n = 2, 3$ and other general cases. First, we solve equation (3) for $n = 2$, and give all solutions of (1), for $n = 2$ and $k = 0, l \neq \pm 2$. In Section 3, we study equation (1) for $n = 3$, $k = l = -1$ and $k \neq l, k, l \neq 0, \pm 1$. Therefore, we solve a part of the problem (i). We also study equation (1) in the case $k = l \neq 0, n \geq 4$. In this case, we give the characteristic of nontrivial solutions of equation (1) or some methods for solving equation (1). See Section 4. In Section 5, for $k = l = 0, n \geq 3$, we give an infinite number of solutions of equation (1). In the last section, we use the results obtained to set some conjectures related to equation (1).

2. The equation $M_0(\mu_{a_1}, \mu_{a_2}) = M_k(\mu_{b_1}, \mu_{b_2})$

Let p be a prime and v_p the standard p -adic valuation normalized defined by $v_p(0) = +\infty$ and

$$v_p\left(\frac{a}{b}\right) = \begin{cases} r, & \text{if } p^r \parallel a, \\ -s, & \text{if } p^s \parallel b, \end{cases}$$

where $a, b, s, t \in \mathbb{Z}$, $s, t \geq 0$, $ab \neq 0$ and $\gcd(a, b) = 1$. Let $q, q_1, q_2 \in \mathbb{Q}$. The following properties on $v_2(q)$ are well-known:

- $v_2(-q) = v_2(q)$, $v_2(q_1 q_2) = v_2(q_1) + v_2(q_2)$;
- $v_2(q_1 + q_2) \geq \min\{v_2(q_1), v_2(q_2)\}$;
- if $v_2(q_1) < v_2(q_2)$, then $v_2(q_1 + q_2) = v_2(q_1)$;
- if $n_1 < n_2$, then $v_2(\mu_{n_1}) > v_2(\mu_{n_2})$.

Now, we recall the following result due to ERDŐS and SELFRIDGE [9] on the product of consecutive integers.

Lemma 2.1. *The equation*

$$n(n+1) \dots (n+k-1) = y^l, \quad (4)$$

in positive integers $n, y, k, l \geq 2$ has no solution.

We prove the following lemma.

Lemma 2.2. *If $x \neq y$, then the equation*

$$\mu_x \mu_y = q^m, \quad m \geq 2, \quad m \in \mathbb{N}, \quad q \in \mathbb{Q} \quad (5)$$

has no solution.

PROOF. As $x \neq y$, without loss of generality, we assume that $x < y$. If $m = 2$, then equation (5) becomes $\frac{\mu_x}{\mu_y} = \left(\frac{q}{\mu_y}\right)^2$. So

$$(2x+1)(2x+2) \dots (2y-1)(2y) = \left(\frac{(2x+1) \dots (2y-1)q}{\mu_y}\right)^2. \quad (6)$$

Hence $\frac{(2x+1) \dots (2y-1)q}{\mu_y} \in \mathbb{N}$ and from Lemma 2.1, equation (6) has no solution.

If $m \geq 3$, from (5) we get

$$\frac{(x+1)(x+2) \dots (2x)}{1 \cdot 2 \cdot \dots \cdot x} \cdot \frac{(y+1)(y+2) \dots (2y)}{1 \cdot 2 \cdot \dots \cdot y} = \left(2^{\frac{2x+2y}{m}} q\right)^m. \quad (7)$$

Notice that $\frac{(x+1)(x+2) \dots (2x)}{1 \cdot 2 \cdot \dots \cdot x}$ and $\frac{(y+1)(y+2) \dots (2y)}{1 \cdot 2 \cdot \dots \cdot y} \in \mathbb{N}$. So $2^{\frac{2x+2y}{m}} q \in \mathbb{N}$. By ERDŐS's proof of Bertrand's Postulate [8], there exists a prime p such that $\lfloor \frac{n}{2} \rfloor < p < 2 \lfloor \frac{n}{2} \rfloor$. Therefore, if $y > 6$, there exists a prime p such that $y+1 \leq p \leq 2y$.

Thus $p \parallel \frac{(y+1)(y+2)\cdots(2y)}{1\cdot 2\cdots y}$ and $v_p\left(\frac{(x+1)(x+2)\cdots(2x)}{1\cdot 2\cdots x}\right) \leq 1$. This is a contradiction to the fact that $m \geq 3$. If $x, y \leq 6$, we directly verify that equation (5) has no solution. $\square$

Using a similar method to that in the proof of Lemma 2.2, it is easy to obtain the following corollary.

Corollary 2.3. *If $n, m \in \mathbb{N}$ with $n < m$, then the equation*

$$\mu_{a_1} \cdots \mu_{a_n} = q^m, \quad q \in \mathbb{Q} \quad (8)$$

has no solution.

We recall here a result obtained by ALZER and FUGLEDE [1].

Lemma 2.4. *Let $k, l \neq 0$ and $a_1, a_2, b_1, b_2 \geq 0$ be integers. If $k = l$, then the equation*

$$M_k(\mu_{a_1}, \mu_{a_2}) = M_l(\mu_{b_1}, \mu_{b_2}) \quad (9)$$

only has the trivial solutions $(a_1, a_2) = (b_1, b_2), (b_2, b_1)$. And if $k \neq l$, then equation (9) holds if and only if $a_1 = a_2 = b_1 = b_2$.

Now we are ready to prove our main result of this section.

Theorem 2.5. *If $k \in \mathbb{Z}$ with $k \neq 0, \pm 2$, then the equation*

$$\sqrt{\mu_{a_1}\mu_{a_2}} = \left(\frac{\mu_{b_1}^k + \mu_{b_2}^k}{2}\right)^{\frac{1}{k}} \quad (10)$$

only has the trivial solutions $a_1 = a_2 = b_1 = b_2$.

PROOF. If $a_1 = a_2$ or $b_1 = b_2$, equation (10) becomes $M_k(\mu_{a_1}, \mu_{a_1}) = M_k(\mu_{b_1}, \mu_{b_2})$ or $M_0(\mu_{a_1}, \mu_{a_2}) = M_0(\mu_{b_1}, \mu_{b_2})$. From Lemma 2.4 and the result of BANG and FUGLEDE [3], equation (10) only has the trivial solutions. Hence, without loss of generality, we assume that $a_1 < a_2$ and $b_1 < b_2$.

If k is odd, we use (10) to deduce the following equation

$$(\mu_{a_1}\mu_{a_2})^{\frac{k-1}{2}} \sqrt{\mu_{a_1}\mu_{a_2}} = \frac{\mu_{b_1}^k + \mu_{b_2}^k}{2}. \quad (11)$$

Then $\sqrt{\mu_{a_1}\mu_{a_2}} = \frac{\mu_{b_1}^k + \mu_{b_2}^k}{2(\mu_{a_1}\mu_{a_2})^{\frac{k-1}{2}}}$, i.e. $\mu_{a_1}\mu_{a_2} = q_1^2$, where $q_1 \in \mathbb{Q}$. From Lemma 2.2, the latter equation has no solution.

If k is even, from (10) we have

$$(\mu_{a_1}\mu_{a_2})^{\frac{k}{2}} = \frac{\mu_{b_1}^k + \mu_{b_2}^k}{2}. \quad (12)$$

Then taking the 2-adic valuation of equation (12), we get

$$\frac{k}{2}v_2(\mu_{a_1}\mu_{a_2}) = v_2\left(\frac{\mu_{b_1}^k + \mu_{b_2}^k}{2}\right) = v_2(\mu_{b_1}^k + \mu_{b_2}^k) - 1 = kv_2(\mu_{b_j}) - 1, \quad (13)$$

where $b_j = b_1$ when $k > 0$ and $b_j = b_2$ when $k < 0$. Then from (13), we have $\frac{k}{2} \mid 1$, since $k \neq \pm 2$. This is impossible. Therefore, the proof of Theorem 2.5 is complete. $\square$

Remark 2.6. In Theorem 2.5, we solve equation (3), for $n = 2$. However, we didn't solve equation (10) when $k = \pm 2$. So we set the following problem: find all solutions of the equations

$$2\mu_{a_1}\mu_{a_2} = \mu_{b_1}^2 + \mu_{b_2}^2 \quad (14)$$

and

$$\frac{2}{\mu_{a_1}\mu_{a_2}} = \frac{1}{\mu_{b_1}^2} + \frac{1}{\mu_{b_2}^2}. \quad (15)$$

3. The equation $M_k(\mu_{a_1}, \mu_{a_2}, \mu_{a_3}) = M_l(\mu_{b_1}, \mu_{b_2}, \mu_{b_3})$

In this section, we first consider equation (1), with $n = 3$ and $k = l = -1$. For the proof of the following theorem, we will use ideas of ALZER and FUGLEDE [1].

Theorem 3.1. *If $0 \leq a_1 \leq a_2 \leq a_3$, $0 \leq b_1 \leq b_2 \leq b_3$, then the equation*

$$\frac{1}{\mu_{a_1}} + \frac{1}{\mu_{a_2}} + \frac{1}{\mu_{a_3}} = \frac{1}{\mu_{b_1}} + \frac{1}{\mu_{b_2}} + \frac{1}{\mu_{b_3}} \quad (16)$$

only has the trivial solutions $a_1 = b_1, a_2 = b_2, a_3 = b_3$.

PROOF. Without loss of generality, we assume that $a_1 \leq b_1$. From (16), we just need to consider two cases: $a_2 \geq b_1$ or $a_2 \leq b_1$.

Case 1: $a_2 \geq b_1$. If $a_1 < b_1$, then

$$\begin{aligned} v_2\left(\frac{1}{\mu_{b_1}}\right) &= \min \left\{ v_2\left(\frac{1}{\mu_{a_2}}\right), v_2\left(\frac{1}{\mu_{a_3}}\right), v_2\left(\frac{1}{\mu_{b_1}}\right), v_2\left(\frac{1}{\mu_{b_2}}\right), v_2\left(\frac{1}{\mu_{b_3}}\right) \right\} \\ &\leq v_2\left(-\frac{1}{\mu_{a_2}} - \frac{1}{\mu_{a_3}} + \frac{1}{\mu_{b_1}} + \frac{1}{\mu_{b_2}} + \frac{1}{\mu_{b_3}}\right) = v_2\left(\frac{1}{\mu_{a_1}}\right) < v_2\left(\frac{1}{\mu_{b_1}}\right). \end{aligned} \quad (17)$$

This is a contradiction. Then $a_1 = b_1$. Thus equation (16) becomes $\frac{1}{\mu_{a_2}} + \frac{1}{\mu_{a_3}} = \frac{1}{\mu_{b_2}} + \frac{1}{\mu_{b_3}}$. By Lemma 2.4, equation (16) only has the trivial solutions.

Case 2: $a_2 \leq b_1$. From the monotony of μ_x , $a_2 \leq b_1$ implies that $a_1 \leq a_2 \leq b_1 \leq b_2 \leq b_3 \leq a_3$. If $a_1 < a_2$, using the method in Case 1, we get also a contradiction. Therefore, $a_1 = a_2$. Moreover, if $a_2 = b_1$, then equation (16) only

has the trivial solutions. So we assume that $a_2 < b_1$. Thus

$$\begin{aligned} v_2\left(\frac{1}{\mu_{a_2}}\right) &< v_2\left(\frac{1}{\mu_{b_1}}\right) \leq v_2\left(\frac{1}{\mu_{b_1}} + \frac{1}{\mu_{b_2}} + \frac{1}{\mu_{b_3}} - \frac{1}{\mu_{a_3}}\right) \\ &= v_2\left(\frac{1}{\mu_{a_1}} + \frac{1}{\mu_{a_2}}\right) = v_2\left(\frac{2}{\mu_{a_1}}\right) = 1 + v_2\left(\frac{1}{\mu_{a_1}}\right). \end{aligned} \quad (18)$$

Hence $v_2\left(\frac{1}{\mu_{b_1}}\right) = 1 + v_2\left(\frac{1}{\mu_{a_1}}\right)$. So $b_1 = a_1 + 1$ and b_1 is odd. In fact, if $b_1 = a_1$, then this is impossible. So $b_1 \geq a_1 + 1$. If $b_1 \geq a_1 + 2$, then $v_2\left(\frac{1}{\mu_{b_1}}\right) \geq v_2\left(\frac{1}{\mu_{a_1}}\right) + 2$, which is also impossible. Moreover, if b_1 is even, then $v_2\left(\frac{1}{\mu_{b_1}}\right) \geq v_2\left(\frac{1}{\mu_{a_1}}\right) + 2$. Therefore, b_1 is odd. Put $b_1 = r$. So we write $a_1 = a_2 = r - 1$.

If $b_2 \geq r + 1$, then from equation (16) we have

$$\begin{aligned} \frac{1}{\mu_{b_3}} - \frac{1}{\mu_{a_3}} &= \frac{2}{\mu_{r-1}} - \frac{1}{\mu_r} - \frac{1}{\mu_{b_2}} = \frac{2r-1}{r} \cdot \frac{1}{\mu_r} - \frac{1}{\mu_{b_2}} \\ &= \left(\frac{2r-1}{r} - \frac{(2r+2) \cdots (2b_2)}{(2r+1) \cdots (2b_2-1)}\right) \frac{1}{\mu_r} = \frac{A}{(2r+1) \cdots (2b_2-1)r} \cdot \frac{1}{\mu_r}, \end{aligned} \quad (19)$$

where $2 \nmid A$. Then

$$v_2\left(\frac{1}{\mu_{b_3}} - \frac{1}{\mu_{a_3}}\right) = v_2\left(\frac{1}{\mu_r}\right) < v_2\left(\frac{1}{\mu_{b_2}}\right) \leq v_2\left(\frac{1}{\mu_{b_3}}\right) \leq v_2\left(\frac{1}{\mu_{b_3}} - \frac{1}{\mu_{a_3}}\right). \quad (20)$$

This leads to a contradiction. Therefore, $b_2 = r \geq 1$ and

$$\frac{1}{\mu_{b_3}} - \frac{1}{\mu_{a_3}} = \frac{2}{\mu_{r-1}} - \frac{2}{\mu_r} = 2\left(\frac{2r-1}{2r} - 1\right) \frac{1}{\mu_r} = -\frac{1}{r} \cdot \frac{1}{\mu_r}. \quad (21)$$

Thus, if $b_3 \geq r + 1$, from (21) and as r is odd, we have

$$v_2\left(\frac{1}{\mu_{b_3}} - \frac{1}{\mu_{a_3}}\right) = v_2\left(\frac{1}{r\mu_r}\right) = v_2\left(\frac{1}{\mu_r}\right) < v_2\left(\frac{1}{\mu_{b_3}}\right) \leq v_2\left(\frac{1}{\mu_{b_3}} - \frac{1}{\mu_{a_3}}\right). \quad (22)$$

This is also a contradiction. Hence, $b_3 = r$ and

$$\frac{1}{\mu_{a_3}} = \frac{3}{\mu_r} - \frac{2}{\mu_{r-1}} = \left(3 - \frac{2r-1}{r}\right) \frac{1}{\mu_r} = \frac{r+1}{r} \cdot \frac{1}{\mu_r}. \quad (23)$$

If $a_3 \geq r + 3$, then $\mu_{a_3} \leq \mu_{r+3} = \frac{(2r+1)(2r+3)(2r+5)}{(2r+2)(2r+4)(2r+6)}\mu_r$. From (23) we get

$$\frac{r}{r+1} \leq \frac{(2r+1)(2r+3)(2r+5)}{(2r+2)(2r+4)(2r+6)}. \quad (24)$$

Therefore, we obtain $r = 1$. Using again equation (23), we get $\mu_{a_3} = \frac{1}{4}$, which is impossible. So $a_3 \leq r + 2$, i.e. $a_3 = r, r + 1, r + 2$. We use each of these values of a_3 to verify that equation (23) has no solutions. Therefore, equation (16) only has the trivial solutions. $\square$

Now, we will study the equation

$$M_k(\mu_{a_1}, \mu_{a_2}, \mu_{a_3}) = M_l(\mu_{b_1}, \mu_{b_2}, \mu_{b_3}), \quad k \neq l, \quad kl \neq 0, \quad k, l \in \mathbb{Z} \quad (25)$$

and prove the following theorem.

Theorem 3.2. *Let $k, l \in \mathbb{Z}$, and $k \neq \pm 1, l \neq \pm 1$. Assume that $\gcd(k, l) = d$, $k = k_1 d, l = l_1 d$, and*

$$\begin{cases} a_1 \leq a_2 \leq a_3, & \text{if } k > 1, \\ a_1 \geq a_2 \geq a_3, & \text{if } k < -1, \end{cases} \quad \text{and} \quad \begin{cases} b_1 \leq b_2 \leq b_3, & \text{if } l > 1, \\ b_1 \geq b_2 \geq b_3, & \text{if } l < -1. \end{cases}$$

Then

- (1) When $2 \nmid k_1 - l_1$, equation (25) only has the trivial solutions $a_1 = a_2 = a_3 = b_1 = b_2 = b_3$.
- (2) When $2 \mid k_1 - l_1$, equation (25) has the trivial solutions and $M_2(\mu_0, \mu_1, \mu_1) = M_{-2}(\mu_1, \mu_0, \mu_0)$. Moreover, if equation (25) has other solutions, these solutions satisfy $a_3 = b_3$ and

$$\min \left\{ kv_2\left(\frac{\mu_{a_2}}{\mu_{a_3}}\right), lv_2\left(\frac{\mu_{b_2}}{\mu_{b_3}}\right) \right\} = v_2(|k_1 - l_1|) + 2 \quad \text{or} \quad v_2(|k_1 - l_1|) + 1. \quad (26)$$

PROOF. From equation (25), we have

$$(\mu_{a_1}^k + \mu_{a_2}^k + \mu_{a_3}^k)^{l_1} \cdot 3^{k_1 - l_1} = (\mu_{b_1}^l + \mu_{b_2}^l + \mu_{b_3}^l)^{k_1}. \quad (27)$$

If $a_1 = a_2 = a_3$, then equation (25) becomes

$$M_k(\mu_{a_1}, \mu_{a_2}, \mu_{a_3}) = M_l(\mu_{a_1}, \mu_{a_1}, \mu_{a_1}) = M_l(\mu_{b_1}, \mu_{b_2}, \mu_{b_3}),$$

which is incompatible with the condition $k \neq l$. If $b_1 = b_2 = b_3$, the same conclusion can be made. Therefore, from the monotony of μ_x , we just need to discuss three cases:

- $a_2 \neq a_3$ and $b_2 \neq b_3$,
- $a_1 \neq a_2 = a_3$ and $b_2 \neq b_3$,
- $a_1 \neq a_2 = a_3$ and $b_1 \neq b_2 = b_3$.

Put $a_3 = m$.

Case 3.1: $a_2 \neq a_3$ and $b_2 \neq b_3$. Without loss of generality, we suppose $k > l$. Since $v_2(\mu_{a_1}^k + \mu_{a_2}^k) > v_2(\mu_{a_3}^k)$, $v_2(\mu_{b_1}^l + \mu_{b_2}^l) > v_2(\mu_{b_3}^l)$, then using equation (27), we get

$$kl_1 v_2(\mu_{a_3}) = k_1 l v_2(\mu_{b_3}). \quad (28)$$

So $a_3 = b_3 = m$. Thus equation (27) implies

$$\left(\frac{\mu_{a_1}^k}{\mu_m^k} + \frac{\mu_{a_2}^k}{\mu_m^k} + 1\right)^{l_1} 3^{k_1-l_1} = \left(\frac{\mu_{b_1}^l}{\mu_m^l} + \frac{\mu_{b_2}^l}{\mu_m^l} + 1\right)^{k_1}. \quad (29)$$

Write

$$\frac{\mu_{a_1}^k}{\mu_m^k} + \frac{\mu_{a_2}^k}{\mu_m^k} = C, \quad \frac{\mu_{b_1}^l}{\mu_m^l} + \frac{\mu_{b_2}^l}{\mu_m^l} = D.$$

The first, we suppose that $k, l > 1$, then

$$C \left(\sum_{j=0}^{l_1-1} \binom{l_1}{j} C^{l_1-j-1} \right) 3^{k_1-l_1} + (3^{k_1-l_1} - 1) = D \left(\sum_{j=0}^{k_1-1} \binom{k_1}{j} D^{k_1-j-1} \right). \quad (30)$$

Now we calculate the value of $v_2(3^q - 1)$, where $q \in \mathbb{N}$. If $2 \nmid q$, then $3^q - 1 \equiv 2 \pmod{8}$. If $q = 2^r q_1$, where $r \in \mathbb{N}$ and $2 \nmid q_1$, since

$$3^q - 1 = 3^{2^r q_1} - 1 = (3^{2^{r-1} q_1} + 1) \cdots (3^{2q_1} + 1)(3^{q_1} + 1)(3^{q_1} - 1),$$

and $3^{2q_1} + 1 \equiv 2 \pmod{8}$, $3^{q_1} + 1 \equiv 4 \pmod{8}$, $3^{q_1} - 1 \equiv 2 \pmod{8}$, then $v_2(3^q - 1) = r + 2$. Therefore, we have

$$v_2(3^q - 1) = \begin{cases} 1, & \text{if } 2 \nmid q, \\ v_2(q) + 2, & \text{if } 2 \mid q. \end{cases} \quad (31)$$

If $2 \mid k_1 l_1$ and as $\gcd(k_1, l_1) = 1$, then $k_1 - l_1$ is odd. So $v_2(3^{k_1-l_1} - 1) = 1$. As $a_2 < m$, $b_2 < m$, one can see that $v_2(C) \geq k$ and $v_2(D) \geq l$. Thus, if k_1 is odd and l_1 is even, from equation (30) we have

$$\begin{aligned} v_2 \left(C \left(\sum_{j=0}^{l_1-1} \binom{l_1}{j} C^{l_1-j-1} \right) 3^{k_1-l_1} \right) &\geq k + v_2 \left(\sum_{j=0}^{l_1-1} \binom{l_1}{j} C^{l_1-j-1} \right) \geq k + 1, \\ v_2 \left(D \left(\sum_{j=0}^{k_1-1} \binom{k_1}{j} D^{k_1-j-1} \right) \right) &= v_2(D) + v_2 \left(\sum_{j=0}^{k_1-1} \binom{k_1}{j} D^{k_1-j-1} \right) \geq 2. \end{aligned}$$

This is impossible. Similarly, if k_1 is even and l_1 is odd, we obtain the same contradiction.

If $2 \nmid k_1 l_1$, then $v_2(3^{k_1-l_1} - 1) = v_2(k_1 - l_1) + 2$. Notice that $v_2(C) \geq k \geq 1$, $v_2(D) \geq l \geq 1$, and $2 \nmid k_1$, $2 \nmid l_1$. Hence we get

$$v_2 \left(C \left(\sum_{j=0}^{l_1-1} \binom{l_1}{j} C^{l_1-j-1} \right) 3^{k_1-l_1} \right) = v_2(C) \geq k,$$

and

$$v_2\left(D\left(\sum_{j=0}^{k_1-1}\binom{k_1}{j}B^{k_1-j-1}\right)\right) = v_2(D).$$

So if $k > l > 1$, $k > v_2(k_1 - l_1) + 2$, then from equation (30) we obtain

$$v_2(D) = v_2(k_1 - l_1) + 2. \quad (32)$$

Since $v_2(D) = v_2\left(\frac{\mu_{b_1}^l}{\mu_m^l} + \frac{\mu_{b_2}^l}{\mu_m^l}\right) = lv_2\left(\frac{\mu_{b_2}}{\mu_{b_3}}\right)$ or $lv_2\left(\frac{\mu_{b_2}}{\mu_{b_3}}\right) + 1$, thus condition (26) holds.

Now, if $k, l < -1$, then from (28) we have

$$\begin{aligned} C\left(\sum_{j=0}^{-l_1-1}\binom{-l_1}{j}C^{-l_1-j-1}\right)3^{k_1-l_1} + (3^{k_1-l_1} - 1) \\ = D\left(\sum_{j=0}^{-k_1-1}\binom{-k_1}{j}D^{-k_1-j-1}\right). \end{aligned} \quad (33)$$

If $k > 1, l < -1$, equation (28) implies $(C+1)^{-l_1}(D+1)^{k_1} = 3^{k_1-l_1}$. So

$$((C+1)^{-l_1} - 1)((D+1)^{k_1} - 1) + ((C+1)^{-l_1} - 1) + ((D+1)^{k_1} - 1) = 3^{k_1-l_1} - 1.$$

Thus, we obtain

$$\begin{aligned} \left(\sum_{j=0}^{-l_1-1}\binom{-l_1}{j}C^{-l_1-j-1}\right)D\left(\sum_{j=0}^{-k_1-1}\binom{-k_1}{j}D^{-k_1-j-1}\right) \\ = C\left(\sum_{j=0}^{-l_1-1}\binom{-l_1}{j}C^{-l_1-j-1}\right) + D\left(\sum_{j=0}^{-k_1-1}\binom{-k_1}{j}D^{-k_1-j-1}\right) + (3^{-k_1+l_1} - 1). \end{aligned} \quad (34)$$

Using an approach similar that of (30), one draws the same conclusions, i.e. when $2 \mid k_1 - l_1$, equation (25) has no other solutions; when $2 \nmid k_1 - l_1$, the other solutions of equation (25) satisfy $a_3 = b_3, \min\{v_2(C), v_2(D)\} = v_2(k_1 - l_1) + 2$ or $v_2(k_1 - l_1) + 1$. Therefore, again condition (26) holds.

Case 3.2: $a_2 = a_3$ and $b_2 \neq b_3$. If $v_2(\mu_{a_1}^k) = v_2(\mu_{a_2}^k + \mu_{a_3}^k) = v_2(2\mu_m^k)$, then $kv_2(\mu_{a_1}) = 1 + kv_2(\mu_m)$. We deduce that $k = \pm 1$. This is impossible.

If $v_2(\mu_{a_1}^k) \neq v_2(2\mu_{a_3}^k)$, then $v_2(\mu_{a_1}^k) > v_2(2\mu_{a_3}^k)$. From the monotony of $v_2(\mu_x)$, equation (27) implies

$$l_1(kv_2(\mu_{a_3}) + 1) = k_1lv_2(\mu_{b_3}). \quad (35)$$

This yields $k_1d = k = \pm 1$, which leads also to a contradiction.

Case 3.3: $a_2 = a_3$ and $b_2 = b_3$. If $v_2(\mu_{a_1}^k) = v_2(2\mu_{a_3}^k)$, then $kv_2(\mu_{a_1}) = kv_2(\mu_{a_3}) + 1$. So we have $k = \pm 1$. If $v_2(\mu_{b_1}^l) = v_2(2\mu_{b_3}^l)$, we get a similar conclusion. Therefore, we suppose that $v_2(\mu_{a_1}^k) \neq v_2(2\mu_{a_3}^k)$ and $v_2(\mu_{b_1}^l) \neq v_2(2\mu_{b_3}^l)$.

As $v_2(\mu_{a_1}^k) \neq v_2(2\mu_{a_3}^k)$ and $v_2(\mu_{b_1}^l) \neq v_2(2\mu_{b_3}^l)$, we deduce that $v_2(\mu_{a_1}^k) > v_2(2\mu_{a_3}^k)$ and $v_2(\mu_{b_1}^l) > v_2(2\mu_{b_3}^l)$. Thus from equation (27), we get

$$l_1(kv_2(\mu_{a_3}) + 1) = k_1(lv_2(\mu_{b_3}) + 1). \quad (36)$$

Therefore, $k_1 \mid l_1$, and $l_1 \mid k_1$, i.e. $k = \pm l$. Since $k \neq l$, $\gcd(k_1, l_1) = 1$, then $k = -l$, and from equation (36) we get $k(v_2(\mu_{a_3}) - v_2(\mu_{b_3})) = -2$. As $k \neq \pm 1$, we have $k = \pm 2$ and $v_2(\mu_{a_3}) - v_2(\mu_{b_3}) = \pm 1$.

If $k = 2$ and $v_2(\mu_{a_3}) - v_2(\mu_{b_3}) = -1$, then $l = -2$, and $b_3 = a_3 - 1 = m - 1$, where m is odd. Thus equation (25) becomes

$$(\mu_{a_1}^2 + 2\mu_m^2) \left(\frac{1}{\mu_{b_1}^2} + \frac{2}{\mu_{m-1}^2} \right) = 9. \quad (37)$$

Note that $a_1 \leq m - 1$ and $b_1 \geq m$. When $b_1 \geq m + 1$, we have

$$\begin{aligned} (\mu_{a_1}^2 + 2\mu_m^2) \left(\frac{1}{\mu_{b_1}^2} + \frac{2}{\mu_{m-1}^2} \right) &\geq (\mu_{m-1}^2 + 2\mu_m^2) \left(\frac{1}{\mu_{m+1}^2} + \frac{2}{\mu_{m-1}^2} \right) \\ &= \frac{288m^6 - 80m^4 + 32m^3 + 12m^2 - 8m + 2}{32m^6 - 16m^4 + 2m^2} > 9. \end{aligned}$$

This contradicts (37). Therefore, $b_1 = m$. If $a_1 \leq m - 2$, similar calculations imply

$$(\mu_{a_1}^2 + 2\mu_m^2) \left(\frac{1}{\mu_{b_1}^2} + \frac{2}{\mu_{m-1}^2} \right) \geq (\mu_{m-2}^2 + 2\mu_m^2) \left(\frac{1}{\mu_m^2} + \frac{2}{\mu_{m-1}^2} \right) > 9.$$

This means that $a_1 = m - 1$. From (37), we get

$$(\mu_{m-1}^2 + 2\mu_m^2) \left(\frac{1}{\mu_m^2} + \frac{2}{\mu_{m-1}^2} \right) = 9.$$

We deduce that $m = 1$. Thus equation (25) has the solution $k = 2, l = -2$,

$$(a_1, a_2, a_3, b_1, b_2, b_3) = (0, 1, 1, 1, 0, 0).$$

Similarly, if $k = -2, l = 2$, one obtains a similar conclusion, i.e. equation (25) has only the solution $(k, l, a_1, a_2, a_3, b_1, b_2, b_3) = (-2, 2, 1, 0, 0, 1, 1)$. This completes the proof of Theorem 3.2. $\square$

Remark 3.3. Using Theorem 3.2, we can find the solutions of equation (25) for some fixed values of k, l . However, we cannot completely solve it. This is the case for examples when $k = 1, l = -1$, or $k = 3, l = 1$. We believe that it may be difficult to completely solve this equation.

4. The equation $M_k(\mu_{a_1}, \mu_{a_2}, \dots, \mu_{a_n}) = M_k(\mu_{b_1}, \mu_{b_2}, \dots, \mu_{b_n})$

Before proving the main theorem of this section, we give the following lemma.

Lemma 4.1. *Let $k \neq 0$, $0 \leq a_1 \leq a_2 \leq a_3$, $0 \leq b_1 \leq b_2 \leq b_3$, and $a_1 \leq b_1$. The equation*

$$M_k(\mu_{a_1}, \mu_{a_2}, \mu_{a_3}) = M_k(\mu_{b_1}, \mu_{b_2}, \mu_{b_3}), \quad k \in \mathbb{Z} \quad (38)$$

only has the trivial solutions, except for $k = 1$, in which case we have the additional solution $(a_1, a_2, a_3, b_1, b_2, b_3) = (1, 3, 3, 2, 2, 2)$.

PROOF. For $k \neq 0, -1$, one can refer to Proposition 2 in [1]. If $k = -1$, from Theorem 3.1, we get the conclusion. $\square$

For $n \geq 4, k \neq 0$, we have the main result of this section.

Theorem 4.2. *Let $n \geq 4, k \neq 0$. We assume that*

$$\begin{cases} a_1 \leq a_2 \leq \dots \leq a_n, b_1 \leq b_2 \leq \dots \leq b_n, & \text{if } k \geq 1, \\ a_1 \geq a_2 \geq \dots \geq a_n, b_1 \geq b_2 \geq \dots \geq b_n, & \text{if } k \leq -1. \end{cases}$$

Then the equation

$$\mu_{a_1}^k + \mu_{a_2}^k + \dots + \mu_{a_n}^k = \mu_{b_1}^k + \mu_{b_2}^k + \dots + \mu_{b_n}^k, \quad k \in \mathbb{Z} \quad (39)$$

only has the trivial solutions $a_j = b_j$ ($1 \leq j \leq n$), except for $k \mid v_2(r)$, where $2 \mid r$ and $2 \leq r \leq n$.

PROOF. For the proof, we will use the mathematical induction on n . Put $a_n = m$. First, we will prove the theorem for $n = 4$.

If $a_3 \neq a_4$, and $b_3 \neq b_4$, then we get

$$v_2(\mu_{a_1}^k + \mu_{a_2}^k + \mu_{a_3}^k) > v_2(\mu_{a_4}^k) \quad \text{and} \quad v_2(\mu_{b_1}^k + \mu_{b_2}^k + \mu_{b_3}^k) > v_2(\mu_{b_4}^k).$$

From equation (39), we have $v_2(\mu_{a_4}^k) = v_2(\mu_{b_4}^k)$. So $a_4 = b_4$ and $\mu_{a_1}^k + \mu_{a_2}^k + \mu_{a_3}^k = \mu_{b_1}^k + \mu_{b_2}^k + \mu_{b_3}^k$. By Lemma 4.1, the equation only has the trivial solutions, except for $k = 1$.

If $a_3 = a_4$ and $b_3 \neq b_4$, we consider four cases.

- If $\mu_{a_1}^k = \mu_{a_2}^k = \mu_{a_3}^k = \mu_{a_4}^k$, then $v_2(\mu_{a_4}^k) + 2 = v_2(\mu_{b_4}^k)$. We deduce that $k = \pm 2$ and $b_4 = m \mp 1$ or $k = \pm 1$ and $|b_4 - m| \geq 1$. This contradicts (39).

- If $\mu_{a_1}^k > \mu_{a_2}^k = \mu_{a_3}^k = \mu_{a_4}^k$, then $v_2(3\mu_{a_4}^k) = v_2(\mu_{b_4}^k)$. So $b_4 = m$. Lemma 4.1 implies that the equation only has the trivial solutions.

• If $\mu_{a_1}^k \geq \mu_{a_2}^k > \mu_{a_3}^k = \mu_{a_4}^k$ and $v_2(\mu_{a_2}^k) = v_2(2\mu_{a_4}^k)$, then $k = \pm 1$, $a_2 = m \mp 1$. Therefore, the theorem holds.

• If $\mu_{a_1}^k \geq \mu_{a_2}^k > \mu_{a_3}^k = \mu_{a_4}^k$ and $v_2(\mu_{a_2}^k) \neq v_2(2\mu_{a_4}^k)$, then we get $v_2(\mu_{a_1}^k + \mu_{a_2}^k) > v_2(2\mu_{a_4}^k)$. Thus we have $v_2(2\mu_{a_4}^k) = v_2(\mu_{b_4}^k)$. We deduce that $k = 1$, $b_4 = m - 1$, and m is odd, or $k = -1$, $b_4 = m + 1$, and m is even. If $k = 1$, $b_4 = m - 1$, equation (39) implies

$$\mu_{a_1} + \mu_{a_2} + \frac{m-1}{m} \cdot \mu_{m-1} = \mu_{b_1} + \mu_{b_2} + \mu_{b_3}. \quad (40)$$

Notice that $b_3 \neq b_4$. So we have $b_3 \leq m - 2$. As

$$\begin{aligned} v_2(\mu_{a_2}) &\geq v_2(\mu_{a_3-1}) = v_2(\mu_{m-2}) = v_2\left(\frac{2m-2}{2m-3} \cdot \mu_{m-1}\right) \\ &= 1 + v_2(m-1) + v_2(\mu_{m-1}) = 1 + v_2\left(\frac{m-1}{m} \cdot \mu_{m-1}\right) \end{aligned}$$

and $v_2(\mu_{b_3}) \geq v_2(\mu_{m-2}) = 1 + v_2\left(\frac{m-1}{m} \cdot \mu_{m-1}\right)$, we get a contradiction to (40). If $k = -1$, $b_4 = m + 1$, using the same method, we come to the same conclusion.

Finally, we suppose that $a_3 = a_4, b_3 = b_4$. If $v_2(\mu_{a_2}^k) \neq v_2(2\mu_{a_4}^k)$ and $v_2(\mu_{b_2}^k) \neq v_2(2\mu_{b_4}^k)$, then by (39) we get $v_2(2\mu_{a_4}^k) = v_2(2\mu_{b_4}^k)$. This implies $a_4 = b_4$. Using Lemma 4.1, one can see that the equation only has the trivial solutions. If $v_2(\mu_{a_2}^k) = v_2(2\mu_{a_4}^k)$ or $v_2(\mu_{b_2}^k) = v_2(2\mu_{b_4}^k)$, then $k = \pm 1$ and $a_2 = m \mp 1$ or $k = \pm 1$ and $b_2 = m \mp 1$. Therefore, the theorem is proved for $n = 4$.

Second, we suppose that Theorem 4.2 holds for $n - 1$ and we will prove that it also holds for n . According to the above discussion, we will also consider three cases: $a_{n-1} \neq a_n, b_{n-1} \neq b_n$ or $a_{n-1} = a_n, b_{n-1} \neq b_n$, or $a_{n-1} = a_n, b_{n-1} = b_n$.

Case 4.1: $a_{n-1} \neq a_n, b_{n-1} \neq b_n$.

Then we obtain $v_2(\mu_{a_1}^k + \cdots + \mu_{a_{n-1}}^k) > v_2(\mu_{a_n}^k)$, $v_2(\mu_{b_1}^k + \cdots + \mu_{b_{n-1}}^k) > v_2(\mu_{b_n}^k)$. From equation (39), we have $v_2(\mu_{a_n}^k) = v_2(\mu_{b_n}^k)$. This implies $a_n = b_n$. Therefore, we obtain $\mu_{a_1}^k + \cdots + \mu_{a_{n-1}}^k = \mu_{b_1}^k + \cdots + \mu_{b_{n-1}}^k$. By the induction hypothesis, the nontrivial solutions of equation (39) satisfy $k \mid v_2(r)$, where $2 \mid r$ and $2 \leq r \leq n - 1$.

Case 4.2: $a_{n-1} = a_n, b_{n-1} \neq b_n$.

Suppose that $\mu_{a_{n-s}}^k > \mu_{a_{n-s+1}}^k = \cdots = \mu_{a_{n-1}}^k = \mu_{a_n}^k$, where $s \geq 2$. If s is odd, then $v_2(s\mu_{a_n}^k) = v_2(\mu_{b_n}^k)$. So we have $a_n = b_n$.

If s is even and $v_2(\mu_{a_{n-s}}^k) \neq v_2(s\mu_{a_n}^k)$, then $v_2(\mu_{a_{n-s}}^k) = v_2(\mu_{b_n}^k)$ or $v_2(s\mu_{a_n}^k) = v_2(\mu_{b_n}^k)$. This implies that $a_{n-s} = b_n$ or $k \mid v_2(s)$, where $2 \leq s \leq n$.

If s is even and $v_2(\mu_{a_{n-s}}^k) = v_2(s\mu_{a_n}^k)$, then $kv_2(\mu_{a_{n-s}}^k) = v_2(s) + kv_2(\mu_{a_n}^k)$. One obtains the same conclusion.

Case 4.3: $a_{n-1} = a_n, b_{n-1} = b_n$.

Assume that $\mu_{a_{n-s}}^k > \mu_{a_{n-s+1}}^k = \cdots = \mu_{a_{n-1}}^k = \mu_{a_n}^k, \mu_{b_{n-t}}^k > \mu_{b_{n-t+1}}^k = \cdots = \mu_{b_{n-1}}^k = \mu_{b_n}^k$, where $s, t \geq 2$.

If $2 \nmid st$, then $v_2(\mu_{a_{n-s}}^k) > v_2(s\mu_{a_n}^k), v_2(\mu_{b_{n-t}}^k) > v_2(t\mu_{b_n}^k)$. So from (39), we get $v_2(s\mu_{a_n}^k) = v_2(t\mu_{b_n}^k)$. Thus, $a_n = b_n$.

If $2 \mid s, 2 \nmid t$, and $v_2(\mu_{a_{n-s}}^k) \neq v_2(s\mu_{a_n}^k)$, then $v_2(\mu_{a_{n-s}}^k) = v_2(t\mu_{b_n}^k) = v_2(\mu_{b_n}^k)$ or $v_2(s\mu_{a_n}^k) = v_2(t\mu_{b_n}^k) = v_2(\mu_{b_n}^k)$. This yields to $a_{n-s} = b_n$ or $k \mid v_2(s)$. If $v_2(\mu_{a_{n-s}}^k) = v_2(s\mu_{a_n}^k)$, we see that $k \mid v_2(s)$.

If $2 \nmid s$ and $2 \mid t$, using a similar method we get the same conclusion.

If $2 \mid s$ and $2 \mid t$, we consider two cases. If $v_2(\mu_{a_{n-s}}^k) \neq v_2(s\mu_{a_n}^k)$ and $v_2(\mu_{b_{n-t}}^k) \neq v_2(t\mu_{b_n}^k)$, then from (39) we get $v_2(\mu_{a_{n-s}}^k) = v_2(t\mu_{b_n}^k)$ or $v_2(s\mu_{a_n}^k) = v_2(t\mu_{b_n}^k)$, or $v_2(\mu_{b_{n-t}}^k) = v_2(s\mu_{a_n}^k)$. Thus we conclude that $k \mid v_2(s)$ or $k \mid v_2(t)$, or $k \mid v_2(s) - v_2(t)$. It clear that $|v_2(s) - v_2(t)| < \max\{v_2(s), v_2(t)\}$. Therefore, Theorem 4.2 also holds for n and this concludes its proof. $\square$

Using the proof of Theorem 4.2, one can easily deduce the following corollary.

Corollary 4.3. *If $k \neq 0, \pm 1$, then the equation*

$$\mu_{a_1}^k + \mu_{a_2}^k + \mu_{a_3}^k + \mu_{a_4}^k = \mu_{b_1}^k + \mu_{b_2}^k + \mu_{b_3}^k + \mu_{b_4}^k, \quad k \in \mathbb{Z} \quad (41)$$

only has the trivial solutions. For $k = \pm 1$, if equation (41) has nontrivial solutions, then these solutions satisfy $a_3 = a_4$ and $a_2 = a_4 \mp 1$ or $b_3 = b_4$ and $b_2 = b_4 \mp 1$.

Again, from Theorem 4.2, we get the following corollary.

Corollary 4.4. *If $k \neq 0, \pm 1, \pm 2$, then the equation*

$$\mu_{a_1}^k + \mu_{a_2}^k + \mu_{a_3}^k + \mu_{a_4}^k + \mu_{a_5}^k = \mu_{b_1}^k + \mu_{b_2}^k + \mu_{b_3}^k + \mu_{b_4}^k + \mu_{b_5}^k, \quad k \in \mathbb{Z} \quad (42)$$

only has the trivial solutions. Moreover, for $k = \pm 2$, if equation (42) has nontrivial solutions, then these solutions satisfy $a_2 = a_3 = a_4 = a_5$ and $b_5 = a_5 \mp 1$ or $b_2 = b_3 = b_4 = b_5$ and $a_5 = b_5 \mp 1$.

PROOF. By Theorem 4.2, the nontrivial solutions of equation (42) satisfy $k = \pm 1, \pm 2$. If $k = \pm 2$, from the proof of Theorem 4.2, we have $\mu_{a_1}^k > \mu_{a_2}^k = \mu_{a_3}^k = \mu_{a_4}^k = \mu_{a_5}^k$ and $\mu_{b_4}^k > \mu_{b_5}^k$ or $\mu_{b_2}^k > \mu_{b_3}^k = \mu_{b_4}^k = \mu_{b_5}^k$, or $b_1 = b_2 = b_3 = b_4 = b_5$. We write $a_5 = m$.

If $k = 2$, then $a_1 > a_2 = a_3 = a_4 = a_5$. Using equation (42), we get $v_2(\mu_{a_1}^2 + 4\mu_{a_5}^2) = v_2(\mu_{b_5}^2)$.

If $v_2(\mu_{a_1}^2) = v_2(4\mu_{a_5}^2) = v_2(\mu_{a_5}^2) + 2$, then we obtain $a_1 = a_5 - 1 = m - 1$ and m is odd. Thus from equation (42) we have

$$\begin{aligned} v_2(\mu_{a_1}^2 + 4\mu_{a_5}^2) &= v_2\left(\frac{4m^2 + 4(2m-1)^2}{(2m-1)^2}\mu_m^2\right) \\ &= 2 + v_2(m^2 + (2m-1)^2) + v_2(\mu_m^2) = v_2(\mu_m^2) + 3 = v_2(\mu_{b_5}^2). \end{aligned}$$

Since $v_2(m^2 + (2m-1)^2) = 1$, then $2v_2(\mu_m) + 3 = 2v_2(\mu_{b_5})$. This leads to a contradiction.

If $v_2(\mu_{a_1}^2) \neq v_2(4\mu_{a_5}^2) = v_2(\mu_{a_5}^2) + 2$ and as $a_1 < a_5$, then $v_2(\mu_{a_1}^2) > v_2(4\mu_{a_5}^2)$. Using equation (42), we get $v_2(4\mu_{a_5}^2) = v_2(\mu_{b_5}^2)$. So $1 + v_2(\mu_{a_5}) = v_2(\mu_{b_5})$. Therefore, we have $b_5 = a_5 - 1 = m - 1$ and m is odd.

If $k = -2$, then $a_1 < a_2 = a_3 = a_4 = a_5$. Using the same method, we obtain $b_5 = a_5 + 1$. This completes the proof of Corollary 4.4. $\square$

5. The equation $\mu_{a_1}\mu_{a_2}\cdots\mu_{a_n} = \mu_{b_1}\mu_{b_2}\cdots\mu_{b_n}$

In this section, we will study the equation

$$M_0(\mu_{a_1}, \dots, \mu_{a_n}) = M_0(\mu_{b_1}, \dots, \mu_{b_n}).$$

As BANG and FUGLEDE [3] have solved the above equation when $n = 2$, we will start with $n = 3$. We obtain the following result.

Theorem 5.1. *Let $a_1 = b_1 - 1$, $a_2 = b_2 - 1$, and $a_3 = b_3 + 1$. Then every solution (b_1, b_2, b_3) of the equation*

$$\mu_{a_1}\mu_{a_2}\mu_{a_3} = \mu_{b_1}\mu_{b_2}\mu_{b_3} \tag{43}$$

can be represented by

$$\begin{cases} b_1 = s(2t + d), \\ b_2 = t\left(2s + \frac{4st - 1}{d}\right), \\ b_3 = 2st - 1, \end{cases} \tag{44}$$

where $s, t, d \in \mathbb{N}$ and $d \mid (4st - 1)$.

PROOF. We write $b_1 = s_1$, $b_2 = t_1$, $b_3 = u_1 - 1$. Then equation (43) becomes

$$\frac{2s_1 - 1}{2s_1} \cdot \frac{2t_1 - 1}{2t_1} = \frac{2u_1 - 1}{2u_1}. \quad (45)$$

So we obtain $2s_1 t_1 (2u_1 - 1) = u_1 (2s_1 - 1)(2t_1 - 1)$. This implies that u_1 is even and $u_1 \mid (2s_1 t_1)$. Put $u_1 = 2v_1$, $v_1 \in \mathbb{N}$. Thus, one can see that $v_1 \mid s_1 t_1$. Therefore, there exist $s, t \in \mathbb{N}$ such that $v_1 = st$ with $s \mid s_1$, and $t \mid t_1$. Set $s_1 = sx$, $t_1 = ty$, where $x, y \in \mathbb{N}$. Hence, from (45), we get

$$2xs + 2yt = xy + 1, \quad (46)$$

from which we conclude that

$$y = \frac{2xs - 1}{x - 2t} = 2s + \frac{4st - 1}{x - 2t}.$$

As y is an integer, it is clear that $(x - 2t) \mid (4st - 1)$. So we put $d = x - 2t$ and we see that $x = 2t + d$, $y = 2s + \frac{4st - 1}{d}$. Therefore, one obtains (44). This completes the proof of Theorem 5.1 $\square$

Examples 5.2. (1) Let $s = k$, $t = 1$, and $d = 1$. From (44), we get

$$b_1 = 3k, b_2 = 6k - 1, b_3 = 2k - 1.$$

This is another example of Remarks (1) in [1].

(2) Let $s = 3k + 1$, $d = 3$, then $t = 1$. From (44), we get

$$b_1 = 15k + 5, b_2 = 10k + 3, b_3 = 6k + 1.$$

Therefore, equation (43) has the solution given by

$$\mu_{15k+4} \mu_{10k+2} \mu_{6k+2} = \mu_{15k+5} \mu_{10k+3} \mu_{6k+1}.$$

For $n \geq 4$ and $k = l = 0$, we will prove the following result.

Theorem 5.3. Let $n \geq 4$, $0 \leq a_1 < a_2 < \cdots < a_n$, and $0 \leq b_1 < b_2 < \cdots < b_n$. Then the equation

$$\mu_{a_1} \mu_{a_2} \cdots \mu_{a_n} = \mu_{b_1} \mu_{b_2} \cdots \mu_{b_n} \quad (47)$$

has infinitely many solutions satisfying $a_i \neq b_j$, for any $1 \leq i, j \leq n$.

PROOF. We use the solutions of equation (43) to construct an infinite number of solutions of equation (47).

Take $t = 1$ in (44), the equation

$$\mu_p \mu_q \mu_r = \mu_u \mu_v \mu_w, \quad p < q < r, \quad u < v < w$$

has the solution

$$(p, q, r; u, v, w) = \left(2s, 2s + \frac{4s-1}{d} - 1, s(2+d) - 1; 2s-1, 2s + \frac{4s-1}{d}, s(2+d) \right). \quad (48)$$

As the order of the indexes is not important, we slightly change the order the indexes. This doesn't affect the result. First, we set $d = d_3 = 1$ in (48). Taking $s = k_1$, we see that

$$a_1 = 2k_1, \quad a_2 = 3k_1 - 1, \quad a_3 = 6k_1 - 2, \quad b_1 = 2k_1 - 1, \quad b_2 = 3k_1, \quad b_3 = 6k_1 - 1$$

is a solution of equation (47) when $n = 3$. Second, we put $d = d_4 = 5$ in (48). In order to have $5 \mid 4s - 1$, we take $s = 5k'_1 + 4$. Therefore,

$$\begin{aligned} a_{41} &= 10k'_1 + 8, & a_{42} &= 14k'_1 + 10, & a_{43} &= 35k'_1 + 27, \\ b_{41} &= 10k'_1 + 7, & b_{42} &= 14k'_1 + 11, & b_{43} &= 35k'_1 + 28 \end{aligned}$$

is a solution of the equation $\mu_{a_{41}} \mu_{a_{42}} \mu_{a_{43}} = \mu_{b_{41}} \mu_{b_{42}} \mu_{b_{43}}$. Put $a_3 = a_{43}$. Since $\gcd(6, 35) = 1$, then the solutions of the equation $6k_1 - 2 = 35k'_1 + 27$ are $k_1 = 35k_2 - 1, k'_1 = 6k_2 - 1$, where $k_2 \in \mathbb{N}$. As

$$\frac{\mu_{a_2}}{\mu_{b_2}} \cdot \frac{\mu_{a_3}}{\mu_{b_3}} = \frac{\mu_{b_1}}{\mu_{a_1}}, \quad \frac{\mu_{a_{42}}}{\mu_{b_{42}}} \cdot \frac{\mu_{a_{43}}}{\mu_{b_{43}}} = \frac{\mu_{b_{41}}}{\mu_{a_{41}}}, \quad (49)$$

$a_3 = a_{43}$, and $b_3 = b_{43}$, then $\frac{\mu_{a_3}}{\mu_{b_3}} = \frac{\mu_{a_{43}}}{\mu_{b_{43}}}$. Hence we get

$$\frac{\mu_{b_{41}}}{\mu_{a_{41}}} \cdot \frac{\mu_{b_{42}}}{\mu_{a_{42}}} \cdot \frac{\mu_{a_2}}{\mu_{b_2}} = \frac{\mu_{b_1}}{\mu_{a_1}}. \quad (50)$$

Therefore, when $n = 4$, equation (47) has the solution

$$(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4) = (a_1, a_2, b_{41}, b_{42}, b_1, b_2, a_{41}, a_{42}).$$

Now let us give two particular solutions of equation (47), for $n = 4$. The first example consists in taking $k_1 = 35k_2 - 1$, $k'_1 = 6k_2 - 1$. Thus one gets the following solution of equation (47)

$$\begin{aligned} a_1 &= 70k_2 - 2, & a_2 &= 105k_2 - 4, & a_3 &= 60k_2 - 3, & a_4 &= 84k_2 - 3, \\ b_1 &= 70k_2 - 3, & b_2 &= 105k_2 - 3, & b_3 &= 60k_2 - 2, & b_4 &= 84k_2 - 4. \end{aligned}$$

For the second example, we take $d = d_5 = 11$ in (48). To satisfy the condition $11 \mid 4s - 1$, we consider $s = 11k_3 + 3$. Therefore,

$$\begin{aligned} a_{51} &= 22k_3 + 6, & a_{52} &= 26k_3 + 6, & a_{53} &= 143k_3 + 38, \\ b_{51} &= 22k_3 + 5, & b_{52} &= 26k_3 + 7, & b_{53} &= 143k_3 + 39 \end{aligned}$$

is a solution of equation (43). Again here, we slightly change the order of the indexes. In the equation

$$\mu_{a_1}\mu_{a_2}\mu_{a_3}\mu_{a_4} = \mu_{b_1}\mu_{b_2}\mu_{b_3}\mu_{b_4}, \quad (51)$$

we take $a_4 = a_{53}$, where $a_4 = b_4 - 1$, i.e. $a_4 = 84k_2 - 4$ and $b_4 = 84k_2 - 3$. Since $\gcd(84, 143) = 1$, then the equation $84k_2 - 4 = 143k_3 + 38$ has a solution of the form $k_2 = 143u + 72$, $k_3 = 84u + 42$. Therefore, from $\frac{\mu_{a_4}}{\mu_{b_4}} = \frac{\mu_{a_{53}}}{\mu_{b_{53}}}$, $\frac{\mu_{a_{53}}}{\mu_{b_{53}}} = \frac{\mu_{b_{51}}\mu_{b_{52}}}{\mu_{a_{51}}\mu_{a_{52}}}$ and (51), one can see that equation (47) has the solution

$$(a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5) = (a_1, a_2, a_3, b_{51}, b_{52}, b_1, b_2, b_3, a_{51}, a_{52}),$$

for $n = 5$.

In general, we use a similar method. Put $d = d_n$ in (48). Then there exist k_n, r_n such that $d_n \mid 4s - 1$, where $s = d_n k_n + r_n$. So $(a_{n1}, a_{n2}, a_{n3}, b_{n1}, b_{n2}, b_{n3})$ is a solution of equation (48). Now, we suppose that equation (47) has a solution $(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n)$, with $a_n = b_n - 1$. The goal is to show that equation (47) has a solution for $n + 1$. Set $d = d_{n+1}$ in (48), where d_{n+1} satisfies

$$\gcd(d_{n+1}, 2 \prod_{j=3}^n d_j(d_j + 2)) = 1. \quad (52)$$

In order to have $d_{n+1} \mid 4s - 1$, we can determine k_{n+1}, r_{n+1} with $s = d_{n+1}k_{n+1} + r_{n+1}$ so that $(a_{n+1,1}, a_{n+1,2}, a_{n+1,3}, b_{n+1,1}, b_{n+1,2}, b_{n+1,3})$ is a solution of equation (48). Taking $a_n = a_{n+1,3}$, from (52), we see that the equation $a_n = a_{n+1,3}$ has a solution. Hence, equation (47) has the solution

$$\begin{aligned} (a_1, a_2, \dots, a_n, a_{n+1}, b_1, b_2, \dots, b_n, b_{n+1}) \\ = (a_1, a_2, \dots, a_{n-1}, b_{n+1,1}, b_{n+1,2}, b_1, b_2, \dots, b_{n-1}, a_{n+1,1}, a_{n+1,2}), \end{aligned}$$

for $n + 1$. Therefore, equation (47) has infinitely many solutions satisfying $a_i \neq b_j$, for any $1 \leq i, j \leq n$. So the proof of Theorem 5.3 is complete. $\square$

6. Conjectures

Using Corollary 2.3 and Theorem 2.5, we believe that the product of n normalized binomial mid-coefficients $\mu_{a_1}\mu_{a_2}\cdots\mu_{a_n}$ cannot be an n -power of rational number, except when $a_1 = a_2 = \cdots = a_n$. So we make the following conjecture.

Conjecture 6.1. *The equation*

$$\mu_{a_1}\mu_{a_2}\cdots\mu_{a_n} = q^n, \quad n \geq 3, \quad n \in \mathbb{N}, \quad q \in \mathbb{Q} \quad (53)$$

only has the trivial solution $a_1 = a_2 = \cdots = a_n$.

For $n = 4$, we have the following conjecture.

Conjecture 6.2. *Equation (39) only has the trivial solution, except for $k = 1$, in which case $(\{a_i, a_j, a_n, a_l\}, \{b_i, b_j, b_n, b_l\}) \in (\{1, 3, 3, m\}, \{2, 2, 2, m\})$, where $m \geq 0$.*

In [1], ALZER and FUGLEDE proved the following result.

Proposition 6.3. *Let $k \neq 0$, $p, q, r \geq 0$ be integers, then the equation $\mu_p^k + \mu_q^k = \mu_r^k$ has only solutions $(k, p, q, r) = (1, 1, 1, 0), (-1, 0, 0, 1)$.*

Let $a_1, a_2, \dots, a_n, k$ be integers. We consider the general equation

$$\mu_{a_1}^k + \cdots + \mu_{a_n}^k = \mu_b^k, \quad k \neq 0, \quad n \geq 3, \quad (54)$$

and we set the following conjecture.

Conjecture 6.4. *Equation (54) has only the solutions given by*

- $\mu_2 + \mu_3 + \mu_3 = \mu_0$, when $n = 3$,
- $2^k \mu_1^k = \mu_0^k$, $2^k \mu_0^{-k} = \mu_1^{-k}$, where $n = 2^k$, $k \geq 2$, $k \in \mathbb{N}$.

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