

Biharmonic maps from Finsler spaces

By NICOLETA VOICU (Brasov)

Abstract. The notions of bienergy of a smooth mapping and of biharmonic map between Riemannian manifolds are extended to the case when the domain is Finslerian. We determine the first and the second variation of the bienergy functional, the equations of Finsler-to-Riemann biharmonic maps and some specific examples. We prove that two notable results in Riemannian geometry concerning the inexistence of nonharmonic biharmonic maps still hold true in this case.

1. Introduction

Biharmonic mappings, as a generalization of harmonic ones, are among the most important mappings in physics; initially appearing from problems of elasticity theory and fluid mechanics, [24], in the latter decades, they proved to be useful also in computer graphics, geometry processing, [12] and radar imaging, [1]. Mathematical arguments, [17], for the use of biharmonic maps include the fact that harmonic maps do not always exist – and biharmonic maps can “succeed where harmonic maps have failed” – together with stability issues. On the other side, Finslerian models seem to gain more and more ground in domains such as: kinematics, elasticity theory, [6], seismic ray theory, [3], [29], [30], gravity theories, [11], [19], [28], geometrical optics, [2], thermodynamics, statistical mechanics, [2], [21], biology, [2], [4].

While in Riemannian geometry, biharmonic mappings have been quite intensively studied (see, for instance, [5], [8], [17], [22], [23]), to our knowledge, in Finsler geometry, only harmonic maps have been considered so far, [14], [15], [16],

Mathematics Subject Classification: 53B40, 53C60, 58E20, 31B30.

Key words and phrases: Finsler space, bienergy, biharmonic map, harmonic map.

[18], [20], [25]. Still, the rich potential of Finslerian geometric models makes us think that such a study is at least necessary.

As a first step in this direction, we study in this paper biharmonic mappings having as domain real Finsler spaces (M, g) and as codomain, Riemannian ones $(\tilde{M}, \tilde{g})$ or, briefly, Finsler-to-Riemann mappings. Our study is a continuation of the one of Finsler-to-Riemann harmonic mappings made by XIAOHUAN MO and collaborators ([14], [15], [16]).

First of all, we extend the concept of bienergy functional for Finsler-to-Riemann mappings and determine its Euler–Lagrange equations, i.e., the equations of Finsler-to-Riemann biharmonic maps. This process points out a generalization of the rough Laplacian from Riemannian geometry.

Any Finsler-to-Riemann harmonic map is biharmonic. Just as in Riemannian geometry, there exist several cases in which the converse is also true; two notable results in Riemannian geometry, due to GUOYING JIANG, [8], and C. ONICIUC, [22], respectively, can be generalized without difficulty to our situation:

- 1) Any biharmonic mapping whose domain is compact and boundaryless and whose codomain has nonpositive sectional curvature, is harmonic.
- 2) Any biharmonic mapping whose codomain has strictly negative sectional curvature, obeying the conditions: a) the norm of its tension is constant and b) its rank is greater or equal to 2 at least at one point of its domain, is harmonic.

Further, we study the biharmonicity of the identity map $\text{id} : (M, g) \rightarrow (M, \tilde{g})$ in two cases of Finsler-to-Riemann transformations of metrics $g \mapsto \tilde{g}$, thus pointing out examples of nonharmonic biharmonic maps. The second case, that of linearized perturbations, is inspired from general relativity; even though we only consider here positive definite metrics, in our opinion, it is illustrative.

In the last section, we determine the second variation of the bienergy functional. Except for the facts that each of the expressions of the tension and of the rough Laplacian gains an extra term and of the use of nonlinear connections on TM , the first and second variation of the bienergy remain formally similar to their Riemannian counterparts.

In the study of a Finsler space (M, g) , there are two major– and equivalent – approaches: the one based on the tangent bundle (TM, π, M) , via horizontal lifts, and the one based on the pullback bundle π^*TM . As noticed in [20], the study of harmonic maps between real Finsler manifolds is usually carried out on π^*TM (as in [14], [25]) while in the case of complex Finsler manifolds, it relies on the geometry of TM . In order to obtain a more unified method, we preferred to work, also in the real case, on TM ; the geometric structures we used are the TM -correspondents of those in [14], [15], [16].

2. Biharmonic maps in Riemannian geometry

In this section, we present in brief some results in [17], [8], [5].

Let (M, g) and $(\tilde{M}, \tilde{g})$ be two $\mathcal{C}^\infty$ -smooth, connected Riemannian manifolds without boundary, of dimensions n and $\tilde{n}$; unless elsewhere specified, we will assume, as in [8], that M is compact and orientable. On the two manifolds, we denote the local coordinates by $(x^i)_{i=\overline{1,n}}$, $(\tilde{x}^\alpha)_{\alpha=\overline{1,\tilde{n}}}$, the Levi-Civita connections by ∇ , $\tilde{\nabla}$ (with coefficients Γ^i_{jk} , $\tilde{\Gamma}^\alpha_{\beta\gamma}$) and by $\Gamma(E)$, $\Gamma(\tilde{E})$, the modules of $\mathcal{C}^\infty$ -smooth sections of any vector bundles $E, \tilde{E}$ over M and $\tilde{M}$. Commas $_{,i}$ and $_{,\alpha}$ will mean partial differentiation with respect to x^i and $\tilde{x}^\alpha$ and ∂_i , $\tilde{\partial}_\alpha$, the natural bases of the modules $\Gamma(TM)$ and $\Gamma(T\tilde{M})$, respectively.

A $\mathcal{C}^\infty$ -smooth mapping $\phi : M \rightarrow \tilde{M}$ is called *harmonic*, if it is a critical point of the *energy functional*

$$E : \mathcal{C}^\infty(M, \tilde{M}) \rightarrow \mathbb{R}, \quad E(\phi) = \frac{1}{2} \int_M \|d\phi\|^2 d\mathcal{V}_g, \quad (1)$$

where $d\phi$ is regarded as a section of the bundle $\phi^{-1}T\tilde{M} \otimes T^*M$, $\|d\phi\|^2 = \text{trace}_g(\phi^*\tilde{g}) = g^{ij}\tilde{g}_{\alpha\beta}\phi^\alpha_{,i}\phi^\beta_{,j}$ is the squared Hilbert-Schmidt norm of $d\phi$ and $d\mathcal{V}_g$ is the Riemannian volume element on M .

Harmonic maps are solutions of the equation $\tau(\phi) = 0$, where, [17],

$$\tau(\phi) = g^{ij}\{\nabla^\phi_{\partial_i} d\phi(\partial_j) - d\phi(\nabla_{\partial_i}\partial_j)\} =: g^{ij}(\nabla^\phi_{\partial_i} d\phi)\partial_j, \quad (2)$$

is a section of the bundle $\phi^{-1}T\tilde{M}$, called the *tension* of ϕ and ∇^ϕ is the connection induced by $\tilde{\nabla}$ in the pullback bundle $\phi^{-1}T\tilde{M}$, [5]. In local writing:

$$\tau^\alpha(\phi) = g^{ij}\{\phi^\alpha_{,ij} + \tilde{\Gamma}^\alpha_{\beta\gamma}\phi^\beta_{,i}\phi^\gamma_{,j} - \Gamma^k_{ij}\phi^\alpha_{,k}\}. \quad (3)$$

The above notion of harmonicity generalizes the usual one for mappings between Euclidean spaces; notable examples include geodesic curves and minimal Riemannian immersions.

Biharmonic maps $\phi \in \mathcal{C}^\infty(M, \tilde{M})$ are defined as critical points of the *bienergy functional*:

$$E_2(\phi) = \frac{1}{2} \int_M \langle \tau(\phi), \tau(\phi) \rangle d\mathcal{V}_g; \quad (4)$$

here $\langle \cdot, \cdot \rangle$ denotes the scalar product on the fibers of $T\tilde{M}$, determined by $\tilde{g}$. The Euler-Lagrange equation attached to the bienergy is, [17]:

$$\tau_2(\phi) = 0, \quad (5)$$

where:

- the *bitension* $\tau_2(\phi)$ of ϕ is the section of the bundle $\phi^{-1}T\tilde{M}$ given by:

$$\tau_2(\phi) = -\Delta^\phi \tau(\phi) - \text{trace}_g(R^{\tilde{\nabla}}(d\phi, \tau(\phi))d\phi); \quad (6)$$

- the operator $\Delta^\phi = -g^{ij}(\nabla_{\partial_i}^\phi \nabla_{\partial_j}^\phi - \nabla_{\nabla_{\partial_i}^\phi \partial_j}^\phi)$ is the *rough Laplacian* (which coincides, up to a sign, with the tensor version of the classical Laplace–Beltrami operator), acting on sections of $\phi^{-1}T\tilde{M}$;
- $R^{\tilde{\nabla}}$ denotes the curvature tensor of the Levi–Civita connection $\tilde{\nabla}$ on the codomain $(\tilde{M}, \tilde{g})$.

Remarks. 1) Equation (5) is the Riemannian generalization of the biharmonic equation in Euclidean spaces, [24].

2) Any harmonic map $\phi : M \rightarrow \tilde{M}$ is biharmonic.

3. Finsler structures

In the following, except for the metric structure on M (and related quantities) we preserve the notations and conventions in Section 2. We denote by TM and $T\tilde{M}$ the tangent bundles of the manifolds M and $\tilde{M}$ and their local coordinates, by $(x, y) := (x^i, y^i)$, $(\tilde{x}, \tilde{y}) := (\tilde{x}^\alpha, \tilde{y}^\alpha)$; dots $\cdot_i$ and $\cdot_\alpha$ will mean partial differentiation with respect to y^i and $\tilde{y}^\alpha$.

A. Metric structure: A *Finsler structure*, [7], [16], on the manifold M is a function $F : TM \rightarrow \mathbb{R}$ with the properties:

- 1) $F(x, y)$ is $\mathcal{C}^\infty$ -smooth for $y \neq 0$ and continuous at $y = 0$.
- 2) $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$, 3) The *Finslerian metric tensor*:

$$g_{ij}(x, y) := \frac{1}{2}(F^2(x, y))_{\cdot_{ij}} \quad (7)$$

is positive definite for any (x, y) with $y \neq 0$.

The arc length of a curve c on the Finsler space (M, g) is given, [7], by:

$$l(c) = \int_c F(x, dx). \quad (8)$$

Condition 2) above ensures the independence of $l(c)$ of the chosen parametrization of c .

B. Nonlinear connection and adapted frame on TM : Ehresmann (or *nonlinear*, [6], Ch. 2) connections on TM , described as splittings $TTM = HTM \oplus$

VTM , with $VTM = \text{Span} \left(\frac{\partial}{\partial y^i} \right)$, help simplify computations in Finsler geometry and obtain geometric objects with simple transformation rules. A typical choice is the *Cartan nonlinear connection*, built as follows, ([7], Section 2.3).

Geodesics of the Finsler space (M, g) are defined as critical points c of the arc length (8); in the natural parametrization, their equations are:

$$\frac{dy^i}{ds} + 2G^i(x, y) = 0, \quad y = \dot{x}, \quad (9)$$

with $2G^i(x, y) = \frac{1}{2}g^{ih}((F^2)_{,h,k}y^k - (F^2)_{,h})$; this defines the local coefficients $G^i_{\cdot j} = G^i_{\cdot j}(x, y)$ of the Cartan nonlinear connection as:

$$G^i_{\cdot j} := G^i_{\cdot j}. \quad (10)$$

The Cartan nonlinear connection gives rise to the adapted basis:

$$\left(\delta_i = \frac{\partial}{\partial x^i} - G^j_{\cdot i}(x, y) \frac{\partial}{\partial y^j}, \quad \dot{\partial}_i = \frac{\partial}{\partial y^i} \right) \quad (11)$$

on $\Gamma(TTM)$ and to its dual $(dx^i, \delta y^i = dy^i + G^i_{\cdot j} dx^j)$.

With respect to coordinate transformations on TM , induced by coordinate transformations $x^{i'} = x^{i'}(x)$ on M , the elements of the adapted basis/cobasis transform by the same rules as vector/covector fields on M ([6] pp. 8, 26).

Any vector field X on TM can be decomposed as: $X = hX + vX$, $hX = X^i \delta_i$, $vX = \hat{X}^i \dot{\partial}_i$; its *horizontal component* hX and its *vertical component* vX are vector fields on TM . This leads to a simple rule of transformation for X^i , $\hat{X}^i$. A similar situation holds for 1-forms $\omega = h\omega + v\omega$, $h\omega = \omega_i dx^i$, $v\omega = \hat{\omega}_i \delta y^i$ and, more generally, for tensors of any rank on TM .

Using the Cartan nonlinear connection, tangent vector fields to lifts $c' := (c, \dot{c})$ to TM of unit speed geodesics of M are always horizontal ([9], Section VIII.2). Another important property ([7], p. 36) is that, for any horizontal vector field $hX \in \Gamma(HTM)$:

$$(hX)F = 0. \quad (12)$$

The adapted basis $\{\delta_i, \dot{\partial}_i\}$ is generally a non-holonomic (non-coordinate) one, i.e., the Lie brackets of its elements do not all vanish. More precisely, these are:

$$[\delta_j, \delta_k] = R^i_{jk}(x, y) \dot{\partial}_i, \quad [\delta_j, \dot{\partial}_k] = G^i_{jk}(x, y) \dot{\partial}_i, \quad [\dot{\partial}_j, \dot{\partial}_k] = 0;$$

where:

$$R^i_{jk}(x, y) = \delta_k G^i_{\cdot j} - \delta_j G^i_{\cdot k}, \quad G^i_{jk}(x, y) := G^i_{\cdot j \cdot k}(x, y). \quad (13)$$

C. As covariant differentiation rule on TM , we will use the one given by the *Chern–Rund affine connection* D on TM ([6], Section 5.6), locally described by:

$$D_{\delta_k} \delta_j = \Gamma^i_{jk} \delta_i, \quad D_{\delta_k} \dot{\delta}_j = \Gamma^i_{jk} \dot{\delta}_i, \quad D_{\dot{\delta}_k} \delta_j = D_{\dot{\delta}_k} \dot{\delta}_j = 0, \quad (14)$$

where $\Gamma^i_{jk} = \frac{1}{2} g^{ih} (\delta_k g_{hj} + \delta_j g_{hk} - \delta_h g_{jk})$ are the “adapted” Christoffel symbols¹ of g . The Chern–Rund connection preserves by parallelism the horizontal and vertical distributions on TTM , i.e.,

$$D_X(hY) = hD_X Y, \quad D_X(vY) = vD_X Y; \quad (15)$$

it is generally, only h -metrical:

$$D_{hX} g = 0, \quad \forall X \in \Gamma(TTM).$$

The Chern–Rund connection D has nontrivial torsion:

$$T = R^i_{jk} \dot{\delta}_i \otimes dx^k \otimes dx^j + P^i_{jk} \dot{\delta}_i \otimes \delta y^k \otimes dx^j, \quad (16)$$

with R^i_{jk} as in (13) and $P^i_{jk} = G^i_{jk} - \Gamma^i_{jk}$; the latter defines a horizontal 1-form:

$$P = P_i dx^i, \quad P_i := P^j_{ij}, \quad (17)$$

which will be used in the following. We notice that the torsion of D has only vertical components, i.e.:

$$hT(X, Y) = 0, \quad \forall X, Y \in \Gamma(TTM). \quad (18)$$

The curvature R of D is locally described by:

$$\begin{aligned} R = & R^i_{jkl} \delta_i \otimes dx^l \otimes dx^k \otimes dx^j + R^i_{jkl} \dot{\delta}_i \otimes dx^l \otimes dx^k \otimes \delta y^j \\ & + P^i_{jk} \delta_i \otimes \delta y^l \otimes dx^k \otimes dx^j + P^i_{jk} \dot{\delta}_i \otimes \delta y^l \otimes dx^k \otimes \delta y^j, \end{aligned} \quad (19)$$

where $R^i_{jkl} = \delta_l \Gamma^i_{jk} - \delta_k \Gamma^i_{jl} + \Gamma^h_{jk} \Gamma^i_{hl} - \Gamma^h_{jl} \Gamma^i_{hk}$ and $P^i_{jk} = \Gamma^i_{jk \cdot l}$.

¹In the Finslerian case, the usual Christoffel symbols $\gamma^i_{jk} = \frac{1}{2} g^{ih} (g_{hj,k} + g_{hk,j} - g_{jk,h})$ do *not* generally represent the coefficients of an affine connection on TM .

We consider the same notions for the Riemannian manifold $(\tilde{M}, \tilde{g})$ and designate them by tildes. In this case: $\tilde{\Gamma}^\alpha_{\beta\gamma} = \tilde{G}^\alpha_{\beta\gamma} = \tilde{\gamma}^\alpha_{\beta\gamma}$, that is:

$$(\tilde{\nabla}_X Y)^{\tilde{h}} = \tilde{D}_{X^{\tilde{h}}} Y^{\tilde{h}}, \quad \forall X, Y \in \Gamma(T\tilde{M}), \quad (20)$$

where the superscript $\tilde{h}$ indicates the horizontal lift of vector fields from $\tilde{M}$ to $T\tilde{M}$; also, $\tilde{G}^\alpha_{\beta\gamma}(\tilde{x}, \tilde{y}) = \tilde{\gamma}^\alpha_{\beta\gamma}(\tilde{x})\tilde{y}^\gamma$, $\tilde{P}^\alpha_{\beta\gamma} = 0$; the Chern–Rund connection becomes “fully” metrical:

$$\tilde{D}_X \tilde{g} = 0, \quad \forall X \in \Gamma(TT\tilde{M}). \quad (21)$$

The only nonzero local component of the curvature tensor $\tilde{R}$ is $\tilde{R}^\alpha_{\beta\gamma\delta}$, i.e.:

$$\tilde{R}(X, Y)Z = \tilde{R}(\tilde{h}X, \tilde{h}Y)Z, \quad \forall X, Y, Z \in \Gamma(TT\tilde{M}); \quad (22)$$

$\tilde{R}^\alpha_{\beta\gamma\delta}$ coincide with the components of the curvature $R^{\tilde{\nabla}}$ of the Levi–Civita connection of $\tilde{g}$ and are thus subject to the same symmetries. Ricci identities of $\tilde{D}$ ([6], p. 106) take the local form:

$$\begin{aligned} \tilde{D}_{\delta_\rho} \tilde{D}_{\delta_\gamma} Z^\alpha - \tilde{D}_{\delta_\gamma} \tilde{D}_{\delta_\rho} Z^\alpha &= \tilde{R}^\alpha_{\beta\gamma\rho} Z^\beta + \tilde{R}^\beta_{\gamma\rho} Z^\alpha{}_{;\beta}; \\ \tilde{D}_{\dot{\delta}_\rho} \tilde{D}_{\delta_\gamma} Z^\alpha - \tilde{D}_{\delta_\gamma} \tilde{D}_{\dot{\delta}_\rho} Z^\alpha &= 0, \quad \forall Z = Z^\alpha \delta_\alpha \in \Gamma(HT\tilde{M}). \end{aligned} \quad (23)$$

Another useful property of $\tilde{D}$ is:

$$\tilde{D}_{\tilde{\delta}_\beta} \tilde{y}^\alpha = 0. \quad (24)$$

The Riemannian metric $\tilde{g}$ gives rise to a scalar product on the fibers on $HT\tilde{M}$, which will be denoted by $\langle \cdot, \cdot \rangle$:

$$\langle X, Y \rangle = \tilde{g}_{\alpha\beta} X^\alpha Y^\beta, \quad \forall X = X^\alpha \tilde{\delta}_\alpha, Y = Y^\alpha \tilde{\delta}_\alpha \in \Gamma(HT\tilde{M}). \quad (25)$$

D. Volume form and integration domain. Consider the Riemannian volume element $d\mathcal{V}_g = \sqrt{\det G} dx \wedge \delta y = \det g dx \wedge \delta y$ on TM , determined by the *Sasaki lift*, ([26], p. 92) $G := g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j$ of g to TM . For an x, y -dependent function $\alpha : TM \rightarrow \mathbb{R}$, we will consider its integral on the total space of the unit ball bundle of M (which is compact, since g is positive definite), divided by the volume $\text{Vol } \mathbb{B}^n$ of the unit ball in the Euclidean space $\mathbb{R}^n$, i.e.,

$$\int_{BM} \alpha(x, y) d\mathcal{V}_g = \frac{1}{\text{Vol } \mathbb{B}^n} \int_M \left(\int_{B_x} \alpha(x, y) \det g(x, y) dy \right) dx, \quad (26)$$

where $B_x = \{y \in T_x M \mid g_{ij}(x, y)y^i y^j \leq 1\}$ (on M , this construction provides a generalization of the Riemannian volume element, called the *Holmes–Thompson volume element*, [26], p. 26).

The divergence $\operatorname{div} X = \frac{1}{\det g} \delta_i (X^i \det g) - G^j_{ij} X^i$, of a horizontal vector field $X = X^i \delta_i$ on TM , [31], can be expressed as:

$$\operatorname{div} X = D_{\delta_i} X^i - P(X);$$

we will also use this relation in the form:

$$g^{ij} \delta_i X_j = \operatorname{div} X + g^{ij} \Gamma^k_{ij} X_k + P_i X^i. \quad (27)$$

4. Some remarks on Finsler-to-Riemann maps

Let $\phi : M \rightarrow \tilde{M}$, $(x^i) \mapsto (\phi^\alpha(x^i))$ be $\mathcal{C}^\infty$ -smooth. Between the tangent bundles TM and $T\tilde{M}$, it acts the differential $\Phi := d\phi$ (regarded as a mapping between manifolds); throughout this section, we will use alternatively the two notations Φ and $d\phi$. The mapping Φ is locally described by:

$$\Phi : \tilde{x}^\alpha = \phi^\alpha(x), \quad \tilde{y}^\alpha = \phi^\alpha_{,j}(x) y^j.$$

Let us make the notation $\phi^{\alpha'}(x, y) := \phi^\alpha_{,j}(x) y^j$ and express in local coordinates the pushforward by Φ of a horizontal vector field $X \in \Gamma(HTM)$, i.e., $\Phi_* X = X(\phi^\alpha) \tilde{\partial}_\alpha + X(\phi^{\alpha'}) \dot{\tilde{\partial}}_\alpha$. Taking $X = X^i \delta_i$ we have, in the adapted bases given by the two Cartan nonlinear connections N (on TM) and $\tilde{N}$ (on $T\tilde{M}$):

$$\Phi_* X = X^i \phi^\alpha_{,i} \tilde{\partial}_\alpha + (X^i \delta_i \phi^{\alpha'} + \tilde{N}^\alpha_\beta X^i \phi^{\beta}_{,i}) \dot{\tilde{\partial}}_\alpha. \quad (28)$$

The horizontal component $\tilde{h}\Phi_* X$ will have a peculiar importance.

Lemma 1. *For any horizontal vector field $X = X^i \delta_i$ on TM :*

$$\tilde{h}\Phi_* X = X^i \phi^\alpha_{,i} \tilde{\partial}_\alpha = d\phi^{\tilde{h}}(X), \quad (29)$$

where $d\phi^{\tilde{h}} := \phi^\alpha_{,i} \tilde{\partial}_\alpha \otimes dx^i$ is the horizontal lift of the vector-valued 1-form $d\phi$.

The connection $\tilde{D}$ on $T\tilde{M}$ determines a connection $D^{d\phi}$ in the pullback bundle $d\phi^{-1}(TT\tilde{M})$:

$$D^{d\phi}_X(\Phi^* Y) := \tilde{D}_{\Phi_* X} Y, \quad \forall X \in \Gamma(TTM), \quad Y \in \Gamma(TT\tilde{M}), \quad (30)$$

where² Φ^* denotes pullback (precomposition) by Φ , i.e., Φ^*Y is the section of $d\phi^{-1}(TT\tilde{M})$ given by $(\Phi^*Y)_{(x,y)} := Y_{\Phi(x,y)}$, $(x,y) \in TM$.

The pulled back connection $D^{d\phi}$ allows us to differentiate vector fields – and further, 1-forms, tensors of arbitrary rank – on the image of $d\phi$ (which is contained in the codomain $T\tilde{M}$) with respect to vector fields on the domain TM . Moreover, setting $D^{d\phi}T := DT$ for tensors T on TM , the action of $D^{d\phi}$ can be naturally extended to sections of arbitrary tensor products of TTM , $TT\tilde{M}$, T^*TM etc. In local writing, taking into account (28) and (14), we will have, for instance:

$$\begin{aligned} D_{\delta_j}^{d\phi} \tilde{\delta}_\beta &= \tilde{D}_{\Phi_*\delta_j} \tilde{\delta}_\beta = \phi^\gamma_{,j} \tilde{\Gamma}^\alpha_{\beta\gamma} \tilde{\delta}_\alpha, & D_{\delta_j}^{d\phi} dx^i &= -\Gamma^i_{jk} dx^k, \\ D_{\delta_j}^{d\phi} (\tilde{\delta}_\beta \otimes dx^i) &= \phi^\gamma_{,j} \tilde{\Gamma}^\alpha_{\beta\gamma} \tilde{\delta}_\alpha \otimes dx^i - \Gamma^i_{jk} \tilde{\delta}_\beta \otimes dx^k. \end{aligned}$$

Taking into account the definition (30) and (15), it follows that the pullback connection $D^{d\phi}$ preserves the distributions generated by the Cartan nonlinear connection $\tilde{N}$ on $T\tilde{M}$:

$$D_X^{d\phi}(\tilde{h}Z) = \tilde{h}D_X^{d\phi}Z, \quad D_X^{d\phi}(\tilde{v}Z) = \tilde{v}D_X^{d\phi}Z, \quad \forall Z \in \Gamma(d\phi^{-1}(TT\tilde{M})). \quad (31)$$

Consider now a 1-parameter variation $f : I_\varepsilon \times M, f = f(\varepsilon, x)$, $f(0, x) = \phi(x)$ of ϕ and:

$$F := df : T(I_\varepsilon \times M) \rightarrow T\tilde{M}.$$

On $T(I_\varepsilon \times M)$, the local coordinates are $(\varepsilon, x, \varepsilon', y)$. Taking on the interval $I_\varepsilon \subset \mathbb{R}$, the Euclidean metric and the product Finsler metric on $I_\varepsilon \times M$, we will obtain a trivial prolongation of the Cartan nonlinear connection to this new manifold, which produces the adapted basis $\{\partial_\varepsilon, \delta_i, \partial_{\varepsilon'}, \dot{\partial}_i\}$ and a trivial prolongation of the Chern connection D (which we will denote again by D). i.e., $D_{\partial_\varepsilon} \delta_i = D_{\partial_\varepsilon} \dot{\partial}_i = 0$, $D_{\delta_i} \partial_\varepsilon = D_{\dot{\partial}_i} \partial_\varepsilon = 0$ etc. We also notice that $[\partial_\varepsilon, \delta_i] = 0$ and $[\partial_\varepsilon, \dot{\partial}_i] = 0$.

The connection D^{df} will be prolonged to $df^{-1}(TT\tilde{M})$, by:

$$D_{\partial_\varepsilon}^{df}(F^*X) := \tilde{D}_{F_*\partial_\varepsilon} X, \quad X \in \Gamma(TT\tilde{M}). \quad (32)$$

Lemma 2. For any $X \in \Gamma(TTM)$, there holds:

$$D_{\partial_\varepsilon}^{df}(df^{\tilde{h}}(X)) = D_X^{df}(df^{\tilde{h}}(\partial_\varepsilon)). \quad (33)$$

PROOF. Equality (18) says that $0 = \tilde{h}\tilde{T}(F_*\partial_\varepsilon, F_*X) = h\{D_{\partial_\varepsilon}^{df}(F_*X) - D_X^{df}(F_*\partial_\varepsilon) - [F_*X, F_*\partial_\varepsilon]\}$. The result follows then from (31) and $[X, \partial_\varepsilon] = 0$. $\square$

²Actually, to be very rigorous, we should have written: $D_X^{d\phi}(\Phi^*Y) := \Phi^*(\tilde{D}_{\Phi_*X}Y)$, thus pointing out that the right hand side is also to be evaluated at points $\Phi(x, y)$, with $(x, y) \in TM$. But, for the simplicity of writing, this composition will be understood, just as in [5], [22], without being explicitly indicated.

5. Bienergy and its first variation

The *energy* of a Finsler-to-Riemann mapping $\phi : M \rightarrow \tilde{M}$, was defined in [14] as: $E(\phi) = \frac{1}{2} \int_{BM} g^{ij} \tilde{g}_{\alpha\beta} \phi^\alpha_{,i} \phi^\beta_{,j} d\mathcal{V}_g$, (note that $\phi = \phi(x)$, hence $\phi^\alpha_{,i} = \delta_i \phi^\alpha$); in our language, this is:

$$E(\phi) = \frac{1}{2} \int_{BM} g^{ij} \langle d\phi^{\tilde{h}}(\delta_i), d\phi^{\tilde{h}}(\delta_j) \rangle d\mathcal{V}_g,$$

with $\langle \cdot, \cdot \rangle$ as in (25). The *tension* of $\phi : M \rightarrow \tilde{M}$, [14], is:

$$\tau(\phi) = g^{ij} \{ D_{\delta_i}^{d\phi} d\phi^{\tilde{h}}(\delta_j) - d\phi^{\tilde{h}}(D_{\delta_i} \delta_j) - P_i d\phi^{\tilde{h}}(\delta_j) \}. \quad (34)$$

Since, on one side, the vector fields $d\phi^{\tilde{h}}(\delta_j)$ and $d\phi^{\tilde{h}}(D_{\delta_i} \delta_j)$ are horizontal and, on the other side, $D_{\delta_i}^{d\phi}$ preserves the distributions generated by $\tilde{N}$ on $T\tilde{M}$, the tension $\tau(\phi)$ is itself horizontal, i.e., it can be regarded as a section of $(d\phi)^{-1}HT\tilde{M}$.

The mapping ϕ is *harmonic* iff its tension vanishes identically. It appears as natural:

Definition 3. The *bienergy* of a Finsler-to-Riemann map $\phi : M \rightarrow \tilde{M}$ is:

$$E_2(\phi) = \frac{1}{2} \int_{BM} \langle \tau(\phi), \tau(\phi) \rangle d\mathcal{V}_g. \quad (35)$$

A critical point of the bienergy (35) is called a *biharmonic map*.

In order to determine the critical points of E_2 , we take variations $f = f(\varepsilon, x)$ of ϕ as above and denote by

$$\mathbf{V} := (df(\partial_\varepsilon))^{\tilde{h}} = df^{\tilde{h}}(\partial_\varepsilon^{\tilde{h}}), \quad V := \mathbf{V}|_{\varepsilon=0}. \quad (36)$$

the horizontal lift of the associated deviation vector field $df(\partial_\varepsilon)$.

Since the Chern–Rund connection $\tilde{D}$ on the codomain $T\tilde{M}$ is metrical, we can write:

$$D_{\partial_\varepsilon}^{df}(\Phi^* \tilde{g}) = \tilde{D}_{\Phi_* \partial_\varepsilon} \tilde{g} = 0. \quad (37)$$

that is,

$$\frac{dE_2}{d\varepsilon}(f) = \frac{1}{2} \frac{d}{d\varepsilon} \langle \tau(f), \tau(f) \rangle d\mathcal{V}_g = \int_{BM} \langle D_{\partial_\varepsilon}^{df} \tau(f), \tau(f) \rangle d\mathcal{V}_g.$$

Let us evaluate the term $D_{\partial_\varepsilon}^{df} \tau(f)$:

$$D_{\partial_\varepsilon}^{df} \tau(f) = D_{\partial_\varepsilon}^{df} \{ g^{ij} (D_{\delta_i}^{df} (df^{\tilde{h}}(\delta_j)) - df^{\tilde{h}}(D_{\delta_i} \delta_j) - P_i df^{\tilde{h}}(\delta_j)) \}$$

$$= g^{ij} \left\{ D_{\partial_\varepsilon}^{df} D_{\delta_i}^{df} (df^{\tilde{h}}(\delta_j)) - D_{\partial_\varepsilon}^{df} (df^{\tilde{h}}(D_{\delta_i} \delta_j)) - P_i D_{\partial_\varepsilon}^{df} (df^{\tilde{h}}(\delta_j)) \right\}$$

(g^{ij} and P_i can be taken in front of the ∂_ε -derivative, since in their expressions $g^{ij} = g^{ij}(x, y)$, $P_i = P_i(x, y)$ the coordinates x, y do not depend on ε). Commuting derivatives by means of the curvature tensor of $\tilde{D}$, taking (22) and $[\delta_i, \partial_\varepsilon] = 0$ into account,

$$D_{\partial_\varepsilon}^{df} D_{\delta_i}^{df} (df^{\tilde{h}}(\delta_j)) = \tilde{R}(\mathbf{V}, df^{\tilde{h}}(\delta_i)) df^{\tilde{h}}(\delta_j) + D_{\delta_i}^{df} D_{\partial_\varepsilon}^{df} (df^{\tilde{h}}(\delta_j)).$$

By (33), the term $D_{\delta_i}^{df} D_{\partial_\varepsilon}^{df} (df^{\tilde{h}}(\delta_j))$ becomes $D_{\delta_i}^{df} D_{\delta_j}^{df} \mathbf{V}$. Using (33) also in the expression $D_{\partial_\varepsilon}^{df} (df^{\tilde{h}}(D_{\delta_i} \delta_j)) - P_i D_{\partial_\varepsilon}^{df} (df^{\tilde{h}}(\delta_j))$ and summing up, we get:

$$D_{\partial_\varepsilon}^{df} \tau(f) = g^{ij} \{ \tilde{R}(\mathbf{V}, df^{\tilde{h}}(\delta_i)) df^{\tilde{h}}(\delta_j) + D_{\delta_i}^{df} D_{\delta_j}^{df} \mathbf{V} - D_{D_{\delta_i} \delta_j}^{df} \mathbf{V} - P_i D_{\delta_j}^{df} \mathbf{V} \}. \quad (38)$$

We notice the operators

$$g^{ij} (-D_{\delta_i}^{df} D_{\delta_j}^{df} + D_{D_{\delta_i} \delta_j}^{df} + P_i D_{\delta_j}^{df}) =: \Delta^{df}, \quad \mathcal{J} = -\Delta^{df} - \text{trace}_g \tilde{R}(df^{\tilde{h}}, \cdot) df^{\tilde{h}}, \quad (39)$$

acting on sections of the bundle $(df)^{-1}(HT\tilde{M})$. With this, we have:

$$D_{\partial_\varepsilon}^{df} \tau(f) = -\Delta^{df} \mathbf{V} - g^{ij} \tilde{R}(df^{\tilde{h}}(\delta_i), \mathbf{V}) df^{\tilde{h}}(\delta_j) = \mathcal{J}(\mathbf{V}). \quad (40)$$

Evaluating at $\varepsilon = 0$ and substituting into the expression of the variation,

$$\frac{dE_2}{d\varepsilon}(f)|_{\varepsilon=0} = \int_{BM} \langle \mathcal{J}(V), \tau(\phi) \rangle d\mathcal{V}_g. \quad (41)$$

It remains to transform the above expression so as to have V in the right hand side of the scalar product. This will be easy using the following lemma.

Lemma 4. *The operators $\Delta^{d\phi}$ and $\mathcal{J}$ are self-adjoint:*

$$\begin{aligned} \int_{BM} \langle \Delta^{d\phi} X, Y \rangle d\mathcal{V}_g &= \int_{BM} \langle X, \Delta^{d\phi} Y \rangle d\mathcal{V}_g, \\ \int_{BM} \langle \mathcal{J} X, Y \rangle d\mathcal{V}_g &= \int_{BM} \langle X, \mathcal{J} Y \rangle d\mathcal{V}_g. \end{aligned} \quad (42)$$

for any $X, Y \in \Gamma(d\phi^{-1}(HT\tilde{M}))$.

PROOF. We start from the left hand side of the first relation in (42); integrating by parts the term $\int_{BM} \langle -g^{ij} D_{\delta_i}^{df} D_{\delta_j}^{df} X, Y \rangle d\mathcal{V}_g$ and applying (27), we get:

$$\int_{BM} \langle \Delta^{d\phi} X, Y \rangle d\mathcal{V}_g = - \int_{BM} g^{ij} \langle D_{\delta_i}^{d\phi} X, D_{\delta_j}^{d\phi} Y \rangle d\mathcal{V}_g. \quad (43)$$

Integrating once again by parts, we obtain the first identity in (42). The self-adjointness of $\mathcal{J}$ follows then from the symmetries of $\tilde{R}$. $\square$

The operator $\Delta^{d\phi}$ is a generalization of the rough Laplacian from Riemannian geometry, built in the same spirit as the horizontal Laplacian acting on differential forms in [31], [32]. Using Lemma 4 in (41), we get:

Proposition 5. a) *The first variation of the bienergy of a mapping $\phi : M \rightarrow \tilde{M}$ from the Finsler space (M, g) to the Riemann space $(\tilde{M}, \tilde{g})$ is:*

$$\frac{dE_2(f)}{d\varepsilon}|_{\varepsilon=0} = \int_{BM} \langle -\Delta^{d\phi}\tau(\phi) - \text{trace}_g \tilde{R}(d\phi^{\tilde{h}}, \tau(\phi))d\phi^{\tilde{h}}, V \rangle d\mathcal{V}_g; \quad (44)$$

b) *The mapping ϕ is biharmonic iff:*

$$\tau_2(\phi) := -\Delta^{d\phi}\tau(\phi) - \text{trace}_g \tilde{R}(d\phi^{\tilde{h}}, \tau(\phi))d\phi^{\tilde{h}} = 0. \quad (45)$$

Remarks. 1) In the above, we considered, as in [8], that M is compact and without boundary. Elsewhere, all the discussion can be made on a compact subset $\mathcal{D}$ of M ; in this case, we assume that, on the boundary of $\mathcal{D}$, the vector field V and the covariant derivatives $D_{\delta_i}^{d\phi}V$ vanish.

2) For any harmonic map ϕ from a Finsler space to a Riemann one, we have $\tau(\phi) = 0$, that is, $E_2(\phi) = 0$. Consequently, any harmonic map is biharmonic, namely, a minimum point for the bienergy functional. A biharmonic map which is not harmonic will be called *proper biharmonic*.

Particular cases: 1) If $\tilde{M} = \mathbb{R}^n$ with the Euclidean metric, then the biharmonic equation (45) becomes:

$$\Delta^{d\phi}\tau(\phi) = 0.$$

2) If $\tilde{M} = \mathbb{S}^n$ is the unit Euclidean sphere, then, using the expression of the Riemann tensor of a space form, we get that $\phi : M \rightarrow \mathbb{S}^n$ is biharmonic iff:

$$\Delta^{d\phi}\tau(\phi) + 2e(\phi)\tau(\phi) - \text{trace}_g \langle d\phi^{\tilde{h}}, \tau(\phi) \rangle d\phi^{\tilde{h}} = 0,$$

where $e(\phi) = \frac{1}{2} \text{trace}_g \langle d\phi^{\tilde{h}}, d\phi^{\tilde{h}} \rangle$ is the energy density of ϕ . The result is similar to the one in the Riemannian case ([5], Section 3.2.2).

3) (*Riemann-to-Riemann maps*) Assume that the domain (M, g) is Riemannian; then, relation (20) also holds true for D and ∇ , the Landsberg tensor P vanishes and thus, the rough Laplacian $\Delta^{d\phi}$ coincides (up to a horizontal lift) with the operator $\Delta^\phi = -g^{ij}(\nabla_{\partial_i}^\phi \nabla_{\partial_j}^\phi - \nabla_{\nabla_{\partial_i} \partial_j}^\phi)$ presented in Section 2:

$$l^{\tilde{h}}(\Delta^\phi U) = \Delta^{d\phi}(l^{\tilde{h}}U), \quad \forall U \in \phi^{-1}(\Gamma(T\tilde{M})).$$

It turns out that, in the case of Riemann-to-Riemann mappings, the bitension of a mapping ϕ obtained by us – let's denote it, for the moment, by $\tau_2^{\text{Finsler}}(\phi)$ – is the horizontal lift of the bitension $\tau_2^{\text{Riem}}(\phi)$ in Section 2: $\tau_2^{\text{Finsler}}(\phi) = (\tau_2^{\text{Riem}}(\phi))^{\tilde{h}}$. In local coordinates, the two versions of the bitension and, accordingly, of the biharmonic equation, coincide.

4) A *weakly Landsberg* manifold (M, g) is defined, ([26], Section 7.2), as a Finsler manifold for which $P = 0$ (this includes, but does not coincide with the class of Riemannian manifolds). If (M, g) is weakly Landsberg, then the local expressions of the tension, rough Laplacian and accordingly, of the biharmonic equation, are still the same as the ones in the Riemannian case – just, this time, depending on the fiber coordinates y^i : $\tau(\phi) = \text{trace}_g D^{d\phi}(d\phi^{\tilde{h}})$, $\Delta^{df} = g^{ij}(-D_{\delta_i}^{df} D_{\delta_j}^{df} + D_{D_{\delta_i} \delta_j}^{df})$.

6. Existence of proper biharmonic maps

The following two results represent generalizations to Finsler-to-Riemann maps of two theorems in [8] and [22] respectively.

Theorem 6. *If (M, g) is a compact Finslerian manifold without boundary and $(\tilde{M}, \tilde{g})$ is Riemannian with nonpositive sectional curvature, then any biharmonic map $\phi : M \rightarrow \tilde{M}$ is harmonic.*

PROOF. The strategy of proof is similar to the one in the Riemannian case, [8]. Let us consider the horizontal Laplace–Beltrami operator $\Delta\psi := -\text{div}(\text{grad}_h \psi)$ (see, for instance, [31]), acting on scalar functions $\psi : TM \rightarrow \mathbb{R}$. In local coordinates, we have $\text{grad}_h \psi := (g^{ij} \delta_j \psi) \delta_i$ and:

$$\Delta\psi = -(D_{\delta_i}(g^{ij} \delta_j \psi) - g^{ij} P_i \delta_j \psi). \quad (46)$$

We apply (46) to the function $\psi := \|\tau(\phi)\|^2$, defined on TM :

$$\begin{aligned} -\frac{1}{2} \Delta \|\tau(\phi)\|^2 &= \frac{1}{2} \{D_{\delta_i}(g^{ij} \delta_j \|\tau(\phi)\|^2) - g^{ij} P_i \delta_j \|\tau(\phi)\|^2\} \\ &= \frac{1}{2} g^{ij} \{\delta_i \delta_j \|\tau(\phi)\|^2 - \Gamma_{ij}^k \delta_k \|\tau(\phi)\|^2 - P_i \delta_j \|\tau(\phi)\|^2\}. \end{aligned} \quad (47)$$

Here, noticing again that the Chern–Rund connection $\tilde{D}$ is metrical, we can express $\delta_j \|\tau(\phi)\|^2$ in terms of $D^{d\phi}$ -covariant derivatives as: $\delta_j \|\tau(\phi)\|^2 = \delta_j \langle \tau(\phi), \tau(\phi) \rangle = 2 \langle D_{\delta_j}^{d\phi} \tau(\phi), \tau(\phi) \rangle$. Substituting the latter expression into (47) and taking into account that $\Gamma_{ij}^k \delta_k = D_{\delta_i} \delta_j$, we obtain, after a brief calculation:

$$-\frac{1}{2} \Delta \|\tau(\phi)\|^2 = -\langle \Delta^{d\phi} \tau(\phi), \tau(\phi) \rangle + g^{ij} \langle D_{\delta_i}^{d\phi} \tau(\phi), D_{\delta_j}^{d\phi} \tau(\phi) \rangle.$$

By means of the biharmonic equation (45), this becomes:

$$-\frac{1}{2}\Delta \|\tau(\phi)\|^2 = \langle \text{trace}_g \tilde{R}(d\phi^{\tilde{h}}, \tau(\phi)) d\phi^{\tilde{h}}, \tau(\phi) \rangle + g^{ij} \langle D_{\delta_i}^{d\phi} \tau(\phi), D_{\delta_j}^{d\phi} \tau(\phi) \rangle. \quad (48)$$

According to the hypothesis that the sectional curvature of $(\tilde{M}, \tilde{g})$ is nonpositive, the curvature term above is nonnegative; since $g^{ij} \langle D_{\delta_i}^{d\phi} \tau(\phi), D_{\delta_j}^{d\phi} \tau(\phi) \rangle$ (as a squared norm) is nonnegative, too, we get: $-\frac{1}{2}\Delta \|\tau(\phi)\|^2 \geq 0$.

On the other side, we have, [31], $\int_{BM} \Delta \|\tau(\phi)\|^2 d\mathcal{V}_g = 0$, hence, $\Delta \|\tau(\phi)\|^2 = 0$; thus, by (48), $g^{ij} \langle D_{\delta_i}^{d\phi} \tau(\phi), D_{\delta_j}^{d\phi} \tau(\phi) \rangle = 0$; as a consequence,

$$D_{\delta_j}^{d\phi} \tau(\phi) = 0. \quad (49)$$

Take the horizontal vector field $X := (g^{ij} \langle \phi_{,i}, \tau(\phi) \rangle) \delta_j$ on TM ; by (49), we get:

$$0 = \int_{BM} \text{div } X d\mathcal{V}_g = \int_{BM} \underbrace{\langle \tau(\phi), \tau(\phi) \rangle}_{\geq 0} d\mathcal{V}_g$$

and therefore, $\langle \tau(\phi), \tau(\phi) \rangle = 0 \Rightarrow \tau(\phi) = 0$, i.e., ϕ is harmonic. $\square$

Dropping any condition upon the compactness or on the boundary of M , we have:

Theorem 7. *Let (M, g) be an arbitrary Finsler space (not necessarily compact), $(\tilde{M}, \tilde{g})$, a Riemannian manifold with strictly negative sectional curvature and $\phi : M \rightarrow \tilde{M}$, a biharmonic map. If ϕ has the properties: 1) $\|\tau(\phi)\| = \text{const.}$ and 2) there exists a point $x_0 \in M$ at which the rank of ϕ is at least 2, then ϕ is harmonic.*

PROOF. The proof is similar to the one in the Riemannian case, [22]. From the hypothesis $\|\tau(\phi)\| = \text{const.}$, in (48), the left hand side is 0; but both terms in the right hand side are nonnegative, hence: $\langle \text{trace}_g \tilde{R}(d\phi^{\tilde{h}}, \tau(\phi)) d\phi^{\tilde{h}}, \tau(\phi) \rangle = 0$.

Let us orthonormalize the basis $\{\delta_i\}_{i=\overline{1,n}}$ of HTM and call this new basis $\{e_i\}_{i=\overline{1,n}}$. In terms of this basis, we can write: $\langle \text{trace}_g \tilde{R}(d\phi^{\tilde{h}}, \tau(\phi)) d\phi^{\tilde{h}}, \tau(\phi) \rangle = \sum_{i=1}^n \langle \tilde{R}(d\phi^{\tilde{h}}(e_i), \tau(\phi)) d\phi^{\tilde{h}}(e_i), \tau(\phi) \rangle$. Since $\text{Riem}_{\tilde{g}} < 0$, the only chance for $\langle \text{trace}_g \tilde{R}(d\phi^{\tilde{h}}, \tau(\phi)) d\phi^{\tilde{h}}, \tau(\phi) \rangle$ to be 0 is that all the (minus) sectional curvatures $\langle \tilde{R}(d\phi^{\tilde{h}}(e_i), \tau(\phi)) d\phi^{\tilde{h}}(e_i), \tau(\phi) \rangle$ be 0. This can, again happen only if: 1) either all $d\phi^{\tilde{h}}(e_i)$ ($i = \overline{1, n}$) are collinear with $\tau(\phi)$ (that is, collinear with one another), or: 2) $\tau(\phi) = 0$. But, at the point x_0 , we have $\text{rank}(\phi) \geq 2$, hence, the only remaining possibility is the second one, i.e., $\tau(\phi)(x_0) = 0$. Using $\|\tau(\phi)\| = \text{const.}$, it follows that $\tau(\phi) \equiv 0$, i.e., ϕ is harmonic. $\square$

7. Biharmonicity of the identity map

Throughout this section, we assume that $M = \tilde{M}$ (not necessarily compact), $\dim M = n$, and denote the coordinates on TM by (x^i, y^i) . Considering on M two metrics: a Riemannian one $\tilde{g}$ and a Finslerian one g , we will explore the biharmonicity of the Finsler-to-Riemann mapping:

$$\text{id} : (M, g) \rightarrow (M, \tilde{g}). \quad (50)$$

In this situation, there appear two adapted bases $(\delta_i, \dot{\partial}_i)$ and $(\tilde{\delta}_i, \dot{\partial}_i)$ on TM , together with the covariant differentiations given by $D, \tilde{D}$ and $D^{d(\text{id})}$. According to [16] (p. 115), the tension of the identity map has the local components

$$\tau^i(\text{id}) = g^{jk}(\tilde{\Gamma}_{jk}^i - G_{jk}^i) \quad (51)$$

(note: our G^i is half the one in [16]).

Let us evaluate (51). We denote $2b := F^2 - \tilde{F}^2$, i.e.:

$$g_{ij}(x, y) = \tilde{g}_{ij}(x) + b_{ij}(x, y), \quad (52)$$

where the function $b = b(x, y)$ is homogeneous of degree 2 in y and $b_{ij} = b_{.ij}$.

In the expression of the spray coefficients (9) of g , i.e.,

$$2G^i(x, y) = \frac{1}{2}g^{ih} (F^2_{,k \cdot h} y^k - F^2_{,h}), \quad (53)$$

the derivative $F^2_{,k}$ can be written in terms of $\tilde{D}_{\tilde{\delta}_k}$ -covariant derivatives (denoted in the following by double bars $\|_k$), as: $F^2_{,k} = F^2_{\|k} + \tilde{G}^l_k F^2_{.l}$. Differentiating the latter relation with respect to y^h , contracting it with y^k and taking into account that $\tilde{G}^l$ are homogeneous of degree 2 in y , we get:

$$F^2_{,k \cdot h} y^k = F^2_{\|k \cdot h} y^k + y^k \tilde{G}^l_{hk} F^2_{.l} + y^k \tilde{G}^l_k F^2_{.lh} = F^2_{\|k \cdot h} y^k + \tilde{G}^l_h F^2_{.l} + 2\tilde{G}^l F^2_{.lh}.$$

We notice that, in the above, the last term is $2\tilde{G}^l F^2_{.lh} = 4\tilde{G}^l g_{lh}$. Expressing also $F^2_{,h}$ in terms of $\tilde{D}_{\tilde{\delta}_h}$ -covariant derivatives and substituting into (53), we are finally led to:

$$2G^i = 2\tilde{G}^i + 2B^i, \quad (54)$$

where:

$$2B^i := \frac{1}{2}g^{ih}(F^2_{\|k \cdot h} y^k - F^2_{\|h}). \quad (55)$$

The tension of id is:

$$\tau^i(\text{id}) = -g^{jk} B_{.j \cdot k}^i. \quad (56)$$

Remarks. 1) Suppose that b is parallel with respect to $\tilde{D}$. Then, according to (12) (applied to $\tilde{F}$ and its Cartan nonlinear connection $\tilde{N}$), we have: $\tilde{F}_{\parallel k}^2 = \tilde{\delta}_k \tilde{F} = 0$. As a consequence, $F_{\parallel k}^2 = \tilde{F}_{\parallel k}^2 + 2b_{\parallel k} = 0$, which entails $B^i = 0 \Rightarrow \tau^i = 0$. We obtain that, in this case, the identity map is harmonic, i.e., also biharmonic.

2) Assuming that g is a Berwald-type metric, i.e., $G_{jk}^i = G_{jk}^i(x)$, then there exists, [13], [27], a Riemannian metric $\tilde{g}$ such that $G_{jk}^i = \tilde{G}_{jk}^i$; in this case, the identity map $\text{id} : (M, g) \rightarrow (M, \tilde{g})$ is, again, harmonic, hence, biharmonic.

We will find in the following two examples of Finslerian perturbations b for which the identity of M is proper biharmonic.

With $\tau^i := \tau^i(\text{id})$, the relation between the $D^{d(\text{id})}$ - and $\tilde{D}$ -covariant derivatives of τ^i is:

$$D_{\delta_j}^{d(\text{id})} \tau^i = \delta_j \tau^i + \tilde{\Gamma}_{jk}^i \tau^k = \tau_{\parallel j}^i - B_{\cdot j}^k \tau_{\cdot k}^i. \quad (57)$$

Example 1. y -independent tension. Assume that the Finslerian function satisfies:

$$F_{\parallel h}^2 = \langle a, y \rangle_g y_h, \quad (58)$$

where $y_h = g_{hl} y^l$, $\langle a, y \rangle_g := g_{ij} a^i y^j$ and $a^i = a^i(x)$ are components of some vector field $a = a^i \partial_i$ on M . A brief calculation leads to: $F_{\parallel k \cdot j}^2 y^k - F_{\parallel j}^2 = a_j F^2$, that is, in (55), $2B^i = \frac{1}{2} a^i(x) F^2$. From (56), we obtain:

$$\tau^i = -\frac{1}{2} n a^i. \quad (59)$$

We notice that, in our case, $\tau^i = \tau^i(x)$; that is, relation (57) becomes simply: $D_{\delta_j}^{d(\text{id})} \tau^i = \tau_{\parallel j}^i$ and the biharmonic equation is written as:

$$g^{jk} (\tau_{\parallel j \parallel k}^i - \tilde{\Gamma}_{jk}^l \tau_{\parallel l}^i - \tilde{R}_{j \parallel k}^i \tau^l) = 0. \quad (60)$$

Here, taking into account that $\tilde{R}_{jhlk} = \tilde{R}_{lkjh}$, Ricci identities (23) for $\tilde{D}$ and the fact that $\tau^i = \tau^i(x)$ does not depend on y , the curvature term $\tilde{R}_{j \parallel k}^i \tau^l$ can be expressed by commuting $\tilde{D}$ -covariant derivatives of τ^i :

$$\tilde{R}_{j \parallel k}^i \tau^l = g^{ih} \tilde{R}_{jhlk} \tau^l = g^{ih} \tilde{R}_{lkjh} \tau^l = g^{ih} g_{km} (\tau_{\parallel j \parallel h}^m - \tau_{\parallel h \parallel j}^m).$$

It turns out that a sufficient condition for the biharmonicity of id is:

$$\tau_{\parallel j}^i = 0. \quad (61)$$

(Note: this statement is always true in the Riemann-to-Riemann case, but generally, not in the Finsler-to-Riemann one, where, as a rule, $\tau^i = \tau^i(x, y)$).

Using (59), we deduce that (61) is identically satisfied if the vector field $a^h = a^i \delta_i$ is parallel with respect to $\tilde{D}$. But, according to (20), this is nothing but: $\tilde{\nabla}_{\partial_i} a = 0$. In other words:

Proposition 8. *If, in (58), the nonzero vector field $a = a^i(x) \partial_i$ is parallel with respect to $\tilde{g}$, then the identity map $\text{id} : (M, g) \rightarrow (M, \tilde{g})$ is proper biharmonic.*

Example 2. Linearized Finslerian perturbations of the Euclidean metric. Assume that $(M, \tilde{g}_{ij}) = (\mathbb{R}^n, \delta_{ij})$ and the perturbation $b_{ij} =: \varepsilon_{ij}(x, y)$ is small (linearly approximable), that is, we may neglect all terms of degree greater than one in ε_{ij} and its derivatives, [10]. In this case, the inverse metric is given by: $g^{ik} = \delta^{ik} - \varepsilon^{ik}$ relation (55) becomes:

$$2B^i = \frac{1}{2} \delta^{ih} (\varepsilon_{hj,k} + \varepsilon_{hk,j} - \varepsilon_{jk,h}) y^j y^k.$$

We notice that the tension τ will be of the same order of smallness as ε ; it means that products of τ with ε and its derivatives can be neglected. For instance, we have: $B_l^h \tau^i_{,h} \simeq 0$, which, substituted into (57), leads to:

$$D_{\delta_j}^{d(\text{id})} \tau^i = \tau^i_{,j}.$$

The biharmonic equation takes the simple form: $\delta^{lm} \tau^i_{,l,m} = 0$. A sufficient condition for biharmonicity is

$$\tau^i_{,l} = 0,$$

(or: $\tau^i = \tau^i(y)$), that is, $\delta^{ih} (\varepsilon_{hj,k,l} + \varepsilon_{hk,j,l} - \varepsilon_{jk,h,l}) y^j y^k = 0$. obtain:

Proposition 9. *Let the Finsler metric g , with $g_{ij}(x, y) = \delta_{ij} + \varepsilon_{ij}(x, y)$, be a linearized perturbation of the Euclidean metric $\tilde{g} = (\delta_{ij})$ on $\mathbb{R}^n$. If the components $\varepsilon_{ij}(x, y)$ are non-constant and affine in x , then $\text{id} : (\mathbb{R}^n, g) \rightarrow (\mathbb{R}^n, \tilde{g})$ is proper biharmonic.*

8. Second variation of the bienergy

Take a biharmonic map $\phi : (M, g) \rightarrow (\tilde{M}, \tilde{g})$ and a smooth 2-parameter variation $f = f(\varepsilon_1, \varepsilon_2, x)$, $f(0, 0, x) = \phi$ of ϕ , with

$$\mathbf{V}_1 = df^{\tilde{h}}(\partial_{\varepsilon_1}), \quad \mathbf{V}_2 = df^{\tilde{h}}(\partial_{\varepsilon_2}), \quad V_1 := \mathbf{V}_1|_{\varepsilon_1=\varepsilon_2=0}, \quad V_2 := \mathbf{V}_2|_{\varepsilon_1=\varepsilon_2=0}$$

(if M has a boundary, then V_1 , V_2 and their δ_i -covariant derivatives are assumed to vanish on ∂M).

The deduction of the second variation of E_2 follows the same steps as in the Riemannian case, with two differences: in the expressions of $\tau(f)$ and of Δ^{df} , there appear extra terms and we have to take into account that $\Phi_*(\partial_{\varepsilon_i})$ is, generally, not horizontal. Fortunately, as we will see below, these will finally not complicate the expression of the variation.

We denote, for simplicity, $\tau := \tau(f)$. According to (44), (45):

$$\frac{\partial E_2(f)}{\partial \varepsilon_1} = \int_{BM} \langle \tau_2(f), \mathbf{V}_1 \rangle d\mathcal{V}_g; \quad (62)$$

differentiating with respect to ε_2 :

$$\frac{\partial^2 E_2(f)}{\partial \varepsilon_1 \partial \varepsilon_2} = \int_{BM} \{ \langle D_{\partial_{\varepsilon_2}}^{df} \tau_2(f), \mathbf{V}_1 \rangle + \langle \tau_2(f), D_{\partial_{\varepsilon_2}}^{df} \mathbf{V}_1 \rangle \} d\mathcal{V}_g.$$

At $\varepsilon_1 = \varepsilon_2 = 0$, since ϕ is biharmonic, the second term in the right hand side will vanish. It is thus enough to evaluate the first one; we have:

$$D_{\partial_{\varepsilon_2}}^{df} \tau_2(f) = -D_{\partial_{\varepsilon_2}}^{df} (\Delta^{df} \tau) - D_{\partial_{\varepsilon_2}}^{df} (\text{trace}_g \tilde{R}(df^{\tilde{h}}, \tau) df^{\tilde{h}}). \quad (63)$$

The covariant derivative of the Laplacian $-\Delta^{df} \tau$ is:

$$T_1 := -D_{\partial_{\varepsilon_2}}^{df} (\Delta^{df} \tau) = g^{ij} D_{\partial_{\varepsilon_2}}^{df} (D_{\delta_i}^{df} D_{\delta_j}^{df} \tau - D_{D_{\delta_i} \delta_j}^{df} \tau - P_i D_{\delta_j}^{df} \tau). \quad (64)$$

Commuting covariant derivatives by means of the curvature $\tilde{R}$ (twice for the term $D_{\delta_i}^{df} D_{\delta_j}^{df} \tau$), taking into account that $[\partial_{\varepsilon}, \delta_i] = 0$ and (22), we find:

$$\begin{aligned} T_1 = & g^{ij} \{ \tilde{R}(\mathbf{V}_2, df^{\tilde{h}}(\delta_i)) D_{\delta_j}^{df} \tau + D_{\delta_i}^{df} (\tilde{R}(\mathbf{V}_2, df^{\tilde{h}}(\delta_j)) \tau + D_{\delta_j}^{df} D_{\partial_{\varepsilon_2}}^{df} \tau) \\ & - \tilde{R}(\mathbf{V}_2, df^{\tilde{h}}(D_{\delta_i} \delta_j)) \tau - D_{D_{\delta_i} \delta_j}^{df} D_{\partial_{\varepsilon_2}}^{df} \tau - P_i \tilde{R}(\mathbf{V}_2, df^{\tilde{h}}(\delta_j)) \tau - P_i D_{\delta_j}^{df} D_{\partial_{\varepsilon_2}}^{df} \tau \}. \end{aligned}$$

The terms in $D_{\partial_{\varepsilon_2}}^{df} \tau$ can be grouped into $-\Delta^{df} (D_{\partial_{\varepsilon_2}}^{df} \tau)$:

$$\begin{aligned} T_1 = & -\Delta^{df} (D_{\partial_{\varepsilon_2}}^{df} \tau) + g^{ij} \{ \tilde{R}(\mathbf{V}_2, df^{\tilde{h}}(\delta_i)) D_{\delta_j}^{df} \tau \\ & + D_{\delta_i}^{df} (\tilde{R}(\mathbf{V}_2, df^{\tilde{h}}(\delta_j)) \tau) - \tilde{R}(\mathbf{V}_2, df^{\tilde{h}}(D_{\delta_i} \delta_j)) \tau - P_i \tilde{R}(\mathbf{V}_2, df^{\tilde{h}}(\delta_j)) \tau \}. \end{aligned}$$

Splitting $D_{\delta_i}^{df} (\tilde{R}(\mathbf{V}_2, df^{\tilde{h}}(\delta_j)) \tau)$ as a sum of derivatives, we recognize in the resulting expression $\tilde{R}(\mathbf{V}_2, \tau) \tau$:

$$T_1 = -\Delta^{df} (D_{\partial_{\varepsilon_2}}^{df} \tau) + \tilde{R}(\mathbf{V}_2, \tau) \tau + g^{ij} \{ (D_{\delta_i}^{df} \tilde{R})(\mathbf{V}_2, df^{\tilde{h}}(\delta_j)) \tau$$

$$+ 2\tilde{R}(\mathbf{V}_2, df^{\tilde{h}}(\delta_i))D_{\delta_j}^{df}\tau + \tilde{R}(D_{\delta_i}^{df}\mathbf{V}_2, df^{\tilde{h}}(\delta_j))\tau\}. \quad (65)$$

The curvature term $T_2 := -D_{\partial_{\varepsilon_2}}^{df}(\text{trace}_g \tilde{R}(df^{\tilde{h}}, \tau)df^{\tilde{h}})$ in (63) is:

$$\begin{aligned} T_2 = & -g^{ij}\{(D_{\partial_{\varepsilon_2}}^{df}\tilde{R})(df^{\tilde{h}}(\delta_i), \tau)df^{\tilde{h}}(\delta_j) + \tilde{R}(D_{\partial_{\varepsilon_2}}^{df}df^{\tilde{h}}(\delta_i), \tau)df^{\tilde{h}}(\delta_j) \\ & + \tilde{R}(df^{\tilde{h}}(\delta_i), D_{\partial_{\varepsilon_2}}^{df}\tau)df^{\tilde{h}}(\delta_j) + \tilde{R}(df^{\tilde{h}}(\delta_i), \tau)D_{\partial_{\varepsilon_2}}^{df}(df^{\tilde{h}}(\delta_j))\}. \end{aligned} \quad (66)$$

Taking into account that $\tilde{R} = \tilde{R}(x)$ only, we obtain $D_{\partial_{\varepsilon_2}}^{df}\tilde{R} = \tilde{D}_{\mathbf{V}_2}\tilde{R}$. Transforming $D_{\partial_{\varepsilon_2}}^{df}df^{\tilde{h}}(\delta_i)$, $D_{\partial_{\varepsilon_2}}^{df}df^{\tilde{h}}(\delta_j)$ by (33) and then using first Bianchi identity in the second term:

$$\begin{aligned} T_2 = & -g^{ij}\{(\tilde{D}_{\mathbf{V}_2}\tilde{R})(df^{\tilde{h}}(\delta_i), \tau)df^{\tilde{h}}(\delta_j) + \tilde{R}(D_{\delta_i}^{df}\mathbf{V}_2, \tau)df^{\tilde{h}}(\delta_j) \\ & + \tilde{R}(df^{\tilde{h}}(\delta_i), D_{\partial_{\varepsilon_2}}^{df}\tau)df^{\tilde{h}}(\delta_j) + \tilde{R}(df^{\tilde{h}}(\delta_i), \tau)\tilde{D}_{\delta_j}\mathbf{V}_2\} \\ = & -g^{ij}\{(\tilde{D}_{\mathbf{V}_2}\tilde{R})(df^{\tilde{h}}(\delta_i), \tau)df^{\tilde{h}}(\delta_j) + 2\tilde{R}(df^{\tilde{h}}(\delta_i), \tau)\tilde{D}_{\delta_j}\mathbf{V}_2 \\ & - \tilde{R}(df^{\tilde{h}}(\delta_j), D_{\delta_i}^{df}\mathbf{V}_2)\tau + \tilde{R}(df^{\tilde{h}}(\delta_i), D_{\partial_{\varepsilon_2}}^{df}\tau)df^{\tilde{h}}(\delta_j)\}. \end{aligned} \quad (67)$$

Second, and then first Bianchi identities for the $(\tilde{D}_{\mathbf{V}_2}\tilde{R})$ -term tell us that:

$$\begin{aligned} -g^{ij}(\tilde{D}_{\mathbf{V}_2}\tilde{R})(df^{\tilde{h}}(\delta_i), \tau)df^{\tilde{h}}(\delta_j) = & g^{ij}\{(\tilde{D}_{\tau}\tilde{R})(\mathbf{V}_2, df^{\tilde{h}}(\delta_i))df^{\tilde{h}}(\delta_j) \\ & - (\tilde{D}_{\delta_i}\tilde{R})(df^{\tilde{h}}(\delta_j), \tau)\mathbf{V}_2 - (\tilde{D}_{\delta_i}\tilde{R})(\mathbf{V}_2, df^{\tilde{h}}(\delta_j))\tau\}. \end{aligned}$$

Substituting into T_2 and adding: $T_1 + T_2 = D_{\partial_{\varepsilon_2}}^{df}\tau_2(f)$, we get:

$$\begin{aligned} D_{\partial_{\varepsilon_2}}^{df}\tau_2(f) = & \mathcal{J}(D_{\partial_{\varepsilon_2}}^{df}\tau) + \tilde{R}(\mathbf{V}_2, \tau)\tau + g^{ij}\{(\tilde{D}_{\tau}\tilde{R})(\mathbf{V}_2, df^{\tilde{h}}(\delta_i))df^{\tilde{h}}(\delta_j) \\ & - (D_{\delta_i}^{df}\tilde{R})(df^{\tilde{h}}(\delta_j), \tau)\mathbf{V}_2 + 2\tilde{R}(\mathbf{V}_2, df^{\tilde{h}}(\delta_i))D_{\delta_j}^{df}\tau - 2\tilde{R}(df^{\tilde{h}}(\delta_i), \tau)D_{\delta_i}^{df}\mathbf{V}_2\}, \end{aligned}$$

with $\mathcal{J}$ as in (39). Using (40) and evaluating at $\varepsilon_1 = \varepsilon_2 = 0$, we get:

Proposition 10. *The second variation of the bienergy of a Finsler-to-Riemann biharmonic map $\phi : M \rightarrow \tilde{M}$ is:*

$$\begin{aligned} \frac{\partial^2 E_2(f)}{\partial \varepsilon_1 \partial \varepsilon_2} \Big|_{\varepsilon_1 = \varepsilon_2 = 0} = & \int_{BM} \langle V_1, \mathcal{J}^2 V_2 + \tilde{R}(V_2, \tau)\tau \\ & + g^{ij}\{(\tilde{D}_{\tau}\tilde{R})(V_2, df^{\tilde{h}}(\delta_i))df^{\tilde{h}}(\delta_j) - (D_{\delta_i}^{df}\tilde{R})(df^{\tilde{h}}(\delta_j), \tau)V_2 \\ & + 2\tilde{R}(V_2, df^{\tilde{h}}(\delta_i))D_{\delta_j}^{df}\tau - 2\tilde{R}(df^{\tilde{h}}(\delta_i), \tau)D_{\delta_i}^{df}V_2\} \rangle d\mathcal{V}_g. \end{aligned} \quad (68)$$

In particular cases (for instance, $\tilde{M} = \mathbb{R}^n$ or S^n), (68) becomes considerably simpler.

The Hessian $\mathcal{H} : (V_1, V_2) \mapsto \mathcal{H}(V_1, V_2) = \frac{\partial^2 E_2(f)}{\partial \varepsilon_1 \partial \varepsilon_2} |_{\varepsilon_1 = \varepsilon_2 = 0}$ of the bienergy is a symmetric bilinear form. A solution ϕ of the biharmonic equation is *stable* if the quadratic form $\mathcal{H}(V, V)$ is nonnegative for any V . As an example, harmonic maps are stable biharmonic maps.

ACKNOWLEDGMENTS. 1) The work was supported by the Sectorial Operational Program Human Resources Development (SOP HRD), financed from the European Social Fund and by Romanian Government under the Project number POSDRU/89/1.5/S/59323.

2) Special thanks to prof. G. MUNTEANU for proofreading the text.

References

- [1] L. E. ANDERSSON, T. ELFVING and G. H. GOLUB, Solution of biharmonic equations with application to radar imaging, *J. Comput. Appl. Math.* **94** (1998), 153–180.
- [2] P. L. ANTONELLI, R. S. INGARDEN and M. MATSUMOTO, The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology, *Kluwer Acad. Publ.*, 1993.
- [3] P. L. ANTONELLI, A. BÓNA and M. A. SLAWIŃSKI, Seismic rays as Finsler geodesics, *Nonlinear Anal.: Real World Appl.* **4** (2003), 711–722.
- [4] P. L. ANTONELLI, S. F. RUTZ and C. E. HIRAKAWA, The mathematical theory of endosymbiosis I, *Nonlinear Anal.: Real World Appl.* **12** (2011), 3238–3251.
- [5] A. BALMUS, Biharmonic Maps and Submanifolds, *Geom. Balkan Press, Bucharest*, 2009.
- [6] I. BUCATARU and R. MIRON, Finsler-Lagrange Geometry – Applications to Dynamical Systems, *Ed. Acad. Romane, Bucharest*, 2007.
- [7] D. BAO, S. S. CHERN and Z. SHEN, An Introduction to Riemann–Finsler Geometry (Graduate Texts in Mathematics; 200), *Springer Verlag*, 2000.
- [8] JIANG GUOYING, 2-harmonic maps and their first and second variational formulas, *Note Mat.* **28** (2009), 209–232.
- [9] R. MIRON and M. ANASTASIEI, The Geometry of Lagrange Spaces: Theory and Applications, *Kluwer Acad. Publ.*, 1994.
- [10] L.D. LANDAU and E. M. LIFSCHIZ, The Classical Theory of Fields, *Elsevier*, 1987, 4th edn..
- [11] XIN LI and ZHE CHANG, Modified Friedmann model in Randers–Finsler space of approximate Berwald type as a possible alternative to dark energy hypothesis, *Phys. Lett. B* **676** (2009), 173–176.
- [12] Y. LIPMAN, R. M. RUSTAMOV and T. A. FUNKHOUSER, Biharmonic distance, *ACM Trans. Graph.* **29** (2010), 27:1–27:11.
- [13] V. S. MATVEEV, Riemannian metrics having common geodesics with Berwald metrics, *Publ. Math. Debrecen* **74** (2009), 405–416.
- [14] XIAOHUAN MO, Harmonic maps from Finsler manifolds, *Illinois J. Math.* **45** (2001), 1331–1345.

- [15] XIAOHUAN MO and YUNYAN YANG, The existence of harmonic maps from Finsler manifolds to Riemannian manifolds, *Sci. China Ser. A* **48** (2005), 115–130.
- [16] XIAOHUAN MO, An Introduction to Finsler Geometry, *World Scientific*, 2006.
- [17] S. MONTALDO and C. ONICIUC, A survey on biharmonic maps between Riemannian manifolds, *Rev. Un. Mat. Argentina* **47** (2006), 1–22.
- [18] H. VON DER MOSEL and S. WINKLMANN, On weakly harmonic maps from Finsler to Riemannian manifolds, *Ann. Inst. Henri Poincaré (C) Nonlinear Anal.* **26** (2009), 39–57.
- [19] G. MUNTEANU and N. ALDEA, A complex Finsler approach of gravity, *Int. J. Geom. Meth. Modern Phys.* **9** (2012).
- [20] S. NISHIKAWA, Harmonic maps of Finsler manifolds, Topics in Differential Geometry, (Mihai Adela et al., eds.), *Ed. Acad. Romane, Bucharest*, 2008, 207–247.
- [21] T. OOTSUKA, Finsler structure in thermodynamics and statistical mechanics, *Acta Math. Acad. Paedagog. Nyhiregyhaziensis* **26** (2010), 377–382.
- [22] C. ONICIUC, Biharmonic maps between Riemannian manifolds, *An. Stiint. Univ. Al. I. Cuza* **48** (2002), 237–248.
- [23] C. ONICIUC, On the second variation formula for biharmonic maps to a sphere, *Publ. Math. Debrecen* **67** (2005), 285–303.
- [24] A. P. S. SELVADURAI, Partial Differential Equations in Mechanics 2: The Biharmonic Equation, Poisson's Equation, *Springer*, 2000.
- [25] YIBING SHEN and YAN ZHANG, Second variation of harmonic maps between Finsler manifolds, *Sci. China Ser. A* **47** (2004), 39–51.
- [26] ZHONGMIN SHEN, Lectures on Finsler Geometry, *World Scientific*, 2001.
- [27] Z. SZABÓ, Positive definite Berwald spaces, *Tensor* **35** (1981), 25–39.
- [28] S. VACARU, Einstein gravity, Lagrange–Finsler geometry, and nonsymmetric metrics, *SIGMA* **4** (2008).
- [29] T. YAJIMA and H. NAGAHAMA, Finsler geometry of seismic ray path in anisotropic media, doi:10.1098/rspa.2008.0453, *Proc. R. Soc. A* **35** (2009).
- [30] T. YAJIMA, K. YAMASAKI and H. NAGAHAMA, Finsler metric and elastic constants for weak anisotropic media, *Nonlinear Anal.: Real World Appl.* **12** (2011), 3177–3184.
- [31] ZHONG CHUNPING and ZHONG TONGDE, Horizontal Laplace operator in real Finsler vector bundles, *Acta Mathematica Scientia* **28B** (2008), 128–140.
- [32] ZHONG CHUNPING, Laplacians on the tangent bundle of Finsler manifold, *Balkan J. Geom. Appl.* **16** (2011), 170–181.

NICOLETA VOICU
 DEPARTMENT OF MATHEMATICS
 AND COMPUTER SCIENCE
 “TRANSILVANIA” UNIVERSITY OF BRASOV
 ROMANIA
E-mail: nico.voicu@unitbv.ro

(Received October 18, 2012; revised August 6, 2013)