

A note on pseudorandom subsets formed by generalized cyclotomic classes

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Abstract. Recently Z. Chen has constructed a family of pseudorandom subsets by generalized cyclotomic classes, and studied the well-distribution and correlation measures. In this paper we further studied the correlation measures of the subsets.

§1. Introduction

Let subset $\mathcal{R} \subset \{1, 2, \dots, N\}$. Define the sequence

$$E_N = E_N(\mathcal{R}) = (e_1, e_2, \dots, e_N) \in \left\{1 - \frac{|\mathcal{R}|}{N}, -\frac{|\mathcal{R}|}{N}\right\}^N$$

by

$$e_n = \begin{cases} 1 - \frac{|\mathcal{R}|}{N} & \text{for } n \in \mathcal{R}, \\ -\frac{|\mathcal{R}|}{N} & \text{for } n \notin \mathcal{R}. \end{cases}$$

C. DARTYGE and A. SÁRKÖZY [2] introduced the following measures of pseudorandomness: the *well-distribution measure* of the subset $\mathcal{R}$ is defined by

$$W(\mathcal{R}, N) = \max_{a, b, t} \left| \sum_{j=0}^{t-1} e_{a+jb} \right|,$$

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where the maximum is taken over all $a, b, t \in \mathbb{N}$ with $1 \leq a \leq a + (t-1)b \leq N$. The *correlation measure of order k* of the subset $\mathcal{R}$ is defined by

$$C_k(\mathcal{R}, N) = \max_{M, D} \left| \sum_{n=1}^M e_{n+d_1} \cdots e_{n+d_k} \right|,$$

where the maximum is taken over all $D = (d_1, \dots, d_k)$ and M with $0 \leq d_1 < \dots < d_k \leq N - M$.

Later many pseudorandom subsets were given and studied. For example, suppose that p, q are two distinct primes satisfying “RSA type” with $2 < p < q < 2p$. Let $N = pq$, $d = (p-1, q-1)$ and $e = [p-1, q-1] = (p-1)(q-1)/d$. There exists a common primitive root g of both p and q . There also exists an integer x satisfying

$$x \equiv g \pmod{p}, \quad x \equiv 1 \pmod{q}.$$

The generalized cyclotomic classes of order d are defined by

$$D_i = \{g^s x^i : s = 0, 1, \dots, e-1\}, \quad i = 0, 1, \dots, d-1.$$

It is obvious that

$$D_i \cap D_j = \emptyset \quad \text{for } i \neq j, \quad |D_i| = e \quad \text{for } 0 \leq i \leq d-1,$$

and

$$\bigcup_{i=0}^{d-1} D_i = \{n : 0 \leq n < pq, (n, pq) = 1\}.$$

Z. CHEN [1] presented a large family of subsets formed by generalized cyclotomic classes, and studied the pseudorandom measures.

Proposition 1.1 (CHEN, 2010). *Let $f(x) = a_l x^l + \dots + a_1 x + a_0 \in \mathbb{Z}[x]$ with $(a_l, N) = 1$ and $0 < l < p < q$. Assume that $f(x)$ as a polynomial over $\mathbb{F}_p$ has no multiple roots in $\overline{\mathbb{F}}_p$ and $f(x)$ as a polynomial over $\mathbb{F}_q$ has no multiple roots in $\overline{\mathbb{F}}_q$. Define*

$$\mathcal{R} = \{n : 1 \leq n \leq N, [f(n)(\bmod N)] \in D_u\}$$

for some fixed u with $1 \leq u \leq d-1$. Then we have

$$W(\mathcal{R}, N) \ll l^2 N^{1/2} (1 + \log N), \quad C_2(\mathcal{R}, N) \ll l^2 N^{3/4} (1 + \log N).$$

In this paper we further study the correlation measures of the subsets. The first purpose of this paper is to give sharper estimates for $C_2(\mathcal{R}, N)$ and $C_3(\mathcal{R}, N)$.

Theorem 1.1. *Define $\mathcal{R}$ as in Proposition 1.1. Then we have*

$$C_2(\mathcal{R}, N) \ll lN^{3/4}, \quad (1.1)$$

$$C_3(\mathcal{R}, N) \ll lN^{3/4}. \quad (1.2)$$

Since $C_2(\mathcal{R}, N) \ll lN^{3/4}$ and $C_3(\mathcal{R}, N) \ll lN^{3/4}$, it is natural to expect that

$$C_k(\mathcal{R}, N) \ll lN^{3/4} \quad \text{for } k \geq 4.$$

However, in Section 4 we shall prove that $C_4(\mathcal{R}, N)$ is very large if p, q are known.

Theorem 1.2. *Define $\mathcal{R}$ as in Proposition 1.1. Then we have*

$$C_4(\mathcal{R}, N) \gg \frac{1}{d^3} N. \quad (1.3)$$

§2. Some lemmas

To complete the proof of theorems, we need the following lemmas.

Lemma 2.1. *Let $k, d \in \mathbb{N}$ and let p be a prime number with $d \mid p-1$. Let $r \leq k$, $0 \leq d_1 < \cdots < d_r < p$, $1 \leq D_1, \dots, D_r < d$ and $(D_1, \dots, D_r) = 1$. Suppose that $f \in \mathbb{F}_p[x]$ is a polynomial of degree l with no multiple roots in $\overline{\mathbb{F}}_p$. Define*

$$F(n) = f(n + d_1)^{D_1} \cdots f(n + d_r)^{D_r}$$

and write

$$F(n) = b(n - x_1)^{u_1} \cdots (n - x_s)^{u_s}$$

in $\overline{\mathbb{F}}_p$, where $x_i \neq x_j$ for $i \neq j$. If one of the following assumptions holds:

- (i) $k = 2$;
- (ii) d is a prime divisor of $p-1$ and $(4k)^l < p$;
- (iii) the polynomial $x^{p-1} + \cdots + x + 1$ is irreducible in $\mathbb{F}_w[x]$ for all prime factors w of d .

Then we have

$$(d, u_1, \dots, u_s) = 1.$$

PROOF. See [3].

□

Lemma 2.2. *Let p be a prime number, and let χ be a non-principal character modulo p of order d . Suppose that $f(x) \in \mathbb{F}_p[x]$ is not the constant multiple of the d -th power of a polynomial over $\mathbb{F}_p$. Then for all $a \in \mathbb{Z}$ we have*

$$\left| \sum_{n \in \mathbb{F}_p} \chi(f(n)) e\left(\frac{an}{p}\right) \right| \leq sp^{1/2},$$

where s denotes the number of distinct zeros of $f(x)$ in $\overline{\mathbb{F}_p}$.

PROOF. See Lemma 2 of [5]. □

§3. Proof of Theorem 1.1

Let $\widehat{Z_N^*}$ be the set of all Dirichlet characters modulo N . For any $\chi \in \widehat{Z_N^*}$ we write $\bar{\chi}$ the inverse of χ . By the definition of $\mathcal{R}$ we have

$$\begin{aligned} n \in \mathcal{R} &\iff [f(n)(\bmod N)] \in D_u \iff \frac{1}{\phi(N)} \sum_{s=0}^{e-1} \sum_{\chi \bmod N} \chi(f(n)) \bar{\chi}(g^s x^u) = 1 \\ &\iff \frac{1}{d} \sum_{\substack{\chi \bmod N \\ \chi(g)=1}} \chi(f(n)) \bar{\chi}(x^u) = 1. \end{aligned}$$

Define

$$\mathcal{H} = \{\chi \bmod N : \chi(g) = 1\} \quad \text{and} \quad G = \{g, g^2, \dots, g^e\}.$$

Then $\mathcal{H}$ is the annihilator of G in $\widehat{Z_N^*}$. By Theorem 5.6 of [4] we know that the order of $\mathcal{H}$ is $\frac{|\widehat{Z_N^*}|}{|G|} = d$. Denote $\mathcal{H}^* = \mathcal{H} \setminus \{\chi_0\}$. It is easy to show that any $\chi \in \mathcal{H}^*$ can be expressed as $\chi = \chi_p \chi_q$, where χ_p is a primitive character modulo p , and χ_q is a primitive character modulo q .

Let

$$\alpha = \frac{|\mathcal{R}|}{N}, \quad \beta = \frac{1}{d} - \alpha.$$

From [1] we know that

$$\alpha = 1/d + O(l^2 N^{-1/2}), \quad \beta = O(l^2 N^{-1/2}),$$

and

$$e_n = \begin{cases} \frac{1}{d} \sum_{\chi \in \mathcal{H}^*} \bar{\chi}(x^u) \chi(f(n)) + O(l^2 N^{-1/2}), & \text{if } (f(n), N) = 1, \\ -\frac{1}{d} + O(l^2 N^{-1/2}), & \text{if } (f(n), N) > 1. \end{cases} \quad (3.1)$$

For $M \in \mathbb{N}$, $d_1, d_2 \in \mathbb{Z}$ with $0 \leq d_1 < d_2 \leq N - M$, by (3.1) we have

$$\begin{aligned}
 & \sum_{n=1}^M e_{n+d_1} e_{n+d_2} \\
 &= \sum_{\substack{n=1 \\ (f(n+d_1)f(n+d_2), N)=1}}^M \left(\frac{1}{d} \sum_{\chi_1 \in \mathcal{H}^*} \bar{\chi}_1(x^u) \chi_1(f(n+d_1)) + O(l^2 N^{-1/2}) \right) \\
 & \quad \times \left(\frac{1}{d} \sum_{\chi_2 \in \mathcal{H}^*} \bar{\chi}_2(x^u) \chi_2(f(n+d_2)) + O(l^2 N^{-1/2}) \right) \\
 &+ O\left(\sum_{\substack{n=1 \\ (f(n+d_1)f(n+d_2), N)>1}}^M 1 \right) \\
 &= \frac{1}{d^2} \sum_{\chi_1 \in \mathcal{H}^*} \bar{\chi}_1(x^u) \sum_{\chi_2 \in \mathcal{H}^*} \bar{\chi}_2(x^u) \sum_{n=1}^M \chi_1(f(n+d_1)) \chi_2(f(n+d_2)) \\
 &+ O\left(\sum_{n=1}^M \frac{1}{d} \sum_{\chi \in \mathcal{H}^*} l^2 N^{-1/2} \right) + O(l^4 N^{-1}) \\
 &+ O\left(\sum_{\substack{n=1 \\ (f(n+d_1)f(n+d_2), N)>1}}^M 1 \right) \\
 &= \frac{1}{d^2} \sum_{\chi_1 \in \mathcal{H}^*} \bar{\chi}_1(x^u) \sum_{\chi_2 \in \mathcal{H}^*} \bar{\chi}_2(x^u) \sum_{n=1}^M \chi_1(f(n+d_1)) \chi_2(f(n+d_2)) \\
 &+ O(l^2 N^{1/2}). \tag{3.2}
 \end{aligned}$$

Noting that $\mathcal{H}$ is cyclic. For $\chi_1, \chi_2 \in \mathcal{H}^*$, there exists $\chi' \in \mathcal{H}^*$ such that

$$\chi_1 = (\chi')^{a_1}, \quad \chi_2 = (\chi')^{a_2}, \quad 1 \leq a_1, a_2 \leq d-1.$$

Define

$$\chi^* = (\chi')^{(a_1, a_2)}, \quad \delta_1 = \frac{a_1}{(a_1, a_2)}, \quad \delta_2 = \frac{a_2}{(a_1, a_2)}.$$

Then

$$\chi_1 = (\chi^*)^{\delta_1}, \quad \chi_2 = (\chi^*)^{\delta_2}, \quad 1 \leq \delta_1, \delta_2 \leq d-1, \quad (\delta_1, \delta_2) = 1,$$

and the order of χ^* is a divisor of d . Therefore the last sum of (3.2) can be reduced that

$$\sum_{n=1}^M \chi_1(f(n+d_1)) \chi_2(f(n+d_2)) = \sum_{n=1}^M \chi^*(f(n+d_1)^{\delta_1} f(n+d_2)^{\delta_2})$$

$$\begin{aligned}
&= \frac{1}{N} \sum_{n=1}^N \chi^*(f(n+d_1)^{\delta_1} f(n+d_2)^{\delta_2}) \sum_{u=1}^M \sum_{|a| < \frac{N}{2}} e\left(\frac{a(n-u)}{N}\right) \\
&= \frac{1}{N} \sum_{|a| < \frac{N}{2}} \sum_{u=1}^M e\left(-\frac{au}{N}\right) \sum_{n=1}^N \chi^*(f(n+d_1)^{\delta_1} f(n+d_2)^{\delta_2}) e\left(\frac{an}{N}\right). \quad (3.3)
\end{aligned}$$

Definition 3.1. For $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $(a, q) = 1$, let $i_q(a)$ denote the unique integer b such that $0 \leq b \leq q-1$ and $ab \equiv 1 \pmod{q}$.

Write $\chi^* = \chi_p \chi_q$ with χ_p is a character modulo p of order $t_p > 1$ and χ_q is a character modulo q of order $t_q > 1$, where t_p, t_q are divisors of d . We have

$$\begin{aligned}
&\sum_{n=1}^N \chi^*(f(n+d_1)^{\delta_1} f(n+d_2)^{\delta_2}) e\left(\frac{an}{N}\right) \\
&= \sum_{u=0}^{q-1} \sum_{v=0}^{p-1} \chi^*(f(up+vq+d_1)^{\delta_1} f(up+vq+d_2)^{\delta_2}) e\left(\frac{a(up+vq)}{N}\right) \\
&= \sum_{u=0}^{q-1} \chi_q(f(up+d_1)^{\delta_1} f(up+d_2)^{\delta_2}) e\left(\frac{au}{q}\right) \\
&\quad \times \sum_{v=0}^{p-1} \chi_p(f(vq+d_1)^{\delta_1} f(vq+d_2)^{\delta_2}) e\left(\frac{av}{p}\right) \\
&= \sum_{u=0}^{q-1} \chi_q(f(u+d_1)^{\delta_1} f(u+d_2)^{\delta_2}) e\left(\frac{ai_q(p)u}{q}\right) \\
&\quad \times \sum_{v=0}^{p-1} \chi_p(f(v+d_1)^{\delta_1} f(v+d_2)^{\delta_2}) e\left(\frac{ai_p(q)v}{p}\right).
\end{aligned}$$

We consider d_1, d_2 in three cases.

Case I: $d_1 \not\equiv d_2 \pmod{p}$ and $d_1 \not\equiv d_2 \pmod{q}$. By Lemma 2.1 and Lemma 2.2 we have

$$\begin{aligned}
&\sum_{u=0}^{q-1} \chi_q(f(u+d_1)^{\delta_1} f(u+d_2)^{\delta_2}) e\left(\frac{ai_q(p)u}{q}\right) \ll lq^{1/2}, \\
&\sum_{v=0}^{p-1} \chi_p(f(v+d_1)^{\delta_1} f(v+d_2)^{\delta_2}) e\left(\frac{ai_p(q)v}{p}\right) \ll lp^{1/2}.
\end{aligned}$$

Then

$$\sum_{n=1}^N \chi^*(f(n+d_1)^{\delta_1} f(n+d_2)^{\delta_2}) e\left(\frac{an}{N}\right) \ll l^2 N^{1/2}. \quad (3.4)$$

Combining (3.2), (3.3) and (3.4) we immediately get

$$\begin{aligned}
 & \sum_{n=1}^M e_{n+d_1} e_{n+d_2} \\
 & \ll \frac{1}{d^2} \sum_{\chi_1 \in \mathcal{H}^*} \sum_{\chi_2 \in \mathcal{H}^*} \frac{1}{N} \sum_{|a| < \frac{N}{2}} \left| \sum_{u=1}^M e\left(-\frac{au}{N}\right) \right| \cdot l^2 N^{1/2} + l^2 N^{1/2} \\
 & \ll \frac{l^2 N^{1/2}}{N} \left(M + \sum_{1 \leq a < \frac{N}{2}} \frac{N}{a} \right) + l^2 N^{1/2} \ll l^2 N^{1/2} \log N. \tag{3.5}
 \end{aligned}$$

Case II: $d_1 \equiv d_2 \pmod{p}$ and $d_1 \not\equiv d_2 \pmod{q}$. According to Lemma 2.1 and Lemma 2.2 we have

$$\sum_{u=0}^{q-1} \chi_q(f(u+d_1)^{\delta_1} f(u+d_2)^{\delta_2}) e\left(\frac{ai_q(p)u}{q}\right) \ll lq^{1/2}.$$

On the other hand, we get

$$\begin{aligned}
 & \sum_{v=0}^{p-1} \chi_p(f(v+d_1)^{\delta_1} f(v+d_2)^{\delta_2}) e\left(\frac{ai_p(q)v}{p}\right) = \sum_{v=0}^{p-1} \chi_p^{\delta_1+\delta_2}(f(v+d_1)) e\left(\frac{ai_p(q)v}{p}\right) \\
 & = \begin{cases} O(lp^{1/2}), & \text{if } t_p \nmid \delta_1 + \delta_2, \\ \sum_{v=0}^{p-1} e\left(\frac{ai_p(q)v}{p}\right) + O(l), & \text{if } t_p \mid \delta_1 + \delta_2, \end{cases} \\
 & = \begin{cases} O(lp^{1/2}), & \text{if } t_p \nmid \delta_1 + \delta_2, \\ O(l), & \text{if } t_p \mid \delta_1 + \delta_2 \text{ and } p \nmid a, \\ p + O(l), & \text{if } t_p \mid \delta_1 + \delta_2 \text{ and } p \mid a. \end{cases} \\
 & = \begin{cases} O(p), & \text{if } t_p \mid \delta_1 + \delta_2 \text{ and } p \mid a, \\ O(lp^{1/2}), & \text{otherwise.} \end{cases}
 \end{aligned}$$

Then we can have

$$\begin{aligned}
 & \sum_{n=1}^N \chi^*(f(n+d_1)^{\delta_1} f(n+d_2)^{\delta_2}) e\left(\frac{an}{N}\right) \\
 & = \begin{cases} O(lq^{1/2}p), & \text{if } t_p \mid \delta_1 + \delta_2 \text{ and } p \mid a, \\ O(l^2 N^{1/2}), & \text{otherwise.} \end{cases} \tag{3.6}
 \end{aligned}$$

Now from (3.2), (3.3) and (3.6) we have

$$\begin{aligned}
\sum_{n=1}^M e_{n+d_1} e_{n+d_2} &\ll \frac{1}{d^2} \sum_{\chi_1 \in \mathcal{H}^*} \sum_{\chi_2 \in \mathcal{H}^*} \frac{1}{N} \sum_{\substack{|a| < \frac{N}{2} \\ p|a}} \left| \sum_{u=1}^M e\left(-\frac{au}{N}\right) \right| \cdot lq^{1/2}p \\
&\quad + \frac{1}{d^2} \sum_{\chi_1 \in \mathcal{H}^*} \sum_{\chi_2 \in \mathcal{H}^*} \frac{1}{N} \sum_{|a| < \frac{N}{2}} \left| \sum_{u=1}^M e\left(-\frac{au}{N}\right) \right| \cdot l^2 N^{1/2} + l^2 N^{1/2} \\
&\ll \frac{lq^{1/2}p}{N} \left(M + \sum_{\substack{1 \leq a < \frac{N}{2} \\ p|a}} \frac{N}{a} \right) + \frac{l^2 N^{1/2}}{N} \left(M + \sum_{1 \leq a < \frac{N}{2}} \frac{N}{a} \right) + l^2 N^{1/2} \\
&\ll lN^{3/4}.
\end{aligned} \tag{3.7}$$

Case III: $d_1 \not\equiv d_2 \pmod{p}$ and $d_1 \equiv d_2 \pmod{q}$. Using the similar methods we have

$$\sum_{n=1}^M e_{n+d_1} e_{n+d_2} \ll lN^{3/4}. \tag{3.8}$$

Now combining (3.5), (3.7) and (3.8) we immediately get

$$C_2(\mathcal{R}, N) = \max_{M, D} \left| \sum_{n=1}^M e_{n+d_1} e_{n+d_2} \right| \ll lN^{3/4}.$$

This proves (1.1).

Let $M \in \mathbb{N}$, $d_1, d_2, d_3 \in \mathbb{Z}$ such that $0 \leq d_1 < d_2 < d_3 < N - M$. From (3.1) we can deduce that

$$\begin{aligned}
&\sum_{n=1}^M e_{n+d_1} e_{n+d_2} e_{n+d_3} \\
&= \sum_{\substack{n=1 \\ (\prod_{j=1}^3 f(n+d_j), N)=1}}^M \prod_{j=1}^3 \left(\frac{1}{d} \sum_{\chi_j \in \mathcal{H}^*} \bar{\chi}_j(x^u) \chi_j(f(n+d_j)) + O(l^2 N^{-1/2}) \right) + O(lN^{1/2}) \\
&= \frac{1}{d^3} \sum_{\chi_1 \in \mathcal{H}^*} \bar{\chi}_1(x^u) \sum_{\chi_2 \in \mathcal{H}^*} \bar{\chi}_2(x^u) \sum_{\chi_3 \in \mathcal{H}^*} \bar{\chi}_3(x^u) \\
&\quad \times \sum_{n=1}^M \chi_1(f(n+d_1)) \chi_2(f(n+d_2)) \chi_3(f(n+d_3))
\end{aligned}$$

$$\begin{aligned}
& + O\left(\sum_{n=1}^M \frac{1}{d^2} \sum_{\chi_1 \in \mathcal{H}^*} \sum_{\chi_2 \in \mathcal{H}^*} l^2 N^{-1/2}\right) + O\left(\sum_{n=1}^M \frac{1}{d} \sum_{\chi \in \mathcal{H}^*} l^4 N^{-1}\right) \\
& + O(l^6 N^{-3/2}) + O(l^2 N^{1/2}) = \frac{1}{d^3} \sum_{\chi_1 \in \mathcal{H}^*} \bar{\chi}_1(x^u) \sum_{\chi_2 \in \mathcal{H}^*} \bar{\chi}_2(x^u) \sum_{\chi_3 \in \mathcal{H}^*} \bar{\chi}_3(x^u) \\
& \times \sum_{n=1}^M \chi_1(f(n+d_1)) \chi_2(f(n+d_2)) \chi_3(f(n+d_3)) + O(l^2 N^{1/2}). \tag{3.9}
\end{aligned}$$

There exists $\chi^* \in \mathcal{H}^*$ such that

$$\begin{aligned}
\chi_1 &= (\chi^*)^{\delta_1}, \quad \chi_2 = (\chi^*)^{\delta_2}, \quad \chi_3 = (\chi^*)^{\delta_3}, \\
1 &\leq \delta_1, \delta_2, \delta_3 \leq d-1, \quad \text{and} \quad (\delta_1, \delta_2, \delta_3) = 1.
\end{aligned}$$

Then

$$\begin{aligned}
& \sum_{n=1}^M \chi_1(f(n+d_1)) \chi_2(f(n+d_2)) \chi_3(f(n+d_3)) \\
&= \sum_{n=1}^M \chi^*(f(n+d_1)^{\delta_1} f(n+d_2)^{\delta_2} f(n+d_3)^{\delta_3}) = \frac{1}{N} \sum_{|a| < \frac{N}{2}} \sum_{u=1}^M e\left(-\frac{au}{N}\right) \\
& \times \sum_{n=1}^N \chi^*(f(n+d_1)^{\delta_1} f(n+d_2)^{\delta_2} f(n+d_3)^{\delta_3}) e\left(\frac{an}{N}\right). \tag{3.10}
\end{aligned}$$

Write $\chi^* = \chi_p \chi_q$. We have

$$\begin{aligned}
& \sum_{n=1}^N \chi^*(f(n+d_1)^{\delta_1} f(n+d_2)^{\delta_2} f(n+d_3)^{\delta_3}) e\left(\frac{an}{N}\right) \\
&= \sum_{u=0}^{q-1} \chi_q(f(u+d_1)^{\delta_1} f(u+d_2)^{\delta_2} f(u+d_3)^{\delta_3}) e\left(\frac{ai_q(p)u}{q}\right) \\
& \times \sum_{v=0}^{p-1} \chi_p(f(v+d_1)^{\delta_1} f(v+d_2)^{\delta_2} f(v+d_3)^{\delta_3}) e\left(\frac{ai_p(q)v}{p}\right).
\end{aligned}$$

We consider d_1, d_2, d_3 in several cases.

I. Suppose that $d_i \not\equiv d_j$ for $1 \leq i < j \leq 3$. By Lemma 2.1 and Lemma 2.2 we easily get

$$\sum_{v=0}^{p-1} \chi_p(f(v+d_1)^{\delta_1} f(v+d_2)^{\delta_2} f(v+d_3)^{\delta_3}) e\left(\frac{ai_p(q)v}{p}\right) \ll lp^{1/2}.$$

II. Assume that $d_1 \equiv d_2 \not\equiv d_3 \pmod{p}$. Then from Lemma 2.1 and Lemma 2.2 we have

$$\sum_{v=0}^{p-1} \chi_p(f(v+d_1)^{\delta_1} f(v+d_2)^{\delta_2} f(v+d_3)^{\delta_3}) e\left(\frac{ai_p(q)v}{p}\right) \ll lp^{1/2}.$$

III. If $d_1 \equiv d_2 \equiv d_3 \pmod{p}$, we get the following results

$$\begin{aligned} & \sum_{v=0}^{p-1} \chi_p(f(v+d_1)^{\delta_1} f(v+d_2)^{\delta_2} f(v+d_3)^{\delta_3}) e\left(\frac{ai_p(q)v}{p}\right) \\ &= \sum_{v=0}^{p-1} \chi_p^{\delta_1+\delta_2+\delta_3}(f(v+d_1)) e\left(\frac{ai_p(q)v}{p}\right) \\ &= \begin{cases} O(lp^{1/2}), & \text{if } t_p \nmid \delta_1 + \delta_2 + \delta_3, \\ O(l), & \text{if } t_p \mid \delta_1 + \delta_2 + \delta_3 \text{ and } p \nmid a, \\ p + O(l), & \text{if } t_p \mid \delta_1 + \delta_2 + \delta_3 \text{ and } p \mid a. \end{cases} \\ &= \begin{cases} O(p), & \text{if } t_p \mid \delta_1 + \delta_2 + \delta_3 \text{ and } p \mid a, \\ O(lp^{1/2}), & \text{otherwise.} \end{cases} \end{aligned}$$

Noting that $d_i \not\equiv d_j \pmod{N}$ for all $1 \leq i < j \leq 3$. We have

$$\begin{aligned} & \sum_{n=1}^N \chi^*(f(n+d_1)^{\delta_1} f(n+d_2)^{\delta_2} f(n+d_3)^{\delta_3}) e\left(\frac{an}{N}\right) \\ &= \begin{cases} O(lN^{3/4}), & \text{if } t_p \mid \delta_1 + \delta_2 + \delta_3, p \mid a, \\ & \text{or } t_q \mid \delta_1 + \delta_2 + \delta_3, q \mid a, \\ O(l^2N^{1/2}), & \text{otherwise.} \end{cases} \quad (3.11) \end{aligned}$$

Then from (3.9), (3.10) and (3.11) we get

$$\begin{aligned} & \sum_{n=1}^M e_{n+d_1} e_{n+d_2} e_{n+d_3} \ll \frac{1}{d^3} \sum_{\chi_1 \in \mathcal{H}^*} \sum_{\chi_2 \in \mathcal{H}^*} \sum_{\chi_3 \in \mathcal{H}^*} \frac{1}{N} \sum_{\substack{|a| < \frac{N}{2} \\ p \mid a}} \left| \sum_{u=1}^M e\left(-\frac{au}{N}\right) \right| \cdot lN^{3/4} \\ &+ \frac{1}{d^3} \sum_{\chi_1 \in \mathcal{H}^*} \sum_{\chi_2 \in \mathcal{H}^*} \sum_{\chi_3 \in \mathcal{H}^*} \frac{1}{N} \sum_{\substack{|a| < \frac{N}{2} \\ q \mid a}} \left| \sum_{u=1}^M e\left(-\frac{au}{N}\right) \right| \cdot lN^{3/4} \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{d^3} \sum_{\chi_1 \in \mathcal{H}^*} \sum_{\chi_2 \in \mathcal{H}^*} \sum_{\chi_3 \in \mathcal{H}^*} \frac{1}{N} \sum_{|a| < \frac{N}{2}} \left| \sum_{u=1}^M e\left(-\frac{au}{N}\right) \right| \cdot l^2 N^{1/2} + l^2 N^{1/2} \\
 & \ll \frac{lN^{3/4}}{N} \left(M + \sum_{\substack{1 \leq a < \frac{N}{2} \\ p|a}} \frac{N}{a} + \sum_{\substack{1 \leq a < \frac{N}{2} \\ q|a}} \frac{N}{a} \right) \\
 & + \frac{l^2 N^{1/2}}{N} \left(M + \sum_{1 \leq a < \frac{N}{2}} \frac{N}{a} \right) + l^2 N^{1/2} \ll lN^{3/4}.
 \end{aligned}$$

Therefore

$$C_3(\mathcal{R}, N) = \max_{M, D} \left| \sum_{n=1}^M e_{n+d_1} e_{n+d_2} e_{n+d_3} \right| \ll lN^{3/4}.$$

This proves (1.2).

§4. Proof of Theorem 1.2

Let $M \in \mathbb{N}$, $d_1, d_2, d_3, d_4 \in \mathbb{Z}$ such that $0 \leq d_1 < d_2 < d_3 < d_4 < N - M$. Suppose that

$$d_1 \equiv d_2 \pmod{p}, \quad d_3 \equiv d_4 \pmod{p}, \quad d_1 \not\equiv d_3 \pmod{p}, \quad (4.1)$$

$$d_1 \equiv d_3 \pmod{q}, \quad d_2 \equiv d_4 \pmod{q}, \quad d_1 \not\equiv d_2 \pmod{q}. \quad (4.2)$$

From (3.1) we can get

$$\begin{aligned}
 & \sum_{n=1}^M e_{n+d_1} e_{n+d_2} e_{n+d_3} e_{n+d_4} \\
 & = \sum_{\substack{n=1 \\ (\prod_{j=1}^4 f(n+d_j), N)=1}}^M \prod_{j=0}^4 \left(\frac{1}{d} \sum_{\chi_j \in \mathcal{H}^*} \bar{\chi}_j(x^u) \chi_j(f(n+d_j)) + O\left(l^2 N^{-1/2}\right) \right) \\
 & + O\left(\sum_{\substack{n=1 \\ (\prod_{j=1}^4 f(n+d_j), N)>1}} 1 \right) \\
 & = \frac{1}{d^4} \sum_{\chi_1 \in \mathcal{H}^*} \bar{\chi}_1(x^u) \sum_{\chi_2 \in \mathcal{H}^*} \bar{\chi}_2(x^u) \sum_{\chi_3 \in \mathcal{H}^*} \bar{\chi}_3(x^u) \sum_{\chi_4 \in \mathcal{H}^*} \bar{\chi}_4(x^u)
 \end{aligned}$$

$$\begin{aligned}
& \times \sum_{n=1}^M \chi_1(f(n+d_1))\chi_2(f(n+d_2))\chi_3(f(n+d_3))\chi_4(f(n+d_4)) \\
& + O(l^2 N^{1/2}).
\end{aligned} \tag{4.3}$$

For $\chi_1, \chi_2, \chi_3, \chi_4 \in \mathcal{H}^*$, there exists $\chi' \in \mathcal{H}^*$ with

$$\chi_i = (\chi')^{a_i}, \quad 1 \leq a_i \leq d-1, \quad i = 1, 2, 3, 4.$$

Define

$$\chi^* = (\chi')^{(a_1, a_2, a_3, a_4)}, \quad \delta_i = \frac{a_i}{(a_1, a_2, a_3, a_4)}, \quad i = 1, 2, 3, 4.$$

Then

$$\chi_1 = (\chi^*)^{\delta_1}, \quad \chi_2 = (\chi^*)^{\delta_2}, \quad \chi_3 = (\chi^*)^{\delta_3}, \quad \chi_4 = (\chi^*)^{\delta_4},$$

and

$$1 \leq \delta_1, \delta_2, \delta_3, \delta_4 \leq d-1, \quad (\delta_1, \delta_2, \delta_3, \delta_4) = 1.$$

Write $\chi^* = \chi_p \chi_q$ with χ_p is a character modulo p of order $t_p > 1$ and χ_q is a character modulo q of order $t_q > 1$, where t_p, t_q are divisors of d . We have

$$\begin{aligned}
& \sum_{n=1}^M \chi_1(f(n+d_1))\chi_2(f(n+d_2))\chi_3(f(n+d_3))\chi_4(f(n+d_4)) \\
& = \sum_{n=1}^M \chi^*(f(n+d_1)^{\delta_1} f(n+d_2)^{\delta_2} f(n+d_3)^{\delta_3} f(n+d_4)^{\delta_4}) \\
& = \frac{1}{N} \sum_{|a| < \frac{N}{2}} \sum_{u=1}^M e\left(-\frac{au}{N}\right) \\
& \quad \times \sum_{n=1}^N \chi^*(f(n+d_1)^{\delta_1} f(n+d_2)^{\delta_2} f(n+d_3)^{\delta_3} f(n+d_4)^{\delta_4}) e\left(\frac{an}{N}\right) \\
& = \frac{1}{N} \sum_{|a| < \frac{N}{2}} \sum_{u=1}^M e\left(-\frac{au}{N}\right) \\
& \quad \times \sum_{u=0}^{q-1} \chi_q(f(u+d_1)^{\delta_1} f(u+d_2)^{\delta_2} f(u+d_3)^{\delta_3} f(u+d_4)^{\delta_4}) e\left(\frac{ai_q(p)u}{q}\right) \\
& \quad \times \sum_{v=0}^{p-1} \chi_p(f(v+d_1)^{\delta_1} f(v+d_2)^{\delta_2} f(v+d_3)^{\delta_3} f(v+d_4)^{\delta_4}) e\left(\frac{ai_p(q)v}{p}\right) \\
& = \frac{1}{N} \sum_{|a| < \frac{N}{2}} \sum_{u=1}^M e\left(-\frac{au}{N}\right)
\end{aligned}$$

$$\begin{aligned}
 & \times \sum_{u=0}^{q-1} \chi_q(f(u+d_1)^{\delta_1+\delta_3} f(u+d_2)^{\delta_2+\delta_4}) e\left(\frac{ai_q(p)u}{q}\right) \\
 & \times \sum_{v=0}^{p-1} \chi_p(f(v+d_1)^{\delta_1+\delta_2} f(v+d_3)^{\delta_3+\delta_4}) e\left(\frac{ai_p(q)v}{p}\right). \tag{4.4}
 \end{aligned}$$

By Lemma 2.1 and Lemma 2.2 we easily have

$$\begin{aligned}
 & \sum_{v=0}^{p-1} \chi_p(f(v+d_1)^{\delta_1+\delta_2} f(v+d_3)^{\delta_3+\delta_4}) e\left(\frac{ai_p(q)v}{p}\right) \\
 & = \begin{cases} p + O(l), & \text{if } t_p \mid \delta_1 + \delta_2, t_p \mid \delta_3 + \delta_4 \text{ and } p \mid a, \\ O(lp^{1/2}), & \text{otherwise,} \end{cases}
 \end{aligned}$$

and

$$\begin{aligned}
 & \sum_{u=0}^{q-1} \chi_q(f(u+d_1)^{\delta_1+\delta_3} f(u+d_2)^{\delta_2+\delta_4}) e\left(\frac{ai_q(p)u}{q}\right) \\
 & = \begin{cases} q + O(l), & \text{if } t_q \mid \delta_1 + \delta_3, t_q \mid \delta_2 + \delta_4 \text{ and } q \mid a, \\ O(lq^{1/2}), & \text{otherwise.} \end{cases}
 \end{aligned}$$

Therefore

$$\begin{aligned}
 & \sum_{n=1}^N \chi^*(f(n+d_1)^{\delta_1} f(n+d_2)^{\delta_2} f(n+d_3)^{\delta_3} f(n+d_4)^{\delta_4}) e\left(\frac{an}{N}\right) \\
 & = \begin{cases} N + O(lN^{1/2}), & \text{if } t_p \mid \delta_1 + \delta_2, t_p \mid \delta_3 + \delta_4, \\ & t_q \mid \delta_1 + \delta_3, t_q \mid \delta_2 + \delta_4, N \mid a, \\ O(lN^{3/4}), & \text{otherwise.} \end{cases} \tag{4.5}
 \end{aligned}$$

Combining (4.3)-(4.5) we get

$$\begin{aligned}
 & \sum_{n=1}^M e_{n+d_1} e_{n+d_2} e_{n+d_3} e_{n+d_4} \\
 & = \frac{1}{d^4} \sum_{\chi_1 \in \mathcal{H}^*} \sum_{\substack{\chi_2 \in \mathcal{H}^* \\ t_p \mid \delta_1 + \delta_2, t_p \mid \delta_3 + \delta_4 \\ t_q \mid \delta_1 + \delta_3, t_q \mid \delta_2 + \delta_4}} \sum_{\chi_3 \in \mathcal{H}^*} \sum_{\chi_4 \in \mathcal{H}^*} \frac{1}{N} \cdot M \left(N + O(lN^{1/2}) \right) \\
 & + O\left(\frac{1}{d^4} \sum_{\chi_1 \in \mathcal{H}^*} \sum_{\chi_2 \in \mathcal{H}^*} \sum_{\chi_3 \in \mathcal{H}^*} \sum_{\chi_4 \in \mathcal{H}^*} \frac{1}{N} \sum_{|a| < \frac{N}{2}} \left| \sum_{u=1}^M e\left(-\frac{au}{N}\right) \right| \cdot lN^{3/4} \right)
 \end{aligned}$$

$$\begin{aligned}
& + O(l^2 N^{1/2}) \\
& = \frac{M}{d^4} \sum_{\chi_1 \in \mathcal{H}^*} \sum_{\substack{\chi_2 \in \mathcal{H}^* \\ t_p | \delta_1 + \delta_2, \\ t_q | \delta_1 + \delta_3,}} \sum_{\substack{\chi_3 \in \mathcal{H}^* \\ t_p | \delta_3 + \delta_4 \\ t_q | \delta_2 + \delta_4}} \sum_{\chi_4 \in \mathcal{H}^*} + O(l N^{3/4} \log N).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\sum_{\substack{\chi_1 \in \mathcal{H}^* \\ t_p | \delta_1 + \delta_2, \\ t_q | \delta_1 + \delta_3,}} \sum_{\substack{\chi_2 \in \mathcal{H}^* \\ t_p | \delta_3 + \delta_4 \\ t_q | \delta_2 + \delta_4}} \sum_{\chi_3 \in \mathcal{H}^*} \sum_{\chi_4 \in \mathcal{H}^*} 1 & \geq \sum_{\alpha=1}^{d-1} \sum_{a_1=1}^{d-1} \sum_{a_2=1}^{d-1} \sum_{a_3=1}^{d-1} \sum_{a_4=1}^{d-1} 1 \\
& \quad \substack{(a_1, a_2, a_3, a_4) = \alpha \\ d | a_1 + a_2, d | a_3 + a_4 \\ d | a_1 + a_3, d | a_2 + a_4} \\
& = \sum_{\alpha=1}^{d-1} \sum_{\substack{a_1=1 \\ (a_1, d-a_1) = \alpha}}^{d-1} 1 = \sum_{\alpha=1}^{d-1} \sum_{\substack{a=1 \\ (a, d) = \alpha}}^{d-1} 1 \gg d.
\end{aligned}$$

Therefore

$$\sum_{n=1}^M e_{n+d_1} e_{n+d_2} e_{n+d_3} e_{n+d_4} \gg \frac{M}{d^3}.$$

Suppose that p, q are known, then we can take

$$d_1 = 0, \quad d_2 = p, \quad d_3 = q, \quad d_4 = p + q, \quad M = N - p - q.$$

It is obvious that d_1, d_2, d_3, d_4 satisfy (4.1) and (4.2). So we have

$$C_4(\mathcal{R}, N) \gg \frac{1}{d^3} N.$$

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References

- [1] Z. CHEN, Large families of pseudo-random subsets formed by generalized cyclotomic classes, *Monatshefte für Mathematik* **161** (2010), 161–172.
- [2] C. DARTYGE and A. SÁRKÖZY, On pseudo-random subsets of the set of the integers not exceeding N , *Periodica Math. Hungar.* **54** (2007), 183–200.
- [3] C. DARTYGE and A. SÁRKÖZY, Large families of pseudorandom subsets formed by power residues, *Uniform Distribution Theory* **2** (2007), 73–88.

- [4] R. LIDL and H. NIEDERREITER, Finite Fields (Encyclopedia of Mathematics and its Applications), *Addison-Wesley, Reading*, 1983.
- [5] C. MAUDUIT and A. SÁRKÖZY, On finite pseudorandom binary sequences I: measure of pseudorandomness, the Legendre symbol, *Acta Arithmetica* **82** (1997), 365–377.

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