

Two estimates concerning classical diophantine approximation constants

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Abstract. In this paper we aim to prove two inequalities involving the classical approximation constants $\lambda_n(\zeta), \widehat{\lambda}_n(\zeta)$ that stem from the simultaneous approximation problem $|\zeta^j x - y_j|, 1 \leq j \leq n$, on the one side and the constants $w_n^*(\zeta), \widehat{w}_n^*(\zeta)$ connected to approximation with algebraic numbers of degree $\leq n$ on the other side. We concretely prove $w_n^*(\zeta)\widehat{\lambda}_n(\zeta) \geq 1$ and $\widehat{w}_n^*(\zeta)\lambda_n(\zeta) \geq 1$. The first result is due to H. Davenport and W. Schmidt, however our method of proving it allows to derive the other inequality as a dual result.

Finally we will discuss estimates of $w_n^*(\zeta), \widehat{w}_n^*(\zeta)$ uniformly in ζ depending on $w_n(\zeta), \widehat{w}_n(\zeta)$ or only on n as an application.

1. Introduction

1.1. Approximation constants $\lambda_n(\zeta), \widehat{\lambda}_n(\zeta)$ and $w_n^*(\zeta), \widehat{w}_n^*(\zeta)$. For a fixed positive integer n and a vector $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) \in \mathbb{R}^n$ define the approximation constants $\lambda_{n,j}(\zeta), 1 \leq j \leq n+1$, as the supremum of all real numbers ν , such that the system

$$|x| \leq X, \quad |\zeta_i x - y_i| \leq X^{-\nu}, \quad 1 \leq i \leq n, \quad (1.1)$$

has j linearly independent solution $(x, y_1, \dots, y_n)$ for certain arbitrary large values of X . Similarly, define $\widehat{\lambda}_{n,j}(\zeta), 1 \leq j \leq n+1$ as the supremum of all ν , such that system (1.1) has j linearly independent solutions for all sufficiently

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large X . Clearly $\widehat{\lambda}_{n,j}(\zeta) \leq \lambda_{n,j}(\zeta)$ for all $1 \leq j \leq n+1$. In the special case $\zeta = (\zeta, \zeta^2, \dots, \zeta^n)$ for some real number ζ , which will be in the focus of our study, let $\lambda_{n,j}(\zeta) := \lambda_{n,j}(\zeta)$ and similarly $\widehat{\lambda}_{n,j}(\zeta) := \widehat{\lambda}_{n,j}(\zeta)$. For convenient writing we further put $\lambda_n(\zeta) := \lambda_{n,1}(\zeta)$, $\widehat{\lambda}_n(\zeta) := \widehat{\lambda}_{n,1}(\zeta)$. In particular $\lambda_n(\zeta)$ (resp. $\widehat{\lambda}_n(\zeta)$) is the supremum of all ν such that (1.1) with $\zeta_i = \zeta^i$ has infinitely many solutions respectively a solution for all sufficiently large X .

The approximation constants $w_n^*(\zeta)$, $\widehat{w}_n^*(\zeta)$ quantify how good a real number can be approximated by algebraic numbers of degree at most n . For a polynomial $P \in \mathbb{Z}[T]$ define $H(P)$ as the largest absolute value of its coefficients and for an algebraic number α define $H(\alpha) := H(P)$ for the minimal polynomial P of α with relatively prime integral coefficients. The constants $w_n^*(\zeta)$ are given by the supremum of all real ν such that

$$|\zeta - \alpha| \leq H(\alpha)^{-\nu-1} \quad (1.2)$$

has infinitely many solutions α algebraic of degree $\leq n$.

The uniform approximation constants $\widehat{w}_n^*(\zeta)$ are defined as the supremum of all real numbers ν , such that

$$|\zeta - \alpha| \leq H(\alpha)^{-1} N^{-\nu} \quad (1.3)$$

has a solution α for any sufficiently large value of N , where α is algebraic of degree $\leq n$ and $H(\alpha) \leq N$. Note that by this definition we immediately see that these quantities $\widehat{w}_n^*(\zeta)$ can be *estimated below* by the supremum of all real ν such that

$$|\zeta - \alpha| \leq N^{-\nu-1} \quad (1.4)$$

has a solution α algebraic of degree $\leq n$ and $H(\alpha) \leq N$ for any sufficiently large N .

Clearly $w_n^*(\zeta)$, $\widehat{w}_n^*(\zeta)$ are monotonically increasing as n increases. For $w_n^*(\zeta)$ it is known that we have

$$\frac{n+1}{2} \leq w_n^*(\zeta) \quad (1.5)$$

for all $\zeta \in \mathbb{R}$ not algebraic of degree $\leq n$ as a consequence of (1.11) and (3.1) we will establish later.

It is however conjectured that even the stronger lower bound $w_n^*(\zeta) \geq n$ holds for any ζ not algebraic of degree $\leq n$. It is well known that n is the optimal possible uniform (in ζ) lower bound in (1.5), as for a generic ζ we have $w_n^*(\zeta) = \widehat{w}_n^*(\zeta) = n$, see Theorem 4.1 in [4] for example.

1.2. Related approximation problems and constants. We will only treat the case $\zeta = (\zeta, \zeta^2, \dots, \zeta^n)$ in the sequel which is sufficient for our concern, although most of the following results in the introduction hold for any vector $\zeta \in \mathbb{R}^n$.

In order to study the values $\lambda_n(\zeta)$, $\widehat{\lambda}_n(\zeta)$ we first introduce a closely connected Diophantine approximation problem. Consider the system

$$\begin{aligned} |x| &\leq Q^{1+\theta} \\ |\zeta x - y_1| &\leq Q^{-\frac{1}{n}+\theta} \\ |\zeta^2 x - y_2| &\leq Q^{-\frac{1}{n}+\theta} \\ &\vdots \quad \vdots \quad \vdots \\ |\zeta^n x - y_n| &\leq Q^{-\frac{1}{n}+\theta}, \end{aligned} \tag{1.6}$$

parametrised by $Q > 1$. Let $\psi_{n,j}(Q)$ be the infimum of all θ such that (1.6) has j linearly independent solutions. The functions $\psi_{n,j}(Q)$ can alternatively be interpreted via successive minima. Consider the lattice $\Lambda = \{(x, \zeta x - y_1, \dots, \zeta^n x - y_n) : x, y_1, \dots, y_n \in \mathbb{Z}\}$ and the convex body (in fact the parallelepiped) $K(Q)$ defined as the set of points $(z_1, z_2, \dots, z_{n+1}) \in \mathbb{R}^{n+1}$ with

$$|z_1| \leq Q \tag{1.7}$$

$$|z_i| \leq Q^{-\frac{1}{n}}, \quad 2 \leq i \leq n+1. \tag{1.8}$$

Now define $\lambda_{n,j}(Q)$ to be the j -th *successive minimum* of $K(Q)$ with respect to Λ , which by definition is the smallest value λ such that $\lambda \cdot K(Q)$ contains (at least) j linearly independent lattice points. (*Note:* using the parameter Q instead of ζ or explicitly talking about successive minima, we will always be able to distinguish the functions $\lambda_{n,j}(Q)$ from the quantities $\lambda_{n,j}(\zeta)$ in the sequel, so no misunderstandings can occur.) Then we have

$$Q^{\psi_{n,j}(Q)} = \lambda_{n,j}(Q).$$

Put $\underline{\psi}_{n,j} := \liminf_{Q \rightarrow \infty} \psi_{n,j}(Q)$, $\overline{\psi}_{n,j} := \limsup_{Q \rightarrow \infty} \psi_{n,j}(Q)$. We have $-1 \leq \underline{\psi}_{n,j}(Q) \leq \frac{1}{n}$ for all $Q > 1$ and $1 \leq j \leq n+1$ (which is implicitly derived rigorously from Section 4 in [12] considering the functions L_j arising from $\psi_{n,j}$) and consequently

$$\begin{aligned} -1 &\leq \underline{\psi}_{n,1} \leq \underline{\psi}_{n,2} \leq \dots \leq \underline{\psi}_{n,n+1} \leq \frac{1}{n} \\ -1 &\leq \overline{\psi}_{n,1} \leq \overline{\psi}_{n,2} \leq \dots \leq \overline{\psi}_{n,n+1} \leq \frac{1}{n}. \end{aligned}$$

A crucial observation for the study of the functions $\psi_{n,j}(Q)$ (a special case of Theorem 1.1 in [12]) is that for ζ not algebraic of degree $\leq n$ and every $1 \leq j \leq n$ there are infinitely many Q with $\psi_{n,j}(Q) = \psi_{n,j+1}(Q)$. Thus in particular

$$\underline{\psi}_{n,j+1} \leq \overline{\psi}_{n,j}, \quad 1 \leq j \leq n. \quad (1.9)$$

An easy generalisation of Theorem 1.4 in [12] states that we have

$$(1 + \lambda_{n,j}(\zeta))(1 + \underline{\psi}_{n,j}) = (1 + \widehat{\lambda}_{n,j}(\zeta))(1 + \overline{\psi}_{n,j}) = \frac{n+1}{n}, \quad 1 \leq j \leq n+1, \quad (1.10)$$

which enables us to easily compute the value $\underline{\psi}_{n,j}$ from $\lambda_{n,j}(\zeta)$ such as $\overline{\psi}_j$ from $\widehat{\lambda}_{n,j}(\zeta)$ and vice versa.

The approximation constants $w_n^*(\zeta), \widehat{w}_n^*(\zeta)$ are in close connection to the dual problem of approximation of a linear form, which in our case is just $x + \zeta y_1 + \dots + \zeta^n y_n$.

First, define the approximation constants $w_{n,j}(\zeta)$ (resp. $\widehat{w}_{n,j}(\zeta)$) as the supremum of all $\nu \in \mathbb{R}$, such that the system

$$|y_i| \leq X, \quad |x + \zeta y_1 + \dots + \zeta^n y_n| \leq X^{-\nu}$$

has j linearly independent solutions $(x, y_1, \dots, y_n)$ for certain arbitrary large values of X (respectively all sufficiently large X) and put $w_n(\zeta) := w_{n,1}(\zeta), \widehat{w}_n(\zeta) := \widehat{w}_{n,1}(\zeta)$. In virtue of Dirichlet's Theorem we have

$$w_n(\zeta) \geq \widehat{w}_n(\zeta) \geq n. \quad (1.11)$$

Given the dual lattice Λ^* and the convex bodies $K^*(Q)$ dual to $K(Q)$, namely

$$\Lambda^* := \{(x + \zeta_1 y_1 + \zeta_2 y_2 + \dots + \zeta_n y_n, y_1, y_2, \dots, y_n) : x, y_1, \dots, y_n \in \mathbb{Z}\} \quad (1.12)$$

$$K^*(Q) := \{\mathbf{x} = (x, y_1, \dots, y_n) : |\langle \mathbf{x}, \mathbf{z} \rangle| \leq 1 \quad \forall \mathbf{z} \in K(Q)\} \quad (1.13)$$

we can define the functions $\lambda_{n,j}^*(Q)$ as the successive minima of $K^*(Q)$ with respect to Λ^* . Put

$$Q^{\psi_{n,j}^*(Q)} = \lambda_{n,j}^*(Q).$$

(Note: The $*$ in $K^*, \Lambda^*, \lambda_{n,j}^*$ is in no direct connection to the $*$ in $w_n^*(\zeta)$ or $\widehat{w}_n^*(\zeta)$.) Furthermore we denote by $\eta_{n,j}^*(Q)$ the successive minima of $K(Q)$ (instead of

$K^*(Q)$ with respect to Λ^* , which correspond to the successive minima of the convex body $K^+(Q)$ with respect to the lattice Λ^+ given by

$$\Lambda^+ := \mathbb{Z}^{n+1} \quad (1.14)$$

$$K^+(Q) := \left\{ \mathbf{x} \in \mathbb{R}^{n+1} : |y_t| \leq Q^{\frac{1}{n}} \ 1 \leq t \leq n, \right. \\ \left. |x + \zeta y_1 + \cdots + \zeta^n y_n| \leq Q^{-1} \right\} \quad (1.15)$$

and define functions $\nu_{n,j}^*(Q)$ by $Q^{\nu_{n,j}^*(Q)} = \eta_{n,j}^*(Q)$. If we put $\underline{\psi}_{n,j}^* := \liminf_{Q \rightarrow \infty} \psi_{n,j}^*(Q)$, $\overline{\psi}_{n,j}^* := \limsup_{Q \rightarrow \infty} \psi_{n,j}^*(Q)$ and similarly define $\underline{\nu}_{n,j}^*$, $\overline{\nu}_{n,j}^*$ the obvious inequalities $K^*(Q) \subset K(Q) \subset (n+1)K^*(Q)$ yield

$$\lim_{Q \rightarrow \infty} \psi_{n,j}^*(Q) - \nu_{n,j}^*(Q) = 0 \quad (1.16)$$

and thus

$$\underline{\psi}_{n,j}^* = \underline{\nu}_{n,j}^*, \quad \overline{\psi}_{n,j}^* = \overline{\nu}_{n,j}^*. \quad (1.17)$$

Combined with Mahler's inequality $\lambda_{n,j} \lambda_{n,n+2-j}^* \asymp 1$ for $1 \leq j \leq n+1$ we have

$$\underline{\psi}_{n,j} = -\overline{\psi}_{n,n+2-j}^* = -\overline{\nu}_{n,n+2-j}^*, \quad \overline{\psi}_{n,j} = -\underline{\psi}_{n,n+2-j}^* = -\underline{\nu}_{n,n+2-j}^*. \quad (1.18)$$

It will be more convenient to work with the functions $\nu_{n,j}^*$ in the sequel. Again by generalizing Theorem 1.4 in [12] we have

$$(w_{n,j}(\zeta) + 1) \left(\frac{1}{n} + \underline{\psi}_{n,j}^* \right) = (\widehat{w}_{n,j}(\zeta) + 1) \left(\frac{1}{n} + \overline{\psi}_{n,j}^* \right) = \frac{n+1}{n} \quad (1.19)$$

for $1 \leq j \leq n+1$, and obviously by (1.17) we can write $\underline{\nu}_j^*$ resp. $\overline{\nu}_j^*$ instead of $\underline{\psi}_j^*$ resp. $\overline{\psi}_j^*$.

A main ingredient in the proof of the Theorem 1.2 will be Minkowski's Theorem, which asserts that for a convex body $K \in \mathbb{R}^{n+1}$, a lattice Λ and corresponding successive minima $\lambda_{n,j}$ we have

$$\frac{2^{n+1}}{(n+1)!} \frac{\det(\Lambda)}{\text{vol}(K)} \leq \lambda_{n,1} \lambda_{n,2} \cdots \lambda_{n,n+1} \leq 2^{n+1} \frac{\det(\Lambda)}{\text{vol}(K)}. \quad (1.20)$$

For a proof see Theorem 1 page 60 and Theorem 2 page 62 in [8].

In our special case the thickness of the parallelepipeds $K^+(Q)$ in direction of the x -axis in every point $(x, y_1, \dots, y_n)$ with $(y_1, y_2, \dots, y_n) \in [-Q^{\frac{1}{n}}, Q^{\frac{1}{n}}]^n$ is $2Q^{-1}$, so we have $\text{vol}(K^+(Q)) = (2Q^{\frac{1}{n}})^n \cdot 2Q^{-1} = 2^{n+1}$ for all $Q > 1$, and

furthermore $\det(\Lambda^+) = \det(\mathbb{Z}^{n+1}) = 1$. Since $\eta_{n,j}^*$ are the successive minima of $K^+(Q)$ with respect to Λ^+ , (1.20) leads to

$$\frac{1}{(n+1)!} \leq \eta_{n,1}^*(Q) \eta_{n,2}^*(Q) \cdots \eta_{n,n+1}^*(Q) \leq 1.$$

Hence by taking logarithms, there is a constant C depending only on n such that

$$\left| \sum_{i=1}^{n+1} \nu_{n,i}^*(Q) \right| \leq \frac{C(n)}{\log(Q)}. \quad (1.21)$$

The constants $w_n^*(\zeta)$, $\widehat{w}_n^*(\zeta)$ are closely linked to the constants $w_{n,j}(\zeta)$, $\widehat{w}_{n,j}(\zeta)$, as already indicated above. For instance we have $\frac{w_n(\zeta)+1}{2} \leq w_n^*(\zeta) \leq w_n(\zeta)$, which by (1.11) implies the bound (1.5) for all ζ mentioned in Subsection 1.1. See also Remark 3 to Corollary 2.2 in Section 2.2. In fact, we will prove

Theorem 1.1. *For any integer $n \geq 1$ and any $\zeta \in \mathbb{R}$ not algebraic of degree $\leq n$ we have*

$$w_n^*(\zeta) \geq w_{n,n+1}(\zeta) \quad (1.22)$$

$$\widehat{w}_n^*(\zeta) \geq \widehat{w}_{n,n+1}(\zeta). \quad (1.23)$$

Note that by (1.10), (1.18), (1.19) one can directly compute $\lambda_{n,n+2-j}(\zeta)$ from $\widehat{w}_{n,j}(\zeta)$ as well as $\widehat{\lambda}_{n,n+2-j}(\zeta)$ from $w_{n,j}(\zeta)$ for $1 \leq j \leq n+1$, which leads to

$$\lambda_{n,n+2-j}(\zeta) = \frac{1}{\widehat{w}_{n,j}(\zeta)}, \quad \widehat{\lambda}_{n,n+2-j}(\zeta) = \frac{1}{w_{n,j}(\zeta)}. \quad (1.24)$$

Combining (1.24) with Theorem 1.1 we immediately obtain

Theorem 1.2. *For any integer $n \geq 1$ and any $\zeta \in \mathbb{R}$ not algebraic of degree $\leq n$ we have*

$$w_n^*(\zeta) \geq \frac{1}{\widehat{\lambda}_n(\zeta)} \quad \widehat{w}_n^*(\zeta) \geq \frac{1}{\lambda_n(\zeta)}.$$

At this point one should point out again (as in the abstract), that the first inequality was found by DAVENPORT and SCHMIDT in [7]. The latter result is new.

It remains to prove Theorem 1.1. A basic observation linking $w_n^*(\zeta)$ with the constants $w_{n,j}(\zeta)$ is the fact that any non-zero polynomial $P \in \mathbb{Z}[T]$ with $P'(\zeta) \neq 0$ has a root α satisfying

$$|\zeta - \alpha| \leq n \left| \frac{P(\zeta)}{P'(\zeta)} \right|, \quad (1.25)$$

since for $P(\zeta) = \prod_{1 \leq i \leq n} (\zeta - \alpha_i)$ with α the closest zero to ζ (i.e. minimizing $|\zeta - \alpha|$) we have $\left| \frac{P'(\zeta)}{P(\zeta)} \right| = \left| \sum_{1 \leq i \leq n} \frac{1}{\zeta - \alpha_i} \right| \leq n \max_{1 \leq i \leq n} \frac{1}{|\zeta - \alpha_i|} = n \frac{1}{|\zeta - \alpha|}$. So in order to get a sequence of good approximation values α for a fixed ζ , we only need to find a sequence of polynomials with small values $\left| \frac{P(\zeta)}{P'(\zeta)} \right|$. Note that the logarithm of $P(\zeta)$ to the basis $H(P)$ (see Section 1.1) is directly connected to $w_n(\zeta), \hat{w}_n(\zeta)$. Putting

$$w_n^*(\zeta, H) := \min_{P: H(P) \leq H} \frac{\log \left| \frac{P(\zeta)}{P'(\zeta)} \right|}{\log H} - 1 \quad (1.26)$$

where for all fixed H , P runs through all polynomials P with $H(P) \leq H$, by virtue of (1.25) and (1.2) (resp. (1.3) and (1.4)) one easily deduces

$$w_n^*(\zeta) \geq \limsup_{H \rightarrow \infty} w_n^*(\zeta, H) \quad (1.27)$$

$$\hat{w}_n^*(\zeta) \geq \liminf_{H \rightarrow \infty} w_n^*(\zeta, H). \quad (1.28)$$

2. Two proofs of Theorem 1.1

We will give two proofs of Theorem 1.1. The first will be quite short, the other one heavily uses the connection of the constants $w_{n,j}(\zeta), \hat{w}_{n,j}(\zeta)$ and the successive minima functions $\psi_{n,j}(Q)$ and is hence a little more technical and longer.

2.1. First proof of Theorem 1.1. Define two hyperplanes of $\mathbb{R}^{n+1}$ as follows:

$$\begin{aligned} \mathcal{B} &:= \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : x_0 + \zeta x_1 + \dots + \zeta^n x_n = 0\} \\ \mathcal{D} &:= \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : x_1 + 2\zeta x_2 + \dots + n\zeta^{n-1} x_n = 0\} \end{aligned}$$

which correspond to the hyperplanes $P(\zeta) = 0$ respectively $P'(\zeta) = 0$. Take arbitrary $w_1 < w_{n,n+1}(\zeta)$ respectively arbitrary $w_2 < \hat{w}_{n,n+1}(\zeta)$. Then for (some) arbitrarily large values of H respectively for all sufficiently large values of H the parallelepiped

$$\Pi_1 := \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : \max_{1 \leq j \leq n} |x_j| \leq H, |x_0 + \zeta x_1 + \dots + \zeta^n x_n| \leq H^{-w_1}\}$$

resp.

$$\Pi_2 := \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : \max_{1 \leq j \leq n} |x_j| \leq H, |x_0 + \zeta x_1 + \dots + \zeta^n x_n| \leq H^{-w_2}\}$$

contains $n + 1$ linearly independent integer points. For simplicity we will write Π in the sequel for either Π_1 or Π_2 as the proofs are exactly the same. Then the dilated parallelepipeds $(n + 1)\Pi$ must contain a fundamental domain with respect to the lattice $\mathbb{Z}^{n+1}$. So for any $\mathbf{z} \in \mathbb{R}^{n+1}$ the shifted parallelepipeds $\mathbf{z} + (n + 1)\Pi$ contains an integer point. Consider a shift $\mathbf{z} \in \mathcal{B}$ such that the distance between $\mathcal{D}$ and $\mathbf{z} + (n + 1)\Pi$ is equal to H but $\mathbf{z} + (n + 1)\Pi$ belongs to a ball with radius $O(H)$ centered in the origin of $\mathbb{Z}^{n+1}$. Then the integer point in $\mathbf{z} + (n + 1)\Pi$, say $(z_1^*, z_2^*, \dots, z_{n+1}^*)$, corresponds to a polynomial

$$P(t) = z_0^* + z_1^* t + \dots + z_n^* t^n$$

such that

$$|P(\zeta)| \ll H^{-w_1}, \quad H(P) \ll H, \quad |P'(\zeta)| \gg H,$$

$$\text{respectively } |P(\zeta)| \ll H^{-w_2}, \quad H(P) \ll H, \quad |P'(\zeta)| \gg H.$$

This gives the bound $w_n^*(\zeta) \geq w_1$ respectively $\widehat{w}_n^*(\zeta) \geq w_2$ in view of (1.27) respectively (1.28). The assertions follow because we can take w_1 respectively w_2 arbitrarily close to $w_{n,n+1}(\zeta)$ respectively $\widehat{w}_{n,n+1}(\zeta)$.

2.2. Strategy of the second proof. In fixed dimension n we consider two successive minima problems. On the one hand the successive minima $\eta_{n,j}^*(Q)$ of the bodies $K^+(Q)$ with respect to the lattice Λ^+ defined in (1.14), (1.15) and the resulting functions $\nu_{n,j}^*(Q)$ arising from $\eta_{n,j}^*(Q)$ from Section 1.2. On the other hand, we compress the bodies $K^+(Q)$ in direction orthogonal to the hyperplane $P'(\zeta) = 0$ by a fixed factor and consider their successive minima with respect to Λ^+ . By applying Minkowski's Theorem (1.20) to both systems we will infer that at least one of the successive minima of these two systems must differ from the corresponding successive minimum of the other system. This will imply the existence of lattice points (in fact points corresponding to a successive minimum!) with relatively “large” values $|P'(\zeta)| \approx H(P)$, which is helpful for lower bounds for $w_n^*(\zeta), \widehat{w}_n^*(\zeta)$ in view of (1.25) (or (1.26)). Assuming this point corresponds to the last (i.e. $(n + 1)$ -st) successive minimum gives a lower estimate for $w_n^*(\zeta), \widehat{w}_n^*(\zeta)$ and together with the very intuitive geometric Lemma 2.1 (although slightly technical to prove) concerning the volume of the compressed bodies, (1.19) and (1.27) resp. (1.28) leads to the estimates (1.22) resp. (1.23) in Theorem 1.1.

2.3. Detailed second proof of Theorem 1.1. We will assume ζ fixed and identify a point $P = (x, y_1, \dots, y_n) \in \mathbb{Z}^{n+1}$ with the polynomial $P(\zeta) = x + \zeta y + \dots + \zeta^n y_n$. For technical reasons we will call the successive minima problem concerning $K^+(Q)$ and Λ^+ *system A* throughout Section 2. We denote it with

superscript A and for simplicity we write $\eta_{n,j}^A(Q) := \eta_{n,j}^*(Q)$, $\nu_{n,j}^A(Q) := \nu_{n,j}^*(Q)$ with $\eta_{n,j}^*(Q), \nu_{n,j}^*(Q)$ as defined in Section 1.2.

Furthermore, define *system B* as the successive minimum problem concerning Λ^+ and the convex body

$$\mathcal{K}(Q) := K^+(Q) \cap c(Q) \cdot W(Q) \quad (2.1)$$

with

$$W(Q) := \{(x, y_1, \dots, y_n) \in \mathbb{R}^{n+1} : |y_1 + 2\zeta y_2 + \dots + n\zeta^{n-1}y_n| \leq Q^{\frac{1}{n}}\}$$

and positive real numbers $c(Q)$ to be chosen later. Observe that $W(Q)$ is just the set of points $P \in \mathbb{R}^{n+1}$ with $|P'(\zeta)| \leq Q^{\frac{1}{n}}$, so the convex bodies $K^+(Q)$ are “somehow compressed” by some factor $c(Q)$ in the direction orthogonal to the hyperplane $P'(\zeta) = 0$ (although this is not quite true as the boundary changes shape). Successive minima functions $\eta_{n,j}^B(Q), \nu_{n,j}^B(Q)$ arise from system B similarly as in system A. Note that by construction we have $\mathcal{K}(Q) \subset K^+(Q)$ and therefore $\eta_{n,j}^B(Q) \leq \eta_{n,j}^A(Q)$ and $\nu_{n,j}^B(Q) \leq \nu_{n,j}^A(Q)$ for every $1 \leq j \leq n+1$ and $Q > 1$.

We now choose the constants $c(Q)$ in (2.1) such that

$$\text{vol}(\mathcal{K}(Q)) = \frac{1}{2(n+1)!} \text{vol}(K^+(Q)) < \frac{1}{(n+1)!} \text{vol}(K^+(Q)) = \frac{2^{n+1}}{(n+1)!}. \quad (2.2)$$

Clearly, this is possible as the volume of $\mathcal{K}(Q)$ with arbitrary $c(Q)$ in (2.1) depends continuously on $c(Q)$ and for sufficiently large $c(Q) \geq c_0(Q)$ we have $K^+(Q) = \mathcal{K}(Q)$, so in particular $\text{vol}(K^+(Q)) = \text{vol}(\mathcal{K}(Q))$, as well as $\text{vol}(\mathcal{K}(Q)) = 0$ for $c(Q) = 0$. By the intermediate value theorem and as the volume increases strictly as $c(Q)$ increases (as long as $\mathcal{K}(Q) \subsetneq K^+(Q)$), there is a unique $c(Q)$ with (2.2) for every $Q > 1$.

By (1.20) we infer that for any $Q > 1$, for at least one $k = k(Q) \in \{1, 2, \dots, n+1\}$ we have strict inequality $\eta_{n,k}^B(Q) < \eta_{n,k}^A(Q)$. By the definition of successive minima and the choice of our convex bodies $\mathcal{K}(Q)$ and $K^+(Q)$ this gives the existence of vectors $\mathbf{d}(Q) \in \mathbb{Z}^{n+1}$ with

$$\mathbf{d}(Q) \in (Q^{\nu_{n,k}^A(Q)} K^+(Q)) \setminus (Q^{\nu_{n,k}^B(Q)} \mathcal{K}(Q)).$$

We will again identify any such $\mathbf{d}(Q) = (x, y_1, \dots, y_n)$ with the corresponding polynomial $P(\zeta) = x + \zeta y_1 + \dots + \zeta^n y_n$. We consider these polynomials $P(\zeta)$ as Q increases and will drop the dependence of P from Q in the notation as no

misunderstandings can occur. Since $\mathcal{K}(Q)$ only differs from $K^+(Q)$ in direction orthogonal to $P'(\zeta) = 0$ we have

$$|P'(\zeta)| > Q^{\nu_{n,k}^A(Q)} \cdot c(Q) Q^{\frac{1}{n}} = c(Q) Q^{\nu_{n,k}^A(Q) + \frac{1}{n}}. \quad (2.3)$$

On the other hand, as $\mathbf{d}(Q) \in Q^{\nu_{n,j}^A(Q)} K^+(Q)$ we have

$$|P(\zeta)| \leq Q^{-1 + \nu_{n,k}^A(Q)}, \quad (2.4)$$

$$H(P) \ll Q^{\frac{1}{n} + \nu_{n,k}^A(Q)}. \quad (2.5)$$

with constants in $\ll$ depending only on n, ζ . More precisely, as by definition we have $|y_t| \leq Q^{\frac{1}{n} + \nu_{n,k}^A(Q)}$ for $1 \leq t \leq k$ and clearly $|P(\zeta)| \leq 1$ for Q sufficiently large, we have the estimates

$$|x| \leq |P(\zeta)| + (1 + |\zeta| + \dots + |\zeta|^n) \max_{1 \leq t \leq n} |y_t| \leq 1 + (1 + |\zeta| + \dots + |\zeta|^n) Q^{\frac{1}{n} + \nu_{n,k}^A(Q)}. \quad (2.6)$$

So we infer $H(P) \leq 1 + (1 + |\zeta| + \dots + |\zeta|^n) Q^{\frac{1}{n} + \nu_{n,k}^A(Q)}$ and thus (2.5).

Note that (2.3), (2.4), (2.5) hold for any large Q and $\nu_{n,k}^A(Q) \leq \nu_{n,n+1}^A(Q)$ as well as $\underline{\nu}_{n,n+1}^A - \epsilon \leq \nu_{n,n+1}(Q) \leq \bar{\nu}_{n,n+1} + \epsilon$ for all $\epsilon > 0$ and $Q \geq Q_0(\epsilon)$. So on the one hand we can choose a sequence of values $(Q_s)_{s \geq 1} \rightarrow \infty$ such that the corresponding polynomials for any $\epsilon > 0$ and sufficiently large $Q \geq Q_0(\epsilon)$ satisfy

$$|P(\zeta)| \leq Q^{-1 + \underline{\nu}_{n,n+1}^A + \epsilon} \quad (2.7)$$

$$|P'(\zeta)| \geq c(Q) Q^{\frac{1}{n} + \underline{\nu}_{n,n+1}^A} \quad (2.8)$$

$$H(P) \ll Q^{\frac{1}{n} + \underline{\nu}_{n,n+1}^A + \epsilon} \quad (2.9)$$

with constants depending only on n, ζ in $\ll$.

On the other hand, for *any* sufficiently large $Q \geq Q_0(\epsilon)$ we clearly have

$$|P(\zeta)| \leq Q^{-1 + \bar{\nu}_{n,n+1}^A + \epsilon} \quad (2.10)$$

$$|P'(\zeta)| \geq c(Q) Q^{\frac{1}{n} + \bar{\nu}_{n,n+1}^A} \quad (2.11)$$

$$H(P) \ll Q^{\frac{1}{n} + \bar{\nu}_{n,n+1}^A + \epsilon}. \quad (2.12)$$

Assume in our present situation, i.e. $c(Q)$ defined by (2.1), (2.2), we already knew

$$\liminf_{Q \rightarrow \infty} \log_Q c(Q) \geq 0, \quad (2.13)$$

which will be shown in Corollary 2.2 from Lemma 2.1. Then a choice of polynomials leading to (2.7), (2.8), (2.9) gives in combination with (1.26), (1.27) and $\epsilon \rightarrow 0$

$$w_n^*(\zeta) + 1 \geq \lim_{H(P) \rightarrow \infty} -\frac{\log\left(\frac{P(\zeta)}{P'(\zeta)}\right)}{\log(H(P))} \geq \frac{n+1}{n} \frac{1}{\frac{1}{n} + \nu_{n,n+1}^A} = w_{n,n+1}(\zeta) + 1, \quad (2.14)$$

the equality on the right being just a variation of (1.19). Similarly, for fixed $H > 0$ implicitly defining Q by the identity $H = Q^{\frac{1}{n} + \bar{\nu}_{n,n+1}^A + \epsilon}$ the induced polynomials satisfying (2.10), (2.11), (2.12) give in combination with (1.26), (1.28), (1.19) and $\epsilon \rightarrow 0$

$$\hat{w}_n^*(\zeta) + 1 \geq \lim_{H \rightarrow \infty} -\frac{\log\left(\frac{P(\zeta)}{P'(\zeta)}\right)}{\log(H)} \geq \frac{n+1}{n} \frac{1}{\frac{1}{n} + \bar{\nu}_{n,n+1}^A} = \hat{w}_{n,n+1}(\zeta) + 1. \quad (2.15)$$

Subtracting one from both sides of (2.14), (2.15) establishes the assertions of Theorem 1.1.

It remains to prove (2.13). In fact, we prove that $c(Q)$ is even bounded below uniformly in the parameter Q .

Lemma 2.1 (Geometric lemma). *Given $n \geq 2, \zeta \in \mathbb{R}$ as well as positive real parameters R and $Q > 1$, consider the sets*

$$\begin{aligned} \chi_A(Q) &:= \{(x, y_1, \dots, y_n) \in \mathbb{R}^{n+1} : |P(\zeta)| \leq Q^{-1}\} \\ \chi_B(R) &:= \{(x, y_1, \dots, y_n) \in \mathbb{R}^{n+1} : |P'(\zeta)| \leq R\} \\ \chi_C(Q) &:= \{(x, y_1, \dots, y_n) \in \mathbb{R}^{n+1} : |y_t| \leq Q^{\frac{1}{n}}, 1 \leq t \leq n\}, \end{aligned}$$

where $P(\zeta) = x + \zeta y_1 + \dots + \zeta^n y_n$.

Then for sufficiently large Q and all R we have

$$\text{vol}(\chi_A(Q) \cap \chi_B(R) \cap \chi_C(Q)) \leq ERQ^{-\frac{1}{n}}$$

with some constant $E = E(n, \zeta)$ independent of Q .

PROOF. Note first that $\chi_A(Q)$ is bounded by the two translates of the fixed hyperplane $P(\zeta) = 0$ by Q^{-1} in direction orthogonal to $P(\zeta) = 0$ to both sides of this hyperplane $P(\zeta) = 0$. In particular $\chi_A(Q)$ converges to $P(\zeta) = 0$ for $Q \rightarrow \infty$, which is not so important, however. Similarly, $\chi_B(R)$ is the space between two hyperplanes parallel to the hyperplane $P'(\zeta) = 0$ with distance R

in both directions from the hyperplane $P'(\zeta) = 0$. Note that with respect to any other *fixed* direction $v \in \mathbb{R}^{n+1}$ with $v \notin \{P : P(\zeta) = 0\}$ resp. $v \notin \{P : P'(\zeta) = 0\}$ (which is equivalent to $P(v) \neq 0$ resp. $P'(v) \neq 0$) $\chi_A(Q)$ resp. $\chi_B(R)$ has width at most $N_1 Q^{-1}$ resp. $N_2 R$ for fixed constants N_1, N_2 depending on v but independent from Q, R . Throughout the proof we will make use of this for some given v determined by n, ζ .

Finally, $\chi_C(Q)$ just bounds the coordinates $(y_1, y_2, \dots, y_n)$ of the vectors $(x, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$ in dependence of Q .

Observe that by these restrictions for $\chi_C(Q)$ and in view of the left hand inequality in (2.6) we can assume $|x| \leq C(n, \zeta)Q^{\frac{1}{n}}$ for some constant $C(n, \zeta)$ independent of Q without loss of generality. Thus every coordinate $(x, y_1, \dots, y_n)$ of a point in $\chi_A(Q) \cap \chi_B(R) \cap \chi_C(Q)$ is bounded by $C(n, \zeta)Q^{\frac{1}{n}}$. Consequently for every $Q > 1$ any rotation of the set $\chi_A(Q) \cap \chi_B(R) \cap \chi_C(Q)$ lies in the centrsymmetric cube with side length $\sqrt{n}C(n, \zeta)Q^{\frac{1}{n}}$ and surfaces parallel to the hyperplanes $x = 0, y_1 = 0, \dots, y_n = 0$ given by

$$\kappa(Q) := \left\{ (x, y_1, \dots, y_n) : |x| \leq \sqrt{n}C(n, \zeta)Q^{\frac{1}{n}}, |y_t| \leq \sqrt{n}C(n, \zeta)Q^{\frac{1}{n}}, 1 \leq t \leq n \right\}.$$

We apply a rotation ϖ on $\mathbb{R}^{n+1}$ in such a way that the hyperplane $P(\zeta) = 0$ is sent to the hyperplane H_0 defined by $x = 0$. Let H_1 be the image of the hyperplane $P'(\zeta) = 0$ under ϖ . Clearly $H_0 \neq H_1$ as $P'(\zeta)$ has lower degree than $P(\zeta)$ and hence the intersection $H_0 \cap H_1$ has dimension $(n-1)$. As rotations preserve volumes,

$$\text{vol}(\varpi(\chi_A(Q) \cap \chi_B(R) \cap \chi_C(Q))) = \text{vol}(\chi_A(Q) \cap \chi_B(R) \cap \chi_C(Q)).$$

Moreover note also that intersecting $\chi_A(Q) \cap \chi_B(R) \cap \chi_C(Q)$ with $\kappa(Q)$ doesn't change the volume as stated (we may replace $\chi_C(Q)$ by $\kappa(Q)$).

More generally, for real numbers a, b define

$$H_{0,a} := \varpi(\{P : P(\zeta) = a\}) = \{(x, y_1, \dots, y_n) \in \mathbb{R}^{n+1} : x = a\},$$

$$H_{1,b} := \varpi(\{P : P'(\zeta) = b\}),$$

such that in particular $H_0 = H_{0,0}, H_1 = H_{1,0}$.

From the preliminary descriptions of $\chi_A(Q), \chi_B(R)$ we easily see

$$\begin{aligned} \varpi(\chi_A(Q)) &= \bigcup_{a \in [-Q^{-1}, Q^{-1}]} H_{0,a}, \\ \varpi(\chi_B(Q)) &= \bigcup_{b \in [-R, R]} H_{1,b}, \end{aligned}$$

and by construction of $\kappa(Q)$ we conclude

$$\begin{aligned} \varpi(\chi_A(Q) \cap \chi_B(R) \cap \chi_C(Q)) \\ \subset \left(\bigcup_{a \in [-Q^{-1}, Q^{-1}]} H_{0,a} \right) \bigcap \left(\bigcup_{b \in [-R, R]} H_{1,b} \right) \bigcap \kappa(Q). \end{aligned} \quad (2.16)$$

In view of (2.16) and since rotations don't change the volume it is sufficient to prove the upper estimate $ERQ^{-\frac{1}{n}}$ for the volume of the right hand side of (2.16), i.e.

$$\text{vol} \left(\left(\bigcup_{a \in [-Q^{-1}, Q^{-1}]} H_{0,a} \right) \bigcap \left(\bigcup_{b \in [-R, R]} H_{1,b} \right) \bigcap \kappa(Q) \right) \leq ERQ^{-\frac{1}{n}}, \quad (2.17)$$

to establish the assertions of the Lemma.

In order to do this we use Fubini's Theorem twice. We first give upper bounds for the $(n-1)$ -dimensional volumes of the intersections $H_{0,a} \cap H_{1,b} \cap \kappa(Q)$, then apply Fubini's Theorem to derive upper bounds for the n -dimensional volumes of $H_{0,a} \cap \bigcup_{b \in [-R, R]} H_{1,b} \cap \kappa(Q)$ for every $a \in [-Q^{-1}, Q^{-1}]$ and then again apply Fubini's Theorem by integrating these n -dimensional volumes along the x -axis to finally derive the required upper bound.

Clearly, the $(n-1)$ -dimensional volume of $H_0 \cap H_1 \cap \kappa(Q)$ is proportional to $Q^{\frac{n-1}{n}}$, let's say $\text{vol}(H_0 \cap H_1 \cap \kappa(Q)) = DQ^{\frac{n-1}{n}}$ for a constant D depending only on the angle between H_0 and H_1 which is determined by n, ζ (in particular independent of Q). Similarly we see that we can find a constant $D_0 = D_0(n, \zeta)$ such that simultaneously for all $a, b \in \mathbb{R}$ we have

$$\text{vol}(H_{0,a} \cap H_{1,b} \cap \kappa(Q)) \leq D_0 Q^{\frac{n-1}{n}}, \quad a, b \in \mathbb{R}, \quad (2.18)$$

as all $H_{0,a} \cap H_{1,b}$ are $(n-1)$ -dimensional subspaces in a n -dimensional cube with side length proportional to $Q^{\frac{1}{n}}$.

Since $\chi_B(R)$ (and so $\varpi(\chi_B(R)) = \bigcup_{b \in [-R, R]} H_{1,b}$ too) has width R in direction orthogonal to $H_{1,b}$, and in view of (2.18), we infer that

$$\text{vol}_n \left(H_{0,a} \bigcap \left(\bigcup_{b \in [-R, R]} H_{1,b} \right) \bigcap \kappa(Q) \right) \leq D_0 Q^{\frac{n-1}{n}} \cdot D_1 R, \quad a \in \mathbb{R} \quad (2.19)$$

with some constant D_1 depending only on the angle between the x -axis and H_1 , which itself is determined by n, ζ . So we have estimated the n -dimensional volume of $\varpi(\chi_B(R) \cap \chi_C(Q)) = \left(\bigcup_{b \in [-R, R]} H_{1,b} \right) \cap \kappa(Q)$ in every hyperplane $H_{0,a}$. Observe that $\varpi(\chi_A(Q) \cap \chi_B(R) \cap \chi_C(Q))$ has width mQ^{-1} in direction of the x -axis with a constant m depending only on the angle between the hyperplane

$P(\zeta) = 0$ and the hyperplane H_0 , which is again determined by n, ζ (so in particular independent of Q). We apply Fubini's Theorem to (2.19) and conclude

$$\begin{aligned} \text{vol} \left(\left(\bigcup_{a \in [-Q^{-1}, Q^{-1}]} H_{0,a} \right) \bigcap \left(\bigcup_{b \in [-R, R]} H_{1,b} \right) \bigcap \kappa(Q) \right) \\ \leq mQ^{-1} \cdot D_0 D_1 Q^{\frac{n-1}{n}} R = mD_0 D_1 Q^{-\frac{1}{n}} R. \end{aligned}$$

Inequality (2.17) and hence the assertion of the Lemma follows with $E = E(n, \zeta) := mD_0 D_1$. $\square$

Corollary 2.2. *In the context of the first part of the proof of Theorem 1.1 (i.e. (2.2) holds) we have $c(Q) \geq B$ for some constant B uniformly in Q , in particular (2.13) holds.*

PROOF. By definition of $c(Q)$ we have $c(Q) = R(Q)Q^{-\frac{1}{n}}$ for $R = R(Q)$ that satisfies

$$\text{vol}(\chi_A(Q) \cap \chi_B(R) \cap \chi_C(Q)) = \frac{2^{n+1}}{2(n+1)!}.$$

We may apply Lemma 2.1 and with respect to the constant $E = E(n, \zeta)$ of this lemma this yields

$$\frac{2^{n+1}}{2(n+1)!} \leq ER(Q)Q^{-\frac{1}{n}},$$

or equivalently $B \cdot Q^{\frac{1}{n}} \leq R(Q) = Q^{\frac{1}{n}} c(Q)$ with $B := \frac{2^{n+1}}{2(n+1)!E}$ for all $Q > 1$ and we conclude $c(Q) \geq B$ uniformly in the parameter Q . $\square$

Thus we have established Theorem 1.1 and consequently Theorem 1.2.

Remarks. 1) One can show $\text{vol}(\chi_A(Q) \cap \chi_B(R) \cap \chi_C(Q)) \geq FRQ^{-\frac{1}{n}}$ for some constant $F = F(n, \zeta)$ independent from Q with a proof similar to the one of Lemma 2.1. So by arguments very similar to those in the proof of Corollary 2.2 we have that in fact $c(Q)$ is also uniformly bounded above by a positive constant and consequently in combination with (2.13) we actually have $\lim_{Q \rightarrow \infty} \log_Q c(Q) = 0$.

2) Observe that all the constants occurring throughout the proof of Lemma 2.1 can be estimated explicitly in dependence of n, ζ so we can write $c(Q) \leq K(n, \zeta)$ with an effective constant $K(n, \zeta)$ in (2.8), (2.11). However, the ϵ -term in the exponent of (2.7), (2.9) as well as (2.10), (2.12) doesn't allow any improvements in (2.14), (2.15) for any given n, ζ .

3) Lemma 15 Chap. 3 of §3 in [14] states that for a polynomial P of degree D and length L and with zero α satisfying $|\zeta - \alpha| \leq 1$

$$|P(\zeta)| \leq |\zeta - \alpha| \cdot LD(1 + |\zeta|)^{D-1}$$

holds, leading to the well known result

$$w_n^*(\zeta) \leq w_n(\zeta) = w_{n,1}(\zeta).$$

So $w_n^*(\zeta)$ can be bounded above in terms of the approximation constants $w_{n,j}(\zeta)$ which reverses the direction of the estimates in Theorem 1.1.

For other inequalities concerning $w_n(\zeta), w_n^*(\zeta)$ we refer to Section 3.4 in [4], especially Theorem 3.4.

3. Estimates for $w_n^*(\zeta), \hat{w}_n^*(\zeta)$ inferred from Theorem 1

3.1. Estimates depending on n only. A famous result of WIRSING [18] states

$$w_n^*(\zeta) \geq \frac{w_n(\zeta) + 1}{2}. \quad (3.1)$$

We combine the results of Theorem 1.1, 1.2 with (3.1) to give lower bounds for $w_n^*(\zeta)$ uniformly in ζ not algebraic of degree $\leq n$. In order to do this, we use the functions $\psi_{n,j}^*$ which have the useful property

$$\left| \sum_{j=1}^{n+1} \psi_{n,j}^*(Q) \right| \leq \frac{C(n)}{\log(Q)}, \quad (3.2)$$

which follows from (1.21) and (1.16), and $\psi_{n,j}^*$ relate to the constants $w_{n,j}(\zeta)$. To get a slightly better result we will also use the dual version of (1.9): indeed from (1.9) and (1.18) it follows that for ζ not algebraic of degree $\leq n$ we have

$$\underline{\psi}_{n,j+1}^* \leq \overline{\psi}_{n,j}^*, \quad 1 \leq j \leq n. \quad (3.3)$$

The following preliminary Proposition is a very easy consequence of (1.16) and (3.3).

Proposition 3.1. *Let $n \geq 2$ be an integer. Then for ζ not algebraic of degree $\leq n$ the relation*

$$\underline{\psi}_{n,1}^* \leq -\frac{2}{n-1} \underline{\psi}_{n,n+1}^*$$

holds for the approximation constants $\psi_{n,1}^, \psi_{n,n+1}^*$ associated to $(\zeta, \zeta^2, \dots, \zeta^n)$.*

PROOF. By definition for any $\epsilon > 0$ and sufficiently large Q we have $\psi_{n,n+1}^*(Q) \geq \underline{\psi}_{n,n+1}^* - \epsilon$. By (3.3) we also have $\underline{\psi}_{n,n+1}^* \leq \overline{\psi}_{n,n}^*$, so that there exist arbitrarily large values of Q such that $\psi_{n,n}^*(Q) \geq \psi_{n,n+1}^*(Q) - \epsilon$. Combining these observations yields arbitrarily large values Q , such that $\psi_{n,n}^*(Q) + \psi_{n,n+1}^*(Q) \geq 2\underline{\psi}_{n,n+1}^*(Q) - 2\epsilon$.

On the other hand, for any Q we have $(n-1)\psi_{n,1}^*(Q) \leq \sum_{j=1}^{n-1} \psi_{n,j}^*(Q)$, in particular for those values Q with the property $\psi_{n,n}^*(Q) + \psi_{n,n+1}^*(Q) \geq 2\underline{\psi}_{n,n+1}^*(Q) - 2\epsilon$. The assertion of the Proposition follows with (3.2) and $\epsilon \rightarrow 0$. $\square$

We give two applications of Proposition 3.1. The first is a new estimate for $w_n^*(\zeta)$ in dependence of $w_n(\zeta)$ derived easily by combining (1.19) and Proposition 3.1 with Theorem 1.1.

Theorem 3.2. *For $n \geq 2$ and ζ not algebraic of degree $\leq n$ we have*

$$w_n^*(\zeta) \geq \frac{w_n(\zeta) + n}{w_n(\zeta) - n + 2}. \quad (3.4)$$

PROOF. Note that one can express $w_n(\zeta)$ by $\underline{\psi}_{n,1}^*$ such as $w_{n,n+1}(\zeta)$ by $\underline{\psi}_{n,n+1}^*$ using (1.19) and vice versa. Now compute straight forward.

For fixed $w_n(\zeta)$ express $\underline{\psi}_{n,1}^*$ in terms of $w_n(\zeta)$ with (1.19), then apply Proposition 3.1 to obtain an upper bound for $\underline{\psi}_{n,n+1}^*$ in terms of $w_n(\zeta)$ and then again use (1.19) to obtain lower bound for $w_{n,n+1}(\zeta)$ in terms of $w_n(\zeta)$. Simplifying this bound gives the right hand side of (3.4). $\square$

Note that Theorem 3.2 improves Wirsing's

$$w_n^*(\zeta) \geq \frac{w_n(\zeta)}{w_n(\zeta) - n + 1}$$

in [18]. There is a better known estimate for $w_n^*(\zeta)$ than Wirsing's though, namely

$$w_n^*(\zeta) \geq \frac{\widehat{w}_n(\zeta)}{\widehat{w}_n(\zeta) - n + 1} \quad (3.5)$$

by LAURENT and BUGEAUD that can be found in [6]. Together with Theorem 3.2 we have

$$w_n^*(\zeta) \geq \max \left\{ \frac{w_n(\zeta) + n}{w_n(\zeta) - n + 2}, \frac{\widehat{w}_n(\zeta)}{\widehat{w}_n(\zeta) - n + 1} \right\}$$

One checks that (3.4) improves (3.5) if and only if

$$\widehat{w}_n(\zeta) \leq w_n(\zeta) < 2\widehat{w}_n(\zeta) - n. \quad (3.6)$$

However, (1.20') in Theorem 1.3 in [12] by SUMMERER and SCHMIDT transferred in our present notation together with (1.18) yields

$$\underline{\psi}_{n,1}^* \leq \frac{(n^2 - 3n + 3)\overline{\psi}_{n,1}^* - (n - 3)n\overline{\psi}_{n,1}^{*2}}{(n - 2)^2 + (2n - 5)n\overline{\psi}_{n,1}^* + n^2\overline{\psi}_{n,1}^{*2}},$$

which in combination with frequent use of (1.19) (similar to the proof of Theorem 3.2) gives

$$w_n(\zeta) \geq (n - 1) \frac{\widehat{w}_n(\zeta)^2 - \widehat{w}_n(\zeta)}{1 + (n - 2)\widehat{w}_n(\zeta)}. \quad (3.7)$$

The lower bound in (3.7) is easily checked to be only smaller than $2\widehat{w}_n(\zeta) - n$ (the necessary condition according to (3.6)) for $n = 2$ and arbitrary $\widehat{w}_n(\zeta) > n$, but larger for $n \geq 3$ and arbitrary $\widehat{w}_n(\zeta) > n$. It can be shown too that Theorem 3.2 also doesn't improve (3.5) for $n = 2$, even provided that there is equality in (3.7). So (3.4) is weaker than (3.5).

However, the bound in Proposition 3.1 is not expected to be good (sharp), so it is likely that using the same method with better estimates one can improve (3.5) for at least some values of n and some $\zeta \in \mathbb{R}$ with not too large values of $w_n(\zeta)$. It is worth mentioning that since for every ζ not algebraic of degree $\leq n$ we have $\widehat{w}_n(\zeta) \leq 2n - 1$ for all ζ not algebraic of degree $\leq n$, see [7], for ζ with large values of $w_n(\zeta)$ the bound in (3.5) will probably be impossible to improve with similar methods (i.e. even if result (3.4) can be improved by an improvement of Proposition 3.1). However Wirsing's bound (3.1) is much better in cases with values of $w_n(\zeta)$ not too close to n anyway.

Note that for $n = 2$ the *extremal numbers* defined in [11] have equality in (3.7) with $w_2(\zeta) = (\frac{2}{\sqrt{5}-1})^3$, $\widehat{w}_2(\zeta) = (\frac{2}{\sqrt{5}-1})^2$. However, as mentioned above the condition (3.6) doesn't hold, so there is no improvement in this case either. Note in this context also that there cannot be equality in (3.7) for any $\zeta \in \mathbb{R}$ with $w_2(\zeta) > (\frac{2}{\sqrt{5}-1})^3$.

Indeed, for $n = 2$ and ζ not algebraic or quadratic irrational we have $\widehat{\lambda}_2(\zeta) \leq \frac{\sqrt{5}-1}{2}$ see ROY [11]. Together with Jarnik's equality in the form $\widehat{\lambda}_2(\zeta) + \lambda_{2,3}(\zeta) = 1$ (which is derived directly from $\overline{\psi}_{2,1} + 2\overline{\psi}_{2,1}\underline{\psi}_{2,3} + \underline{\psi}_{2,3} = 0$, which is Theorem 1.5 in [12], and (1.10)) and observing $\lambda_{2,3}(\zeta) = \frac{1}{\widehat{w}_2(\zeta)}$ by (1.24) we deduce the bound $\widehat{w}_2(\zeta) \leq (\frac{2}{\sqrt{5}-1})^2$. Combined with the assumed equality in (3.7) this leads to $w_2(\zeta) \leq (\frac{2}{\sqrt{5}-1})^3$. (However there is a good chance that there is equality for some (ζ_1, ζ_2) together with 1 linearly independent over $\mathbb{Q}$ with $\zeta_2 \neq \zeta_1^2$ and $\widehat{w}_n(\zeta_1, \zeta_2) > (\frac{2}{\sqrt{5}-1})^3$.)

Remark. Using $\underline{\psi}_{n,n+1}^* \leq -n\overline{\psi}_{n,1}^*$, which is dual to (1.10) in [13], instead of Proposition 3.1, and proceeding as in the proof of Theorem 3.2 we obtain the bounds

$$w_n^*(\zeta) \geq \frac{n}{\widehat{w}_n(\zeta) - n + 1}.$$

This is weaker than (3.5) as $\widehat{w}_n(\zeta) \geq n$, however the bound $\underline{\psi}_{n,n+1}^* \leq -n\overline{\psi}_{n,1}^*$ isn't expected to be any sharp.

Applying Proposition 3.1 directly to Wirsing's result (3.1) yields

Corollary 3.3. *Let $n \geq 2$ be an integer. Then for ζ not algebraic of degree $\leq n$ the approximation constant $w_n^*(\zeta)$ of $(\zeta, \zeta^2, \dots, \zeta^n)$ is bounded below as follows*

$$w_n^*(\zeta) \geq \frac{1}{4} \left(n + 1 + \sqrt{n^2 + 10n - 7} \right) =: \mathcal{U}(n). \quad (3.8)$$

For $n \rightarrow \infty$ we have the asymptotical behaviour $\mathcal{U}(n) = \frac{n}{2} + \frac{3}{2} + o(1)$.

PROOF. From (1.19) we have

$$\underline{\psi}_{n,n+1}^* = \frac{n - w_{n,n+1}(\zeta)}{n(w_{n,n+1}(\zeta) + 1)}. \quad (3.9)$$

Dividing the right hand side of (3.9) by $(n-1)$ and combining it with the estimate from Proposition 3.1 gives an upper bound for $\underline{\psi}_{n,1}^*$ in terms of $w_{n,n+1}(\zeta)$. Using this expression in (1.19) and applying Wirsing's result (3.1) leads to

$$w_n^*(\zeta) \geq \frac{w_n(\zeta) + 1}{2} \geq \frac{n+1}{2} \cdot \frac{1}{1 - \frac{2}{n-1} \frac{n - w_{n,n+1}(\zeta)}{w_{n,n+1}(\zeta) + 1}}.$$

On the other hand we have the lower bound $w_{n,n+1}(\zeta)$ for $w_n^*(\zeta)$ by Theorem 1.1, so

$$w_n^*(\zeta) \geq \max \left\{ \frac{n+1}{2} \cdot \frac{1}{1 - \frac{2}{n-1} \frac{n - w_{n,n+1}(\zeta)}{w_{n,n+1}(\zeta) + 1}}, w_{n,n+1}(\zeta) \right\}.$$

It's not hard to see that the left hand term in the maximum decreases as $w_{n,n+1}(\zeta)$ increases, so the minimum is attained at the value $w_{n,n+1}(\zeta) > 0$ where both expressions in the maximum coincide. This leads to a quadratic equation and after basic simplifications finally yields $\mathcal{U}(n)$ as the minimum lower bound for $w_n^*(\zeta)$. Checking the asymptotics for $\mathcal{U}(n)$ is a standard calculation. $\square$

Remarks. 1) Lemma 1 on page 46 in [10] states that in the present case of simultaneous approximation of $(\zeta, \zeta^2, \dots, \zeta^n)$ of ζ not algebraic of degree $\leq \lceil \frac{n}{2} \rceil$ the approximation constant $\hat{\lambda}_n(\zeta)$ is bounded above by $\lceil \frac{n}{2} \rceil^{-1}$. Applying this to (1.24) and Theorem 1.1 we immediately derive

$$\lambda_n(\zeta) \geq \left\lceil \frac{n}{2} \right\rceil$$

and slight refinements in combination with Wirsing's result can be derived similarly as in Corollary 3.3. However, the results of Corollary 3.3 are a little stronger.

2) Bernik and Tishchenko established slightly better results than Corollary 3.3 on the approximation constant $w_n^*(\zeta)$, for example

$$w_n^*(\zeta) \geq \frac{n}{4} + \frac{\sqrt{n^2 + 16n - 8}}{4}$$

with right hand side $\frac{n}{2} + 2 + o(1)$ as $n \rightarrow \infty$, see [3] or [15] for instance.

We now turn our focus to the approximation constant $\hat{w}_n^*(\zeta)$. Note that there exists no nontrivial upper bound for the value $\lambda_n(\zeta)$ (as for $\hat{\lambda}_n(\zeta)$ in Remark 1) even in the present special case of simultaneous approximation of $(\zeta, \zeta^2, \dots, \zeta^n)$. Indeed, we can have $\lambda_n(\zeta) = \infty$ (which is equivalent to $\hat{w}_{n,n+1}(\zeta) = 0$ by (1.24)), taking $\zeta = \sum_{l \geq 1} 10^{-l!}$ for example. So we cannot use Theorems 1.1, 1.2 to give nontrivial bounds for the approximation constant $\hat{w}_n^*(\zeta)$ as easily as above.

Moreover, no analogue of (3.1) for $\hat{w}_n^*(\zeta)$ seems to be known. BUGEAUD and LAURENT established in Theorem 2.1 in [6] for ζ not algebraic of degree $\leq n$ the inequality

$$\hat{w}_n^*(\zeta) \geq \frac{w_n(\zeta)}{w_n(\zeta) - n + 1}, \quad (3.10)$$

which we can combine with Theorem 1.1 to get

$$\hat{w}_n^*(\zeta) \geq \max \left\{ \frac{w_n(\zeta)}{w_n(\zeta) - n + 1}, \hat{w}_{n,n+1}(\zeta) \right\}. \quad (3.11)$$

However, any number ζ with $\lambda_n(\zeta) = \infty$ (for instance again $\zeta = \sum_{l \geq 1} 10^{-l!}$) automatically yields $w_n(\zeta) = \infty$ (which follows easily from the definition of $w_n(\zeta)$, $\lambda_n(\zeta)$ or alternatively from Khinchin's transference principle $w_n(\zeta) \geq n\lambda_n(\zeta) + n - 1$, see [9]). For such ζ , by (1.24) we also have $\hat{w}_{n,n+1}(\zeta) = 0$ though, so in this case (3.11) only leads to the very weak bound $\hat{w}_n^*(\zeta) \geq 1$.

3.2. Finding ζ for which (3.11) is an improvement. As a last point we want to investigate (3.11) a bit further. Our result (1.23) or equivalently (1.2) improves the result (3.10) of Bugeaud and Laurent only in the case of numbers ζ satisfying

$$\frac{1}{\lambda_n(\zeta)} > \frac{w_n(\zeta)}{w_n(\zeta) - n + 1}. \quad (3.12)$$

So the question naturally arises if such numbers exist and what we can tell about them.

Firstly, as the right hand side of (3.12) is bounded below by 1, it only makes sense to look for numbers ζ satisfying $\lambda_n(\zeta) \in (\frac{1}{n}, 1)$. Also note that for $n \geq 2$ and all $w_n > n$ we have that the upper bound $\frac{(n-1)w_n+n}{w_n}$ for the value $\frac{1}{\lambda_n(\zeta)}$ arising from Khinchin's transference principle is easily checked to be larger than $\frac{w_n(\zeta)}{w_n(\zeta)-n+1}$, which enables (3.12). For $n \geq 3$, just looking at the transference principle even the case

$$w_n(\zeta) = \infty, \quad \lambda_n(\zeta) = \frac{1}{n-1}$$

is reasonable, where $w_n(\zeta) = \infty, \lambda_n(\zeta) < 1$ would already be sufficient for (3.12). So the existence follows if the bounds in the transference principle are "somehow sharp", especially for large values of $w_n = w_n(\zeta)$.

In order to give further reasons that numbers satisfying (3.12) do exist we first quote two metrical results.

Theorem 3.4. *Let $\tau \geq 1$ be a real number. Then the Hausdorff-dimension of the set*

$$\{\zeta \in \mathbb{R} : w_n(\zeta) = \tau(n+1) - 1\} \quad (3.13)$$

is $\frac{1}{\tau}$.

This is Theorem 5.8 in [4].

Furthermore it is known due to BERESNEVICH [1] that

Theorem 3.5. *Let $n \geq 2$ be an integer. Let λ be a real number satisfying $\frac{1}{n} \leq \lambda < \frac{3}{2n-1}$. Then*

$$\dim(\{\zeta \in \mathbb{R} : \lambda_n(\zeta) \geq \lambda\}) \geq \frac{n+1}{\lambda+1} - (n-1). \quad (3.14)$$

BERESNEVICH, DICKINSON, VAUGHAN and VELANI [2], [16] further prove that there is actually equality in (3.14) for $n = 2$ and Beresnevich conjectures this to be true for all $n \geq 2$. Under the assumption that this is true we can prove the existence of uncountably many ζ satisfying (3.12) for $n \geq 3$.

Lemma 3.6. *If for some $n \geq 3$ there is equality in (3.14), then for this n the set of values ζ such that (3.12) holds has Hausdorff-dimension 1.*

PROOF. Let n be fixed. For a real number $x \geq n$ to be specified later consider the set of all $\zeta \in \mathbb{R}$ such that $w_n(\zeta) = x$. Basic rearrangements of Theorem 3.4 lead to

$$\dim(\{\zeta \in \mathbb{R} : w_n(\zeta) = x\}) = \frac{n+1}{x+1}. \quad (3.15)$$

On the other hand, assuming $x \in (n, n + \epsilon)$ for sufficiently small $\epsilon > 0$ (precisely such that $\frac{x-n+1}{x} < \frac{3}{2n-1}$) we may apply Theorem 3.5 with the additive equality condition for $\lambda := \frac{x-n+1}{x}$ and infer

$$\dim\left(\left\{\zeta \in \mathbb{R} : \lambda_n(\zeta) \geq \frac{x-n+1}{x}\right\}\right) = \frac{(n+1)x}{2x-n+1} - (n-1).$$

Defining $\kappa_n(t) := \frac{n+1}{t+1} - [\frac{(n+1)t}{2t-n+1} - (n-1)]$ one readily checks $\kappa_n(n) = 0$ and $\kappa'_n(n) = \frac{n-2}{n+1} > 0$ for $n \geq 3$, such that we have $\kappa_n(t) > 0$ in an interval $t \in (n, n + \epsilon)$ for sufficiently small $\epsilon > 0$. This means for $x \in (n, n + \epsilon)$ we have

$$\dim\left(\left\{\zeta \in \mathbb{R} : \lambda_n(\zeta) \geq \frac{x-n+1}{x}\right\}\right) < \dim(\{\zeta \in \mathbb{R} : w_n(\zeta) = x\}). \quad (3.16)$$

For those values of x we infer that $\{\zeta \in \mathbb{R} : w_n(\zeta) = x\} \not\subset \{\zeta \in \mathbb{R} : \lambda_n(\zeta) \geq \frac{x-n+1}{x}\}$, so there are numbers ζ with $w_n(\zeta) = x$ and $\lambda_n(\zeta) < \frac{x-n+1}{x} = \frac{w_n(\zeta)-n+1}{w_n(\zeta)}$, and consequently these ζ satisfy (3.12).

In view of (3.15), (3.16) and the fact and the Hausdorff-dimension of a finite union of sets is the maximum of the Hausdorff-dimensions we get that for any $x \in (n, n + \epsilon)$ the Hausdorff-dimension of the set satisfying (3.12) and $w_n(\zeta) = x$ is at least $\frac{n+1}{x+1}$. Letting $x - n = \epsilon \rightarrow 0$ we infer that the Hausdorff-dimension of the set satisfying (3.12) is arbitrarily close to 1, so it equals 1. $\square$

Finally an idea how to *construct* numbers satisfying (3.12) for $n \geq 3$ (and similarly for $n = 2$).

Let $n \geq 3$ be fixed and $\eta := \sum_{k \geq 1} 10^{-k!}$. For any rational number $a \in \mathbb{Q}$ to be fixed later consider $\zeta := (a + \eta)^{\frac{1}{n}}$. By definition of the approximation constants w_n , their stability under birational transformations and our choice of ζ , clearly $w_n(\zeta) \geq w_1(\zeta^n) = w_1(\eta) = \infty$. So the right hand side of (3.12) equals 1 for these values ζ . So, if $\lambda_n(\zeta) < 1$ for some a , we would be done. Numerical computations suggest this could be true.

In the case $n = 2$ (where $\lambda_2 < 1$ is not possible by transference principle), one could consider $\eta := \sum_{k \geq 1} 10^{-b^k}$ with some fixed large b and proceed similarly.

Remark. Dually to the Remark preceding Corollary 3.3 in Section 3.1 (where we treated $w_n^*(\zeta)$), one can use $\overline{\psi}_{n,n+1}^* \leq -n\psi_{n,1}^*$, which is again dual to (1.10) in [12], to infer

$$\widehat{w}_n^*(\zeta) \geq \frac{n}{w_n(\zeta) - n + 1}.$$

This is again weaker than (3.10) as $w_n(\zeta) \geq n$, note however that the bound $\overline{\psi}_{n,n+1}^* \leq -n\psi_{n,1}^*$ isn't expected to be good.

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