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Some remarks on Rizza–Kähler manifolds

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Dedicated to the 90th birthday of Professor Lajos Tamássy

Abstract. In the present paper we prove that if an almost complex structure J on a Finsler manifold (M, L) is parallel with respect to the Berwald connection D of (M, L), then (M, L) is a Berwald space. Furthermore, in this case, the Berwald connection D is induced from the Levi–Civita connection of a Kähler metric on M.

1. Introduction

Let M be an n-dimensional smooth manifold, and $\pi : TM \to M$ its tangent bundle. We denote by $V := \ker\{d\pi : TTM \to TM\}$ the *vertical subbundle* over TM. Since the quotient bundle TTM/V is isomorphic to the pull-back bundle π^*TM, we obtain the following short exact sequence of vector bundles:

$$\mathbb{O} \longrightarrow V \xrightarrow{\iota} TTM \xrightarrow{\widetilde{d\pi}} \pi^*TM \longrightarrow \mathbb{O}, \tag{1.1}$$

where $\iota : V \hookrightarrow TTM$ is the inclusion, and $\widetilde{d\pi} := (\pi, d\pi)$.

Let $y \in T_xM$ be a tangent vector at $x \in M$, where $T_xM = \pi^{-1}(x)$ is the tangent space at $x \in M$. Then the pair $v = (x, y)$ denotes a point in TM. Since every subspace of T_vTM complementary to the fibre V_v at $v \in TM$ is mapped isomorphically onto the tangent space $T_{\pi(v)}M$, there is no canonical choice of a subspace H_v complementary to V_v. Thus we shall fix a complementary subspace at each point $v \in TM$. An *Ehresmann connection* for TM is a subbundle $H \subset TTM$ complementary to V.

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Definition 1.1. An Ehresmann connection H of a vector bundle TM is called a *nonlinear connection* for TM if it satisfies the following conditions:

(1) The distribution $H : TM \ni v \longmapsto H_v \subset T_vTM$ is smooth on $TM\backslash\{0_M\}$ and is continuous on the whole of TM, where 0_M is the zero section of TM.

(2) The distribution H is invariant under the action m of $\mathbb{R}$ on TM defined by $m_\lambda(v) := (x, \lambda \cdot y)$, i.e.,
$$dm_\lambda(H_v) = H_{m_\lambda(v)} \tag{1.2}$$
for any $\lambda \in \mathbb{R}$ and $v = (x, y) \in TM$.

Remark 1.1. If H is smooth on the whole of TM, then it is called *linear*.

Alternatively, a non-linear connection is defined as a V-valued 1-form θ on TM satisfying $\theta(\mathcal{Z}) = \mathcal{Z}$ for any section $\mathcal{Z}$ of V. Thus θ is a splitting of the exact sequence (1.1):
$$\mathbb{O} \longrightarrow V \underset{\theta}{\overset{\iota}{\rightleftarrows}} TTM \xrightarrow{\widetilde{d\pi}} \pi^*TM \longrightarrow \mathbb{O}.$$
The Ehresmann connection $H = \ker(\theta)$ is also called a *horizontal subbundle* of TTM.

The action m of $\mathbb{R}$ on TM induces the so-called *Liouville vector field* $\mathcal{E}$ by
$$\mathcal{E}_v(f) := \frac{d}{d\lambda}\Big|_{\lambda=0} f(m_{e^\lambda}(v)), \quad f \in C^\infty(TM). \tag{1.3}$$
Considering $\mathcal{E}$ as a section of V, we call it the *tautological section* of V.

Let θ be a nonlinear connection for TM. A vector field X in M is parallel along a regular curve $c : [a, b] \to M$ with respect to θ if it satisfies the ordinary differential equation
$$(X \circ c)^*\theta = 0, \tag{1.4}$$
or, equivalently, its velocity vector field $(X \circ c)'$ is always horizontal, i.e., $(X \circ c)'(t) \in H_{(X\circ c)(t)}$ for all $t \in [a, b]$. Equation (1.4) has a unique solution X_v for each initial value $v \in T_{(c(a)}M$, on which it depends smoothly. The parallel transport $P_{c(t)} : T_{c(a)}M \to T_{c(t)}M$ defined by
$$P_{c(t)}(v) = X_v(t) \tag{1.5}$$
has the homogeneity property
$$P_{c(t)}(\lambda \cdot v) = \lambda \cdot P_{c(t)}(v) \tag{1.6}$$
for any $v \in TM_{c(a)}$ and $\lambda \in \mathbb{R}$, where we write $\lambda \cdot v := m_\lambda(v)$ for simplicity. The parallel transport $P_{c(t)}$ is a diffeomorphism between the fibres, but not a linear isomorphism in general. A nonlinear connection is linear if and only if $P_{c(t)}$ is a linear isomorphism between the fibres for every curve $c : [a, b] \to M$ and all $t \in [a, b]$.

2. Canonical connection

Let in the sequel $A^k(F)$ be the space of all k-forms with values in a vector bundle F. Then, in particular, $A^0(F) = \varGamma(F)$ denotes the space of all smooth sections of F.

If a nonlinear connection θ is specified in TM, then there exists a *partial connection* $\delta : \varGamma(V) \to \varGamma(V \otimes H^*)$ along $H = \ker(\theta)$ in the bundle V, where H^* is the dual bundle of H. Moreover, any partial connection δ can be extended to a connection $D : \varGamma(V) \to \varGamma(V \otimes T^*TM)$ so that the diagram

$$\begin{tikzcd}
\varGamma(V) \arrow[r, "D"] \arrow[dr, "\delta"'] & \varGamma(V \otimes T^*TM) \arrow[d, "1\otimes p"] \arrow[dl, phantom, "\circlearrowright"] \\
& \varGamma(V \otimes H^*)
\end{tikzcd}$$

is commutative, where $p : T^*TM \to H^*$ is the natural projection and T^*TM is the dual bundle of TTM (see [Ba-Bo]).

We suppose that a nonlinear connection θ is given in TM. Since we do not assume the differentiability of θ over 0_M, the parallel translation P_c along any curve c in M is compatible only with the scalar multiplication, but not with the addition in general. Therefore we can not define a connection ∇ on TM from a nonlinear connection θ in general, however, we can show that any θ induces a connection D on the vertical subbundle V as the extension of a partial connection δ.

A *connection* D in the bundle V is usually defined to be a covariant derivative in V, i.e., as a homomorphism $D : \varGamma(V) \to A^1(V)$ satisfying the *Leibniz rule*

$$D(f \cdot \mathcal{Z}) = df \otimes \mathcal{Z} + fD\mathcal{Z}$$

for all $f \in C^\infty(TM)$ and $\mathcal{Z} \in \varGamma(V)$. We shall now introduce a connection D associated with a given nonlinear connection θ.

Since the vertical subbundle V is isomorphic to the induced bundle π^*TM, any vector field X in M is naturally lifted to a section $X^V \in \varGamma(V)$. The section X^V is defined as the vector field which is tangent to the curve $c(t) = (x, y+tX(x))$ in the fiber T_xM at $t = 0$. The map $T_xM \ni X(x) \longmapsto X^V(v) \in V_v$ is an isomorphism. The vector field X^V is called the *vertical lift* of X. In the sequel, we use the superscript V for the vertical lifts of vector fields on M.

On the other hand, for any vector field X in M, there exists a section X^H of H such that $d\pi_v(X^H) = X_{\pi(v)}$ at any point $v \in TM$. The vector field X^H on the total space TM is called the *horizontal lift* of X. In the sequel, we use the superscript H for the horizontal lifts of vector fields on M.

Since any vector field Y on M is a smooth map $Y : M \to TM$ such that $\pi \circ Y = \mathrm{id}$, its derivative $dY_x : T_xM \to T_{Y(x)}TM$ satisfies

$$d\pi\left(dY\left(\frac{dc}{dt}\right) - \left(\frac{dc}{dt}\right)^H\right) = 0$$

for any regular curve c in M. Then it is easy to check that

$$dY\left(\frac{dc}{dt}\right) = \mathcal{L}_{(dc/dt)^H}Y^V + \left(\frac{dc}{dt}\right)^H$$

holds, where $\mathcal{L}_{X^H}$ denotes the Lie derivative by X^H. Then, since $H = \ker(\theta)$, we have

$$(Y \circ c)^*\theta\left(\frac{d}{dt}\right) = \theta\left(\mathcal{L}_{(dc/dt)^H}Y^V + \left(\frac{dc}{dt}\right)^H\right) = \theta\left(\mathcal{L}_{(dc/dt)^H}Y^V\right),$$

and thus Y is parallel vector field on M with respect to θ if and only if

$$\theta\left(\mathcal{L}_{X^H}Y^V\right) = 0$$

for all $X \in \Gamma(TM)$. Hence, it is natural to define a partial connection $\delta : \Gamma(V) \to \Gamma(V \otimes H^*)$ by

$$\delta_{\mathcal{X}}\mathcal{Z} := \theta\left(\mathcal{L}_{\mathcal{X}}\mathcal{Z}\right) = \theta([\mathcal{X}, \mathcal{Z}]) \tag{2.1}$$

for all $\mathcal{Z} \in \Gamma(V)$ and $\mathcal{X} \in \Gamma(H)$.

Since the vertical subbundle V is relatively flat, the partial connection δ may be extended to a connection D of V so that the covariant derivative along V is flat, i.e.,

$$D_{\mathcal{Z}}X^V = 0 \tag{2.2}$$

for all $\mathcal{Z} \in \Gamma(V)$ and $X \in \Gamma(TM)$.

Definition 2.1. The connection $D : \Gamma(V) \to \Gamma(V \otimes T^*TM) := A^1(V)$ defined by (2.1) and (2.2) is called the *canonical connection* on V associated with the given nonlinear connection θ.

From the definition of $\mathcal{E}$ and (2.2), we have $D_{\mathcal{Z}}\mathcal{E} = \mathcal{Z}$ for all $\mathcal{Z} \in \Gamma(V)$, and the homogeneity condition (1.2) implies $D_{\mathcal{X}}\mathcal{E} = \theta\left(\mathcal{L}_{\mathcal{X}}\mathcal{E}\right) = \theta\left(-\mathcal{L}_{\mathcal{E}}\mathcal{X}\right) = 0$ for all $\mathcal{X} \in \Gamma(H)$. Therefore the given nonlinear connection θ is recovered by D.

Proposition 2.1. *The canonical connection D associated with θ satisfies*

$$D\mathcal{E} = \theta \tag{2.3}$$

for the tautological section $\mathcal{E}$ of V.

3. Finsler manifolds and Berwald connections

In the sequel of this paper, we use the chart $(\pi^{-1}(U),(x^i,y^i)_{1\leq i\leq n})$ in TM induced by a chart $(U,(x^i))_{1\leq i\leq n}$ in M, where $y^1,\dots,y^n$ are the fibre coordinates in each T_pM, $p\in U$.

Definition 3.1. A function $L: TM\to\mathbb{R}$ is called a (real) *Finsler metric* if it satisfies

(1) L is continuous on the total space TM, and is smooth on the slit tangent bundle $TM\backslash\{0_M\}$,

(2) $L(v)\geq 0$ for every $v\in TM$, and the equality holds if and only if $v=0$,

(3) $L(\lambda\cdot v)=\lambda L(v)$ for every $v\in TM$ and $\lambda\in\mathbb{R}^+$,

(4) L is strongly convex, i.e., the Hessian (G_{ij}) defined by

$$G_{ij}=\frac{1}{2}\frac{\partial^2 L^2}{\partial y^i\partial y^j} \tag{3.1}$$

is positive definite at each point of $\pi^{-1}(U)$.

Then the pair (M,L) is called a *Finsler manifold.* The *Minkowski norm* of $v\in TM$ is measured by $\|v\|=L(v)$.

The equation of a geodesic in (M,L) is given by

$$\frac{d^2x^i}{ds^2}+2G^i\left(x,\frac{dx}{ds}\right)=0, \tag{3.2}$$

where s is the arc-length with respect to the Finsler metric L, and

$$G^i=\frac{1}{4}\sum G^{im}\left(\frac{\partial G_{jm}}{\partial x^k}+\frac{\partial G_{mk}}{\partial x^j}-\frac{\partial G_{jk}}{\partial x^m}\right)y^jy^k.$$

Then the velocity vector field of the natural lift $\widetilde{c}=(c,c')$ of a geodesic c is given by

$$\widetilde{c}'=\sum y^j\left[\frac{\partial}{\partial x^j}-\sum\frac{\partial G^i}{\partial y^j}\left(x,\frac{dx}{ds}\right)\frac{\partial}{\partial y^i}\right](c,c').$$

We define a nonlinear connection θ so that $\widetilde{c}'$ is horizontal, i.e., by

$$\theta=\sum\frac{\partial}{\partial y^i}\otimes\theta^i:=\sum\frac{\partial}{\partial y^i}\otimes\left(dy^i+\sum N^i_j(x,y)\,dx^j\right), \tag{3.3}$$

where the coefficients N^i_j are given by

$$N^i_j:=\frac{\partial G^i}{\partial y^j}=\frac{1}{2}\sum G^{im}\left(\frac{\partial G_{jm}}{\partial x^k}+\frac{\partial G_{mk}}{\partial x^j}-\frac{\partial G_{jk}}{\partial x^m}\right)y^k. \tag{3.4}$$

Definition 3.2. The nonlinear connection θ defined by (3.3) is called the *Berwald nonlinear connection* of (M, L). The canonical connection D associated with the Berwald nonlinear connection θ is called the *Berwald connection* of (M, L).

The connection forms ω^i_j of D with respect to the local frame field $(\partial/\partial y^i)_{1\leq i\leq m}$ of V are given by $\omega^i_j = \sum \Gamma^i_{jk} dx^k$ with the coefficients

$$\Gamma^i_{jk} = \frac{\partial N^i_k}{\partial y^j}. \tag{3.5}$$

In fact, from (2.1)

$$D_{(\partial/\partial x^i)^H} \frac{\partial}{\partial y^j} = \theta\left[\left(\frac{\partial}{\partial x^i}\right)^H, \frac{\partial}{\partial y^j}\right] = \sum \frac{\partial N^h_i}{\partial y^j}\frac{\partial}{\partial y^h}.$$

In the case of TM, both V and H are isomorphic to the bundle π^*TM induced from TM via π. Therefore the derivative $d\pi$ of π can also be considered as the projection from $T(TM)$ onto V with $\ker(d\pi) = V$, and thus $d\pi$ can be interpreted as a section of $A^1(V)$ given locally by

$$d\pi = \sum \frac{\partial}{\partial y^i} \otimes dx^i.$$

From (3.4) and (3.5) we obtain

Proposition 3.1. *The Berwald connection D satisfies*

$$Dd\pi \equiv 0. \tag{3.6}$$

Any Finsler metric L defines a Riemannian structure G on the vertical subbundle V by

$$G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = G_{ij}. \tag{3.7}$$

The homogeneity assumption for L implies $L^2 = G(\mathcal{E}, \mathcal{E})$ for the tautological section $\mathcal{E}$. Then, by the definition of the Berwald nonlinear connection θ, we have $\mathcal{X}(L^2) = 0$ for all $\mathcal{X} \in \Gamma(H)$. Thus L is constant along the horizontal subbundle H, i.e.,

$$\mathcal{X}(L) \equiv 0 \tag{3.8}$$

for all $\mathcal{X} \in \Gamma(H)$.

Since the vertical lift X^V is related to the horizontal lift X^H by

$$X^V = d\pi(X^H),$$

$[X^H, Y^H] - [X, Y]^H \in \Gamma(V)$ implies $d\pi([X^H, Y^H]) = d\pi([X, Y]^H) = [X, Y]^V$, and thus (3.6) implies

$$D_{X^H} Y^V - D_{Y^H} X^V = [X, Y]^V \tag{3.9}$$

for all $X, Y \in \Gamma(TM)$.

4. Landsberg spaces and Berwald spaces

The specific goal of this section is to recall some facts on Landsberg spaces and Berwald spaces which we need later on (see, e.g., [Ai3], [Ai-Ko], [Ic1], [Sz] for details).

Let (M, L) be a Finsler manifold, and let (φ_t) be the local flow generated by a vector field X in M , and (φ_t^H) the flow of the horizontal lift X^H of X with respect to the Berwald nonlinear connection θ. From (3.8), we have $\frac{d}{dt}\big|_{t=0}(\varphi_t^H)^*L = 0$. Therefore φ_t^H preserves the indicatrix $I_x := \{y \in T_xM \mid L(x,y) = 1\}$ for all $x \in M$:

$$I_{\varphi_t(x)} = \varphi_t^H(I_x). \tag{4.1}$$

Since each fibre $V_{(x,y)}$ over $(x, y) \in TM$ is the tangent space $T_y(T_xM)$ of T_xM at $y \in T_xM$, each fibre T_xM is a Riemannian space endowed the metric $G_x := G \restriction T_xM$. A Finsler manifold (M, L) is called a *Landsberg space* if the parallel transport $P_{c(t)}$ along any curve c in M is an isometry from the initial Riemannian space $(T_{c(a)}M, G_{c(a)})$ to $(T_{c(t)}M, G_{c(t)})$ for all $t \in [a, b]$. Thus (M, L) is a Landsberg space if and only if

$$\mathcal{L}_{X^H} G = \frac{d}{dt}\Big|_{t=0}(\varphi_t^H)^*G = 0 \tag{4.2}$$

for any $X \in \Gamma(TM)$. By the definition of D, this condition is equivalent to

$$D_{X^H} G = 0 \tag{4.3}$$

for all $X \in \Gamma(TM)$ (see [Ai-Ko]).

On the other hand, a Finsler manifold (M, L) is called a *Berwald space* if the parallel translation $P_{c(t)}$ is an isometry between the normed tangent spaces, i.e.,

$$\|v - w\| = \left\|P_{c(t)}(v) - P_{c(t)}(w)\right\| \tag{4.4}$$

is satisfied for all $v, w \in T_{c(a)}M$ and $t \in [a, b]$ (see [Ic1]). Then, by a well-known theorem due to Szabó [Sz], the Berwald connection D of a Berwald space (M, L) is induced from the Levi–Civita connection ∇^g of a Riemannian metric g on M, i.e.,

$$D_{X^H} Y^V = \left(\nabla^g_X Y\right)^V \tag{4.5}$$

for all $X, Y \in \Gamma(TM)$.

Since we have $G(\mathcal{E}, \mathcal{E}) = L^2$, the tautological section $\mathcal{E}$ is a unit vector at every point $y \in I_x$. Further, the gradient vector field of the level hypersurface

$I_x \subset T_xM$ is given by

$$\begin{aligned}\sum G^{im}\frac{\partial L}{\partial y^m}\left(\frac{\partial}{\partial y^i}\right) &= \frac{1}{2L}\sum G^{im}\frac{\partial L^2}{\partial y^m}\left(\frac{\partial}{\partial y^i}\right)\\ &= \frac{1}{L}\sum G^{im}G_{lm}y^l\left(\frac{\partial}{\partial y^i}\right) = \frac{1}{L}\sum y^i\frac{\partial}{\partial y^i}\end{aligned}$$

at each point of I_x. Thus $\mathcal{E}$ may be considered as the outward-pointing unit normal vector field of the indicatrix I_x. Hence, for the volume form $d\mu = \sqrt{\det G}\ dy^1 \wedge \cdots \wedge dy^n$ on each tangential Riemannian space (T_xM, G_x), the $(n-1)$-form

$$d\mu_I = \iota(\mathcal{E})d\mu = \sum(-1)^{j-1}y^j\sqrt{\det G}\ dy^1 \wedge \cdots \wedge \check{dy^j} \wedge \cdots dy^n \tag{4.6}$$

defines a volume form of each indicatrix I_x, and the volume $\mathrm{vol}(I_x)$ of I_x is given by $\mathrm{vol}(I_x) = \int_{I_x} d\mu_I$.

The *averaged Riemannian metric* of G is a Riemannian metric g on M defined by

$$g(X,Y) = \frac{1}{\mathrm{vol}(I_x)}\int_{I_x} G(X^V, Y^V)\ d\mu_I, \tag{4.7}$$

and the *averaged connection* of D is a linear connection ∇ on TM defined by

$$g(\nabla_X Y, Z) = \frac{1}{\mathrm{vol}(I_x)}\int_{I_x} G\left(D_{X^H}Y^V, Z^V\right)\ d\mu_I \tag{4.8}$$

for all $X, Y, Z \in \Gamma(TM)$, respectively (see [Ma-Ra-Tr-Ze], [To-Et]). Since $\mathcal{E}$ satisfies (2.3), we have

Lemma 4.1 (cf. [Ai-Ko]). *If (M, L) is a Landsberg space, then*

$$\mathcal{L}_{X^H} d\mu_I = 0 \tag{4.9}$$

for every $X \in \Gamma(TM)$, and therefore the volume of the indicatrix I_x is constant.

From Lemma 4.1 we conclude that

$$X\left(\int_{I_x} f d\mu_I\right) = \int_{I_x}\mathcal{L}_{X^H}(f d\mu_I) = \int_{I_x} X^H(f)\ d\mu_I$$

for all $X \in \Gamma(TM)$ and $f \in C^\infty(TM)$ if (M, L) is a Landsberg space. This identity implies that ∇ is compatible with the averaged metric g. Further, (3.9) implies that ∇ is torsion-free.

Theorem 4.1 ([Ai3]). *If (M, L) is a Landsberg space, then the averaged connection ∇ of D is the Levi–Civita connection of the averaged Riemannian metric g of G.*

In particular, if (M, L) is a Berwald space, we have the the following well-known result.

Theorem 4.2. ([Sz], [Vi]) *If (M, L) is a Berwald space, then the Berwald connection D is induced by the Levi–Civita connection ∇^g of the averaged Riemannian metric g of G, i.e., D is given by (4.5) for all $X, Y \in \Gamma(TM)$.*

5. Rizza–Kähler manifolds

In the sequel of this paper we assume that (M, L) is a $2n$-dimensional Finsler manifold which admits an almost complex structure J, i.e., an endomorphism J of TM such that $J \circ J = -I$, where I is the identity morphism of TM.

Definition 5.1 ([Ri1], [Ic3]). A Finsler metric L is called a *complex Finsler metric* or *Rizza metric* if it satisfies

$$L \circ (aI + bJ)X = \sqrt{a^2 + b^2}\, L \circ X \tag{5.1}$$

for all $X \in \Gamma(TM)$ and $a, b \in \mathbb{R}$. Then the triplet (M, J, L) is called a *Rizza manifold.*

Example 5.1. Let h be a Hermitian metric on an almost complex manifold (M, J). For any $X \in \Gamma(TM)$, we put $L \circ X = \sqrt{h(X, X)}$. Then, since $h(JX, X) + h(X, JX) = 0$ is satisfied, it is easily checked that L satisfies (5.1). Thus Rizza manifolds are natural generalizations of Hermitian manifolds.

Let ϕ_θ be the endmorphism of TM defined by $\phi_\theta = \cos\theta \cdot I + \sin\theta \cdot J$ for each $\theta \in \mathbb{R}$. Then we can write assumption (5.1) as $L \circ \phi_\theta X = L \circ X$ for any $\theta \in \mathbb{R}$. By direct calculation, we have $\phi_{\theta_1} \circ \phi_{\theta_2} = \phi_{\theta_1 + \theta_2}$ for all $\theta_1, \theta_2 \in \mathbb{R}$. Using this fact, we obtain

Theorem 5.1 ([Ic3], [Ri1]). *For any Finsler metric $\overline{L}$ on an almost complex manifold (M, J), the function $L \circ X = \left(\frac{1}{2\pi}\int_0^{2\pi} \overline{L}(\phi_\theta X)^2 d\theta\right)^{1/2}$ defines a Rizza metric on (M, J).*

The almost complex structure J of TM is lifted to that of V:

$$J^V X^V := (JX)^V \tag{5.2}$$

for any $X \in \Gamma(TM)$. Suppose that J^V is parallel with respect to the Berwald connection D:

$$DJ^V = 0. \tag{5.3}$$

From the definitions of D and J^V, it is obvious that $D_{\mathcal{Z}} J^V = 0$ for all $\mathcal{Z} \in \Gamma(V)$. Thus this assumption is equivalent to

$$D_{\mathcal{X}} J^V = 0 \tag{5.4}$$

for all $\mathcal{X} \in \Gamma(H)$. Since the Kähler condition in [Ic4] implies (5.3), the class of Rizza manifolds satisfying (5.3) includes the class of *Kaehlerian Finsler manifolds* in [Ic4]. To distinguish our Kählerity from that of [Ic1] or [Ab-Pa], we use a new terminology:

Definition 5.2. A Rizza manifold (M, J, L) is said to be *Rizza–Kähler* if (5.3) is satisfied.

Remark 5.1. Since the Kähler condition in [Le-Wo] implies the corresponding condition in [Ic4], our Rizza–Kähler manifolds are defined in wider sense than that of [Le-Wo]. A complex manifold M with a normal (a, b, f)-metric L discussed in [Ic-Ha] is an example of Rizza–Kähler manifolds.

The integrability tensor for J is the Nijenhuis tensor field N_J given by $N_J(X, Y) := [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$ for all $X, Y \in \Gamma(TM)$. From (3.9), the assumption (5.3) implies

$$\begin{aligned}
(N_J(X,Y)^V &= ([X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]^V \\
&= [X,Y]^V + J^V[JX,Y]^V + J^V[X,JY]^V - [JX,JY]^V \\
&= D_{X^H}Y^V - D_{Y^H}X^V + J^V\left(D_{(JX)^H}Y^V - D_{Y^H}(JX)^V\right) \\
&\quad + J^V(D_{X^H}(JY)^V - D_{(JY)^H}X^V) - D_{(JX)^H}(JY)^V + D_{(JY)^H}(JX)^V \\
&= D_{X^H}Y^V - D_{Y^H}X^V + J^V D_{(JX)^H}Y^V + D_{Y^H}X^V - D_{X^H}Y^V \\
&\quad - J^V D_{(JY)^H}X^V - J^V D_{(JX)^H}Y^V + J^V D_{(JY)^H}X^V = 0
\end{aligned}$$

for all $X, Y \in \Gamma(TM)$. Thus $N_J \equiv 0$, therefore J is integrable.

Proposition 5.1 ([Ic4]). *If the Berwald connection D satisfies* (5.3)*, then J is integrable, i.e., (M, J) is a complex manifold.*

Remark 5.2. This proposition was first proved by Ichijyō [Ic4] in terms of the *Cartan connection* of (M, L). The essential fact we need in the proof above is the condition (3.9) of D.

We suppose that (5.3) is satisfied. Since (M,J) is a complex manifold, we identify TM with the holomorphic tangent bundle over (M,J). Then each fibre T_xM over $x \in M$ is a complex manifold with complex structure J_x. Since $\mathcal{L}_{X^H}(JY)^V$, $J^V(\mathcal{L}_{X^H}Y^V) \in \Gamma(V)$ for all $X,Y \in \Gamma(TM)$, we obtain $(D_{X^H}J^V)(Y^V) = D_{X^H}(J^VY^V) - J^V(D_{X^H}Y^V) = (\mathcal{L}_{X^H}J^V)Y^V$.

Proposition 5.2. *If (M,J,L) is a Rizza–Kähler manifold, then*

$$\mathcal{L}_{X^H}J^V = 0 \tag{5.5}$$

for all $X \in \Gamma(TM)$.

A real vector field X on a complex manifold (M,J) is real holomorphic if $X^{1,0} := (X - \sqrt{-1}JX)/2$ is a holomorphic vector field on (M,J). A vector field X is real holomorphic if and only if the flow (φ_t) generated by X is a holomorphic map of (M,J), i.e., $d\varphi_t \circ J_x = J_{\varphi_t(x)} \circ d\varphi_t$ is satisfied for every $x \in M$. Thus X is real holomorphic if and only if $\mathcal{L}_X J = 0$, i.e., X is an infinitesimal automorphism of J.

Let X be a real holomorphic vector field on a Rizza–Kähler manifold (M,J,L). Then, since (5.5) implies $d\varphi_t^H \circ J_x^V = J_{\varphi_t(x)}^V \circ d\varphi_t^H$, the flow $\varphi_t^H : T_xM\backslash\{0\} \to T_{\varphi_t(x)}M\backslash\{0\}$ generated by the horizontal lift X^H of X is a holomorphic map for every $x \in M$.

Theorem 5.2. *If (M,J,L) is a Rizza–Kähler manifold, then (M,L) is a Berwald space.*

Proof. Let $\varphi_t^H : T_xM\backslash\{0\} \to T_{\varphi_t(x)}M\backslash\{0\}$ be the flow generated by the horizontal lift X^H of any real holomorphic vector field X. By Proposition 5.2, each φ_t^H is a holomorphic map. Since we are always concerned with M of $\dim_{\mathbb{C}} M \geq 2$, the isolated singularity $\{0\}$ of φ_t^H is removable by Hartogs' theorem (see, e.g., [Hu]), and thus each φ_t^H may be extended to a holomorphic map on the whole of T_xM for every $x \in M$.

Let $\eta^a = y^a + \sqrt{-1}y^{(a)}$ $(1 \leq a \leq m,\ (a) = m+a)$ be the complex coordinates on each fiber of the holomorphic tangent bundle over (M,J) naturally induced from the given local complex coordinate system $(z^1,\dots,z^m)$ $(n = 2m)$ on M. Denoting by $\mathcal{N}_b^a$ the coefficients of the Berwald nonlinear connection H in the complex coordinate system (z^a,η^a) in TM, the horizontal lifts $(\partial/\partial z^b)^H$ of the members of the local frame field $(\partial/\partial z^b)_{1\leq b\leq m}$ are given by

$$\left(\frac{\partial}{\partial z^b}\right)^H = \frac{\partial}{\partial z^b} - \sum \mathcal{N}_b^a \frac{\partial}{\partial \eta^a},$$

where the coefficients $\mathcal{N}^a_b$ are holomorphic in $\eta = (\eta^1, \dots, \eta^m)$. Further, the relations between the coefficients N^i_j and $\mathcal{N}^a_b$ are given by $\mathcal{N}^a_b := N^a_b + \sqrt{-1}N^{(a)}_b$ and

$$\left(N^i_j\right) = \begin{pmatrix} N^a_b & N^{(a)}_b \\ -N^{(a)}_b & N^a_b \end{pmatrix}, \quad 1 \leq i,\ j \leq n = 2m.$$

The power series expansions of $\mathcal{N}^a_b(z, \eta)$ with respect to $(\eta^1, \dots, \eta^m)$ are of the form

$$\mathcal{N}^a_b(z, \eta) = \sum_{c_1, \dots, c_m \geq 0} \mathcal{N}^a_{bc_1 \cdots c_m}(z)(\eta^1)^{c_1} \cdots (\eta^m)^{c_m},$$

and thus

$$\begin{aligned} N^a_b(x, y) &+ \sqrt{-1}N^{(a)}_b(x, y) \\ &= \sum \mathcal{N}^a_{bc_1 \cdots c_m}(z) \left(y^1 + \sqrt{-1}y^{(1)}\right)^{c_1} \cdots \left(y^m + \sqrt{-1}y^{(m)}\right)^{c_m}. \end{aligned}$$

Since the real coefficients N^i_j satisfy the homogeneity condition $N^i_j \circ m_\lambda = \lambda N^i_j$ for all $\lambda > 0$, the surviving terms in the RHS of the above relation are given by $c_1 + \cdots + c_m = 1$:

$$N^a_b(x, y) + \sqrt{-1}N^{(a)}_b(x, y) = \sum \mathcal{N}^a_{bc}(z) \left(y^c + \sqrt{-1}y^{(c)}\right).$$

If we put $\mathcal{N}^a_{bc}(z) = \Gamma^a_{bc}(x) + \sqrt{-1}\Gamma^{(a)}_{bc}(x)$, then we obtain

$$\begin{aligned} \sum \mathcal{N}^a_{bc}(z) \left(y^c + \sqrt{-1}y^{(c)}\right) &= \sum \left(\Gamma^a_{bc}(x)y^c - \Gamma^{(a)}_{bc}(x)y^{(c)}\right) \\ &\quad + \sqrt{-1} \sum \left(\Gamma^a_{bc}(x)y^{(c)} + \Gamma^{(a)}_{bc}(x)y^c\right). \end{aligned}$$

Consequently we have

$$N^a_b = \sum \left(\Gamma^a_{bc}y^c - \Gamma^{(a)}_{bc}y^{(c)}\right), \quad N^{(a)}_b = \sum \left(\Gamma^a_{bc}y^{(c)} + \Gamma^{(a)}_{bc}y^c\right)$$

This shows that the real coefficients N^i_j are of the forms $N^i_j = \sum \Gamma^i_{jk}y^k$, and the coefficients N^i_j of H are polynomials of degree one in $(y^1, \dots, y^n)$. This shows that (M, L) is a Berwald space by SZABÓ's theorem[Sz]. □

6. Kähler metrics associated with Rizza–Kähler metrics

Let (M, J, L) be a Rizza manifold. If the metric G on V defined by (3.7) satisfies the Hermitian condition

$$G(J^V \mathcal{Z}, J^V \mathcal{W}) = G(\mathcal{Z}, \mathcal{W}) \tag{6.1}$$

for all $\mathcal{Z}, \mathcal{W} \in \Gamma(V)$, then L is the norm function $L(x, y) = \sqrt{\sum g_{ij} y^i y^j}$ of certain Riemannian metric $g = \sum g_{ij} dx^i \otimes dx^j$ (see [He], [Ic4]). Thus, in [Ic4], the following Riemannian structure K on V has been introduced:

$$K(\mathcal{Z}, \mathcal{W}) := \frac{1}{2} \left[G(\mathcal{Z}, \mathcal{W}) + G(J^V \mathcal{Z}, J^V \mathcal{W}) \right], \tag{6.2}$$

where $\mathcal{Z}, \mathcal{W} \in \Gamma(V)$. Obviously, K satisfies the Hermitian condition, but K is never obtained from a Finsler metric. Hence K is a *generalized Finsler structure*. If (M, J, L) is a Rizza–Kähler manifold, then (4.3) and (5.3) show that the Berwald connection D of (M, J, L) satisfies

$$D_{X^H} K = 0 \tag{6.3}$$

for all $X \in \Gamma(TM)$. The aim of this section is to show that D is induced from the Levi–Civita connection of a Kähler metric on (M, J).

Let (M, J, L) be a Rizza–Kähler manifold. Then, from Theorem 5.2, (M, L) is a Berwald space, i.e., the Berwald connection D is induced from the Levi–Civita connection ∇^g of the averaged Riemannian metric g defined by (4.7). Further, from (4.5), we obtain

$$[(\nabla^g_X J)Y]^V = (D_{X^H} J^V) Y^V = 0$$

for all $X, Y \in \Gamma(TM)$, and thus ∇^g is a complex connection of (M, J).

Let k be the averaged Riemannian metric of K:

$$k(X, Y) := \frac{1}{\mathrm{vol}(I_x)} \int_{I_x} K\left(X^V, Y^V\right) d\mu_I. \tag{6.4}$$

Theorem 6.1. *Let (M, J, L) be a Rizza–Kähler manifold. Then (M, L) is a Berwald space, and its Berwald connection D is induced from the Levi–Civita connection of the averaged Kähler metric k on M.*

PROOF. First we show that the metric k is a Hermitian metric on (M, J). Indeed, from the definitions of K and k, we have

$$\begin{aligned} k(JX, JY) &= \frac{1}{\mathrm{vol}(I_x)} \int_{I_x} K\left(J^V X^V, J^V Y^V\right) d\mu_I \\ &= \frac{1}{\mathrm{vol}(I_x)} \int_{I_x} K\left(X^V, Y^V\right) d\mu_I = k(X, Y). \end{aligned}$$

Furthermore (4.5) implies

$$\begin{aligned} k(\nabla_Z X, Y) &= \frac{1}{\mathrm{vol}(I_x)} \int_{I_x} K\left((\nabla^g_Z X)^V, Y^V\right) d\mu_I \\ &= \frac{1}{\mathrm{vol}(I_x)} \int_{I_x} K\left(D_{Z^H} X^V, Y^V\right) d\mu_I. \end{aligned}$$

Then from (6.3) we obtain

$$\begin{aligned} (\nabla^g_Z k)(X, Y) &= Z\left(k(X, Y)\right) - k(\nabla^g_Z X, Y) - k(X, \nabla^g_Z Y) \\ &= \frac{1}{\mathrm{vol}(I_x)} \int_{I_x} \left[Z^H K(X^V, Y^V) - K(D_{Z^H} X^V, Y^V) - K(X, D_{Z^H} Y^V)\right] d\mu_I \\ &= \frac{1}{\mathrm{vol}(I_x)} \int_{I_x} (D_{Z^H} K)\left(X^V, Y^V\right) d\mu_I = 0. \end{aligned}$$

Since ∇^g is torsion-free, k is a Kähler metric on M. □

Therefore, if (M, J, L) is a Rizza–Kähler manifold, then (M, J) is a Kähler manifold. As is well-known, since a Hopf manifold never admits any Kähler metric, there exists no Rizza–Kähler metric on such a manifold. However, any Hopf manifold admits a locally conformal Kähler metric ([Va]). Hence, in the next section, we shall consider conformal changes of Rizza metrics.

7. Locally conformal Rizza–Kähler manifolds

First define the operator $d_\nabla : L \to d_\nabla L$ for a Finsler metric L by

$$i(X) d_\nabla L = X^H(L), \quad X \in \Gamma(TM), \tag{7.1}$$

where X^H is the horizontal lift of X with respect to a linear connection ∇. A Finsler manifold (M, L) is said to be *locally conformal Berwald* (*l.c. Berwald* in short) if there exists a torsion-free linear connection ∇ on TM such that

$$d_\nabla L = \beta \otimes L \tag{7.2}$$

for a closed 1-form β on M ([Ai1], [Ai3]). Such a space is a special type of the so-called *Wagner spaces* (see [Ha-Ic]).

Let (M, J, L) be a Rizza manifold. We consider a conformal change

$$L^\alpha := e^{\sigma_\alpha} L \tag{7.3}$$

of L by a local function σ_α defined on an open subset $U_\alpha \subset M$

Definition 7.1. A Rizza manifold (M, J, L) is called a *locally conformal Rizza–Kähler manifold* (*l.c. Rizza–Kähler manifold* in short) if there exists an open cover $(U_\alpha)_{\alpha\in A}$ of M and a family $(\sigma_\alpha)_{\alpha\in A}$ of functions $\sigma_\alpha : U_\alpha \to \mathbb{R}$ such that all L^α's are Rizza–Kähler metrics on U_α.

From Theorem 5.2, the Finsler metric L^α is a Berwald metric on U_α. Since the Berwald connection D^α of L^α satisfies $D^\alpha J^V = 0$, Proposition 5.1 implies that J is integrable, and thus (M, J) is a complex manifold.

For the Hermitian metric K on V defined by (6.2), we denote by K^α the Hermitian metric on $V \restriction \pi^{-1}(U_\alpha)$ obtained by the conformal change $K^\alpha = e^{2\sigma_\alpha} K$. Then the averaged Riemannian metric k^α of K^α is defined by

$$k^\alpha(X, Y) = \frac{1}{\mathrm{vol}(I_x^\alpha)} \int_{I_x^\alpha} K^\alpha\left(X^V, Y^V\right) d\mu_{I^\alpha}$$

for all $X, Y \in \varGamma(TM)$, where $I_x^\alpha = e^{-\sigma_\alpha} I_x$ is the indicatrix at $x \in M$ with respect to L^α, and $d\mu_{I^\alpha}$ is the volume form on I_x^α:

$$\begin{aligned} d\mu_{I^\alpha} &= \sum (-1)^{i-1} \sqrt{\det G^\alpha}\; w^i dw^1 \wedge \cdots \wedge \check{dw^i} \wedge \cdots \wedge dw^n \\ &= e^{n\sigma_\alpha(x)} \sum (-1)^{i-1} \sqrt{\det G}\; w^i dw^1 \wedge \cdots \wedge \check{dw^i} \wedge \cdots \wedge dw^n \end{aligned}$$

at $w = (w^1, \ldots, w^n) \in I_x^\alpha$, where G^α is the metric obtained by the conformal change $G^\alpha = e^{2\sigma_\alpha} G$. Since the isomorphism $\psi_\alpha : (T_x M, G_x) \ni y \longrightarrow \psi_\alpha(y) = e^{-\sigma_\alpha} y \in (T_x M, G_x^\alpha)$ is an isometry, we have

$$\begin{aligned} \psi_\alpha^*(d\mu_{I^\alpha}) &= e^{n\sigma_\alpha(x)} \sum (-1)^{i-1} \sqrt{\det G \circ \psi_\alpha}\; e^{-n\sigma_\alpha(x)} y^i dy^1 \wedge \cdots \wedge \check{dy^i} \wedge \cdots \wedge dy^n \\ &= \sum (-1)^{i-1} \sqrt{\det G}\; y^i dy^1 \wedge \cdots \wedge \check{dy^i} \wedge \cdots \wedge dy^n = d\mu_I, \end{aligned}$$

which implies $\mathrm{vol}(I_x^\alpha) = \int_{I_x^\alpha} d\mu_{I^\alpha} = \int_{I_x} \psi_\alpha^*(d\mu_{I^\alpha}) = \int_{I_x} d\mu_I = \mathrm{vol}(I_x)$. Thus the averaged Riemannian metric k^α obtained from K^α is given by

$$k^\alpha(X, Y) = \frac{1}{\mathrm{vol}(I_x^\alpha)} \int_{I_x^\alpha} K^\alpha(X^V, Y^V) d\mu_{I^\alpha}$$

$$= \frac{1}{\mathrm{vol}(I_x)} \int_{I_x} \left(K^\alpha (X^V, Y^V) \circ \psi_\alpha \right) d\mu_I$$
$$= \frac{e^{2\sigma_\alpha}}{\mathrm{vol}(I_x)} \int_{I_x} K\left(X^V, Y^V\right) d\mu_I = e^{2\sigma_\alpha} k(X, Y)$$

for all $X, Y \in \Gamma(TM)$, i.e., k^α is given by the conformal change

$$k^\alpha = e^{2\sigma_\alpha} k \tag{7.4}$$

for the Hermitian metric k defined by (6.4). From Theorem 6.1, the metric k^α is a local Kähler metric for all $\alpha \in A$. Since the Kähler form Ω^α of k^α is given by $\Omega^\alpha = e^{2\sigma_\alpha}\Omega$ for the one Ω of k, we obtain $d\Omega = -2d\sigma_\alpha \wedge \Omega$ and, hence $(d\sigma_\alpha - d\sigma_\beta) \wedge \Omega = 0$. By the non-degeneracy of Ω, we have $d\sigma_\alpha = d\sigma_\beta$ on the intersection $U_\alpha \cap U_\beta \neq \phi$. Therefore the family $(d\sigma_\alpha)_{\alpha \in A}$ of exact local one-forms $d\sigma_\alpha$ glues up to a global 1-form β_L on M which implies $d\Omega = -2\beta_L \wedge \Omega$. Thus (M, J, k) is an l.c. Kähler manifold ([Va]).

Theorem 7.1. *Let (M, J, L) be an l.c. Rizza–Kähler manifold. Then the associated Hermitian manifold (M, J, k) is an l.c. Kähler manifold.*

Let ∇^α be the averaged connection determined by the local connection D^α with respect to the local metric G^α by the formula (4.8). Since each local connection ∇^α is compatible with the local Kähler metric k^α, we have

$$0 = \nabla^\alpha k^\alpha = e^{2\sigma_\alpha} \left(2d\sigma_\alpha \otimes k + \nabla^\alpha k\right),$$

and thus we obtain

$$\nabla^\alpha k = -2\beta_L \otimes k.$$

Therefore ∇^α is the Weyl connection of (M, k) with the Lee form β_L of the conformal class represented by k. The uniqueness of the Weyl connection implies that the local connections ∇^α glue up to a global torsion-free linear connection ∇. Since each local connection D^α is induced from ∇^α, the connection ∇^α satisfies $d_{\nabla^\alpha} L^\alpha = 0$, i.e., $d_{\nabla^\alpha} L = -d\sigma_\alpha \otimes L$. Hence we obtain

$$d_\nabla L = -\beta_L \otimes L \tag{7.5}$$

on M. Therefore we have

Theorem 7.2. *If (M, J, L) is an l.c. Rizza–Kähler manifold, then the underlying real Finsler manifold (M, L) is l.c. Berwald.*

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