

On monotonicity of some combinatorial sequences

By QING-HU HOU (Tianjin), ZHI-WEI SUN (Nanjing) and HAOMIN WEN (Philadelphia)

Abstract. We confirm Sun's conjecture that $(\sqrt[n+1]{F_{n+1}}/\sqrt[n]{F_n})_{n \geq 4}$ is strictly decreasing to the limit 1, where $(F_n)_{n \geq 0}$ is the Fibonacci sequence. We also prove that the sequence $(\sqrt[n+1]{D_{n+1}}/\sqrt[n]{D_n})_{n \geq 3}$ is strictly decreasing with limit 1, where D_n is the n -th derangement number. For m -th order harmonic numbers $H_n^{(m)} = \sum_{k=1}^n 1/k^m$ ($n = 1, 2, 3, \dots$), we show that $(\sqrt[n+1]{H_{n+1}^{(m)}}/\sqrt[n]{H_n^{(m)}})_{n \geq 3}$ is strictly increasing.

1. Introduction

A challenging conjecture of Firoozbakht states that

$$\sqrt[n]{p_n} > \sqrt[n+1]{p_{n+1}} \quad \text{for every } n = 1, 2, 3, \dots,$$

where p_n denotes the n -th prime. Note that $\lim_{n \rightarrow \infty} \sqrt[n]{p_n} = 1$ by the Prime Number Theorem. In [4] the second author conjectured further that for any integer $n > 4$ we have the inequality

$$\frac{\sqrt[n+1]{p_{n+1}}}{\sqrt[n]{p_n}} < 1 - \frac{\log \log n}{2n^2},$$

which has been verified for all $n \leq 3.5 \times 10^6$. Motivated by this and [3], Sun [4, Conjecture 2.12] conjectured that the sequence $(\sqrt[n+1]{S_{n+1}}/\sqrt[n]{S_n})_{n \geq 7}$ is strictly

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increasing, where S_n is the sum of the first n positive squarefree numbers. Moreover, he also posed many conjectures on monotonicity of sequences of the type $(\sqrt[n+1]{a_{n+1}}/\sqrt[n]{a_n})_{n \geq N}$ with $(a_n)_{n \geq 1}$ a familiar combinatorial sequence of positive integers.

Throughout this paper, we set $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$.

Let A and B be integers with $\Delta = A^2 - 4B \neq 0$. The Lucas sequence $u_n = u_n(A, B)$ ($n \in \mathbb{N}$) is defined as follows:

$$u_0 = 0, \quad u_1 = 1, \quad \text{and} \quad u_{n+1} = Au_n - Bu_{n-1} \quad \text{for } n = 1, 2, 3, \dots$$

It is well known that $u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ for all $n \in \mathbb{N}$, where

$$\alpha = \frac{A + \sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{A - \sqrt{\Delta}}{2}$$

are the two roots of the characteristic equation $x^2 - Ax + B = 0$. The sequence $F_n = u_n(1, -1)$ ($n \in \mathbb{N}$) is the famous Fibonacci sequence, see [1, p. 46] for combinatorial interpretations of Fibonacci numbers.

Our first result is as follows.

Theorem 1.1. *Let $A > 0$ and $B \neq 0$ be integers with $\Delta = A^2 - 4B > 0$, and set $u_n = u_n(A, B)$ for $n \in \mathbb{N}$. Then there exists an integer $N > 0$ such that the sequence $(\sqrt[n+1]{u_{n+1}}/\sqrt[n]{u_n})_{n \geq N}$ is strictly decreasing with limit 1. In the case $A = 1$ and $B = -1$ we may take $N = 4$.*

Remark 1.1. Under the condition of Theorem 1.1, by [2, Lemma 4] we have $u_n < u_{n+1}$ unless $A = n = 1$. Note that the second assertion in Theorem 1.1 confirms a conjecture of the second author [4, Conjecture 3.1] on the Fibonacci sequence.

For $n \in \mathbb{Z}^+$ the n -th derangement number D_n denotes the number of permutations σ of $\{1, \dots, n\}$ with $\sigma(i) \neq i$ for no $i = 1, \dots, n$. It has the following explicit expression (cf. [1, p. 67]):

$$D_n = \sum_{k=0}^n (-1)^k \frac{n!}{k!}.$$

Our second theorem is the following result conjectured by the second author [4, Conjecture 3.3].

Theorem 1.2. *The sequence $(\sqrt[n+1]{D_{n+1}}/\sqrt[n]{D_n})_{n \geq 3}$ is strictly decreasing with limit 1.*

Remark 1.2. It follows from Theorem 1.2 that the sequence $(\sqrt[n]{D_n})_{n \geq 2}$ is strictly increasing.

For each $m \in \mathbb{Z}^+$ those $H_n^{(m)} = \sum_{k=1}^n 1/k^m$ ($n \in \mathbb{Z}^+$) are called harmonic numbers of order m . The usual harmonic numbers (of order 1) are those rational numbers $H_n = H_n^{(1)}$ ($n = 1, 2, 3, \dots$).

Our following theorem confirms Conjecture 2.16 of Sun [4].

Theorem 1.3. *For any $m \in \mathbb{Z}^+$, the sequence $(\sqrt[n+1]{H_{n+1}^{(m)}} / \sqrt[n]{H_n^{(m)}})_{n \geq 3}$ is strictly increasing.*

We will prove Theorems 1.1–1.3 in Sections 2–4 respectively. It seems that there is no simple form for the generating function $\sum_{n=0}^{\infty} \sqrt[n]{a_n} x^n$ with $a_n = u_n, D_n, H_n^{(m)}$. Note also that the set of those sequences $(a_n)_{n \geq 1}$ of positive numbers with $(\sqrt[n+1]{a_{n+1}} / \sqrt[n]{a_n})_{n \geq 1}$ decreasing (or increasing) is closed under multiplication.

2. Proof of Theorem 1.1

PROOF OF THEOREM 1.1. Set

$$\alpha = \frac{A + \sqrt{\Delta}}{2}, \quad \beta = \frac{A - \sqrt{\Delta}}{2}, \quad \text{and} \quad \gamma = \frac{\beta}{\alpha} = \frac{A - \sqrt{\Delta}}{A + \sqrt{\Delta}}.$$

Then

$$\log u_n = \log \frac{\alpha^n(1 - \gamma^n)}{\alpha - \beta} = n \log \alpha + \log(1 - \gamma^n) - \log \sqrt{\Delta}$$

for any $n \in \mathbb{Z}^+$. Note that

$$\log \frac{\sqrt[n+1]{u_{n+1}}}{\sqrt[n]{u_n}} = \frac{\log u_{n+1}}{n+1} - \frac{\log u_n}{n} = \frac{\log(1 - \gamma^{n+1})}{n+1} - \frac{\log(1 - \gamma^n)}{n} + \frac{\log \sqrt{\Delta}}{n(n+1)}.$$

Since

$$\lim_{n \rightarrow \infty} \frac{\log(1 - \gamma^n)}{n} = \lim_{n \rightarrow \infty} \frac{-\gamma^n}{n} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{n(n+1)} = 0,$$

we deduce that

$$\lim_{n \rightarrow \infty} \log \frac{\sqrt[n+1]{u_{n+1}}}{\sqrt[n]{u_n}} = 0, \quad \text{i.e.,} \quad \lim_{n \rightarrow \infty} \frac{\sqrt[n+1]{u_{n+1}}}{\sqrt[n]{u_n}} = 1.$$

For any $n \in \mathbb{Z}^+$, clearly

$$\begin{aligned} \frac{\sqrt[n+1]{u_{n+1}}}{\sqrt[n]{u_n}} &> \frac{\sqrt[n+2]{u_{n+2}}}{\sqrt[n+1]{u_{n+1}}} \iff \frac{\log u_{n+1}}{n+1} - \frac{\log u_n}{n} > \frac{\log u_{n+2}}{n+2} - \frac{\log u_{n+1}}{n+1} \\ &\iff \Delta_n := \frac{2 \log u_{n+1}}{n+1} - \frac{\log u_n}{n} - \frac{\log u_{n+2}}{n+2} > 0. \end{aligned}$$

Observe that

$$\begin{aligned} \Delta_n &= 2 \log \alpha + \frac{2 \log(1 - \gamma^{n+1})}{n+1} - \frac{2 \log \sqrt{\Delta}}{n+1} \\ &\quad - \left(2 \log \alpha + \frac{\log(1 - \gamma^n)}{n} + \frac{\log(1 - \gamma^{n+2})}{n+2} - \frac{\log \sqrt{\Delta}}{n} - \frac{\log \sqrt{\Delta}}{n+2} \right) \\ &= \frac{\log \Delta}{n(n+1)(n+2)} + \frac{2}{n+1} \log(1 - \gamma^{n+1}) - \frac{\log(1 - \gamma^n)}{n} - \frac{\log(1 - \gamma^{n+2})}{n+2}. \end{aligned}$$

The function $f(x) = \log(1+x)$ on the interval $(-1, +\infty)$ is concave since $f''(x) = -1/(x+1)^2 < 0$. Note that $|\gamma| < 1$. If $-|\gamma| \leq x \leq 0$, then $t = -x/|\gamma| \in [0, 1]$ and hence

$$f(x) = f(t(-|\gamma|) + (1-t)0) \geq t f(-|\gamma|) + (1-t)f(0) = qx,$$

where $q = -\log(1 - |\gamma|)/|\gamma| > 0$. Note also that $\log(1+x) < x$ for $x > 0$. So we have

$$\begin{aligned} \log(1 - \gamma^{n+1}) &\geq \log(1 - |\gamma|^{n+1}) \geq -q|\gamma|^{n+1}, \\ \log(1 - \gamma^n) &\leq \log(1 + |\gamma|^n) < |\gamma|^n, \\ \log(1 - \gamma^{n+2}) &\leq \log(1 + |\gamma|^{n+2}) < |\gamma|^{n+2}. \end{aligned}$$

Therefore

$$\Delta_n > \frac{\log \Delta}{n(n+1)(n+2)} - |\gamma|^n \left(\frac{2q|\gamma|}{n+1} + \frac{1}{n} + \frac{|\gamma|^2}{n+2} \right)$$

and hence

$$\begin{aligned} n(n+1)(n+2)\Delta_n &> \log \Delta - |\gamma|^n (2q|\gamma|n(n+2) + (n+1)(n+2) + |\gamma|^2 n(n+1)). \quad (1) \end{aligned}$$

Since $\lim_{n \rightarrow \infty} n^2 |\gamma|^n = 0$, when $\Delta > 1$ we have $\Delta_n > 0$ for large n .

Now it remains to consider the case $\Delta = 1$. Clearly $\gamma = (A-1)/(A+1) > 0$. Recall that $\log(1-x) < -x$ for $x \in (0, 1)$. As

$$\frac{d}{dx}(\log(1-x) + x + x^2) = -\frac{1}{1-x} + 1 + 2x = \frac{x(1-2x)}{1-x} > 0 \quad \text{for } x \in (0, 0.5),$$

we have $\log(1-x) + x + x^2 > \log 1 + 0 + 0^2 = 0$ for $x \in (0, 0.5)$. If n is large enough, then $\gamma^n < 0.5$ and hence

$$\Delta_n = \frac{2}{n+1} \log(1 - \gamma^{n+1}) - \frac{\log(1 - \gamma^n)}{n} - \frac{\log(1 - \gamma^{n+2})}{n+2} > w_n,$$

where

$$w_n := \frac{2}{n+1}(-\gamma^{n+1} - \gamma^{2n+2}) + \frac{\gamma^n}{n} + \frac{\gamma^{n+2}}{n+2}.$$

Note that

$$\lim_{n \rightarrow \infty} \frac{nw_n}{\gamma^n} = -2\gamma + 1 + \gamma^2 = (1 - \gamma)^2 > 0.$$

So, for sufficiently large n we have $\Delta_n > w_n > 0$.

Now we show that $n \geq 4$ suffices in the case $A = 1$ and $B = -1$. Note that $\Delta = 5$ and $\gamma \approx -0.382$. The sequence $(|\gamma|^n(n+1)(n+2))_{n \geq 1}$ is decreasing since

$$|\gamma| \frac{(n+2)(n+3)}{(n+1)(n+2)} < \frac{1}{2} \left(1 + \frac{2}{n+1} \right) \leq 1$$

for $n \geq 1$. It follows that $|\gamma|^n(n+1)(n+2) \leq \gamma^6 \times 7 \times 8 < 1/3$ for $n \geq 6$. In view of (1), if $n \geq 6$ then

$$\begin{aligned} n(n+1)(n+2)\Delta_n &> \log 5 - |\gamma|^n(n+1)(n+2) (2q|\gamma| + 1 + |\gamma|^2) \\ &> \log 5 - \frac{1 + 1 + \gamma^2}{3} > \log 5 - 1 > 0. \end{aligned}$$

It is easy to verify that Δ_4 and Δ_5 are positive. So $(\sqrt[n+1]{F_{n+1}} / \sqrt[n]{F_n})_{n \geq 4}$ is strictly decreasing.

In view of the above, we have completed the proof of Theorem 1.1. $\square$

3. Proof of Theorem 1.2

PROOF OF THEOREM 1.2. Let $n \geq 3$. It is well known that $|D_n - n!/e| \leq 1/2$ (cf. [1, p. 67]). Applying the Intermediate Value Theorem in calculus, we obtain

$$\left| \log D_n - \log \left(\frac{n!}{e} \right) \right| \leq \left| D_n - \frac{n!}{e} \right| \leq 0.5.$$

Set $R_0(n) = \log D_n - \log n!$. Then $|R_0(n)| \leq 1.5$.

Since $\lim_{n \rightarrow \infty} R_0(n)/n = 0$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{\log D_{n+1}}{n+1} - \frac{\log D_n}{n} \right) &= \lim_{n \rightarrow \infty} \left(\frac{\log((n+1)!)}{n+1} - \frac{\log(n!)}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{n \log(n+1) + n \log(n!) - (n+1) \log(n!)}{n(n+1)} \\ &= \lim_{n \rightarrow \infty} \frac{n \log n + n \log(1+1/n) - \log(n!)}{n(n+1)} \\ &= \lim_{n \rightarrow \infty} \frac{\log(n^n/n!)}{n(n+1)}. \end{aligned}$$

As $n! \sim \sqrt{2\pi n}(n/e)^n$ (i.e., $\lim_{n \rightarrow \infty} n!/(\sqrt{2\pi n}(n/e)^n) = 1$) by Stirling's formula, we have $\log(n^n/n!) \sim n$ and hence

$$\lim_{n \rightarrow \infty} \left(\frac{\log D_{n+1}}{n+1} - \frac{\log D_n}{n} \right) = 0.$$

Thus $\lim_{n \rightarrow \infty} \sqrt[n+1]{D_{n+1}}/\sqrt[n]{D_n} = 1$.

From the known identity $D_n/n! = \sum_{k=0}^n (-1)^k/k!$, we have the recurrence $D_n = nD_{n-1} + (-1)^n$ for $n > 1$. Thus, if $n \geq 3$ then

$$R_0(n) - R_0(n-1) = \log \frac{D_n}{n!} - \log \frac{D_{n-1}}{(n-1)!} = \log \frac{D_n}{nD_{n-1}} = \log \left(1 + \frac{(-1)^n}{nD_{n-1}} \right).$$

Fix $n \geq 4$. If n is even, then

$$0 < R_0(n) - R_0(n-1) = \log \left(1 + \frac{1}{nD_{n-1}} \right) < \frac{1}{nD_{n-1}} = \frac{1}{D_n - 1} \leq \frac{3}{D_n + 0.5}.$$

If n is odd, then

$$0 > R_0(n) - R_0(n-1) = \log \left(1 - \frac{1}{nD_{n-1}} \right) > \frac{-2}{nD_{n-1}} = \frac{-2}{D_n + 1} \geq \frac{-3}{D_n + 0.5}$$

since $\log(1-x) + 2x > 0$ for $x \in (0, 0.5)$. So

$$|R_0(n) - R_0(n-1)| < \frac{3}{D_n + 0.5} \leq \frac{3e}{n!}$$

and hence

$$\left| \frac{R_0(n-1) - R_0(n)}{n-1} \right| < \frac{3e}{n!(n-1)} \leq \frac{3e}{n(n-1)(n+1)}.$$

Similarly, we also have

$$\left| \frac{R_0(n+1) - R_0(n)}{n+1} \right| < \frac{3e}{n!(n+1)} \leq \frac{3e}{n(n-1)(n+1)}.$$

Therefore,

$$\begin{aligned} & \left| \frac{R_0(n+1)}{n+1} - \frac{2R_0(n)}{n} + \frac{R_0(n-1)}{n-1} - \frac{2R_0(n)}{n(n-1)(n+1)} \right| \\ &= \left| \frac{R_0(n+1) - R_0(n)}{n+1} + \frac{R_0(n-1) - R_0(n)}{n-1} \right| \leq \frac{6e}{n(n-1)(n+1)} \end{aligned}$$

and hence

$$\left| \frac{R_0(n+1)}{n+1} - \frac{2R_0(n)}{n} + \frac{R_0(n-1)}{n-1} \right| \leq \frac{2|R_0(n)| + 6e}{n(n-1)(n+1)} \leq \frac{6e+3}{n(n-1)(n+1)}.$$

Thus $|R_1(n)| \leq 6e+3$, where

$$R_1(n) := n(n-1)(n+1) \left(\frac{R_0(n+1)}{n+1} - \frac{2R_0(n)}{n} + \frac{R_0(n-1)}{n-1} \right).$$

Since

$$\begin{aligned} \log((n-1)!) &= \sum_{k=1}^{n-1} \int_k^{k+1} (\log k) dx < \sum_{k=1}^{n-1} \int_k^{k+1} \log x dx \\ &= \int_1^n \log x dx = n \log n - n + 1 < \sum_{k=1}^{n-1} \int_k^{k+1} (\log(k+1)) dx = \log(n!), \end{aligned}$$

we have

$$n \log n - n < \log(n!) = \log((n-1)!) + \log n < n \log n - n + \log n + 1$$

and so $\log(n!) = n \log n - n + R_2(n)$ with $|R_2(n)| < \log n + 1$.

Observe that

$$\begin{aligned} & \frac{\log D_{n+1}}{n+1} - \frac{2}{n} \log D_n + \frac{\log D_{n-1}}{n-1} \\ &= \frac{\log((n+1)!)}{n+1} - \frac{2 \log(n!)}{n} + \frac{\log((n-1)!)}{n-1} + \frac{R_1(n)}{(n-1)n(n+1)} \\ &= \frac{2 \log(n!)}{(n-1)n(n+1)} - \frac{\log n}{n-1} + \frac{\log(n+1)}{n+1} + \frac{R_1(n)}{(n-1)n(n+1)} \end{aligned}$$

$$\begin{aligned}
&= -\frac{2n}{(n-1)n(n+1)} + \frac{\log(n+1) - \log n}{n+1} + \frac{2R_2(n) + R_1(n)}{(n-1)n(n+1)} \\
&\leq -\frac{2n}{(n-1)n(n+1)} + \frac{n-1}{(n-1)n(n+1)} + \frac{2R_2(n) + R_1(n)}{(n-1)n(n+1)} \\
&= -\frac{n+1 - 2R_2(n) - R_1(n)}{(n-1)n(n+1)}.
\end{aligned}$$

If $n \geq 27$, then $n+1 - 2R_2(n) - R_1(n) > n - 2\log n - 1 - 6e - 3 > 0$, and hence we get

$$\log \frac{\sqrt[n]{D_n}}{\sqrt[n-1]{D_{n-1}}} > \log \frac{\sqrt[n+1]{D_{n+1}}}{\sqrt[n]{D_n}}.$$

By a direct check via computer, the last inequality also holds for $n = 4, \dots, 26$. Therefore, the sequence $(\sqrt[n+1]{D_{n+1}}/\sqrt[n]{D_n})_{n \geq 3}$ is strictly decreasing. This ends the proof. $\square$

4. Proof of Theorem 1.3

Lemma 4.1. *For $x > 0$ we have*

$$\log(1+x) > x - \frac{x^2}{2}. \quad (2)$$

PROOF. As

$$\frac{d}{dx} \left(\log(1+x) - x + \frac{x^2}{2} \right) = \frac{x^2}{1+x},$$

we see that $\log(1+x) - x + x^2/2 > \log 1 - 0 + 0^2/2 = 0$ for any $x > 0$. $\square$

Lemma 4.2. *Let $m, n \in \mathbb{Z}^+$ with $n \geq 3$. If $m \geq 11$ or $n \geq 30$, then*

$$H_n^{(m)} \log H_n^{(m)} > 4 \left(\frac{2}{n+2} \right)^{m-1}. \quad (3)$$

PROOF. Recall that H_n refers to $H_n^{(1)}$. If $n \geq 30$, then

$$H_n \log H_n \geq H_{30} \log H_{30} > 4,$$

and hence (3) holds for $m = 1$.

Below we assume that $m \geq 2$. As $n \geq 3$, we have

$$H_n^{(m)} \log H_n^{(m)} \geq H_3^{(m)} \log H_3^{(m)}.$$

So it suffices to show that

$$\left(\frac{n+2}{2}\right)^{m-1} H_3^{(m)} \log H_3^{(m)} > 4 \quad (4)$$

whenever $m \geq 11$ or $n \geq 30$. By Lemma 4.1,

$$\begin{aligned} \log H_3^{(m)} &= \log(1 + 2^{-m} + 3^{-m}) > 2^{-m} + 3^{-m} - \frac{(2^{-m} + 3^{-m})^2}{2} \\ &> 2^{-m} + 3^{-m} - \frac{(2^{1-m})^2}{2} = \frac{1}{2^m} + \frac{1}{3^m} - \frac{2}{4^m}. \end{aligned}$$

If $m \geq 3$, then $(4/3)^m \geq (4/3)^3 > 2$ and hence $\log H_3^{(m)} > 1/2^m$. Note also that $H_3^{(2)} \log H_3^{(2)} > 1/4$. So we always have

$$H_3^{(m)} \log H_3^{(m)} > \frac{1}{2^m}.$$

If $m \geq 11$, then $1.25^m \geq 1.25^{11} > 10$ and hence

$$\frac{1}{2^m} > \frac{4}{2.5^{m-1}} \geq \frac{4}{((n+2)/2)^{m-1}},$$

therefore (4) holds. When $n \geq 30$, we have

$$\frac{1}{2^m} \geq \frac{1}{2^{4m-6}} = \frac{4}{16^{m-1}} \geq 4 \left(\frac{2}{n+2}\right)^{m-1}$$

and hence (4) also holds. $\square$

PROOF OF THEOREM 1.3. Let $m \geq 1$ and $n \geq 3$. Set

$$\Delta_n(m) := \log \frac{\sqrt[n+1]{H_{n+1}^{(m)}}}{\sqrt[n]{H_n^{(m)}}} - \log \frac{\sqrt[n+2]{H_{n+2}^{(m)}}}{\sqrt[n+1]{H_{n+1}^{(m)}}} = \frac{2 \log H_{n+1}^{(m)}}{n+1} - \frac{\log H_n^{(m)}}{n} - \frac{\log H_{n+2}^{(m)}}{n+2}.$$

It suffices to show that $\Delta_n(m) < 0$. This can be easily verified by computer if $m \in \{1, \dots, 10\}$ and $n \in \{3, \dots, 29\}$.

Below we assume that $m \geq 11$ or $n \geq 30$. Recall (2) and the known fact that $\log(1+x) < x$ for $x > 0$. We clearly have

$$\log \frac{H_{n+1}^{(m)}}{H_n^{(m)}} = \log \left(1 + \frac{1}{(n+1)^m H_n^{(m)}} \right) < \frac{1}{(n+1)^m H_n^{(m)}}$$

and

$$\log \frac{H_{n+2}^{(m)}}{H_n^{(m)}} > \log \left(1 + \frac{2}{(n+2)^m H_n^{(m)}} \right) > \frac{2}{(n+2)^m H_n^{(m)}} - \frac{2}{(n+2)^{2m} (H_n^{(m)})^2}.$$

It follows that

$$\begin{aligned} \Delta_n(m) &= \left(\frac{2}{n+1} - \frac{1}{n} - \frac{1}{n+2} \right) \log H_n^{(m)} + \frac{2}{n+1} \log \frac{H_{n+1}^{(m)}}{H_n^{(m)}} - \frac{1}{n+2} \log \frac{H_{n+2}^{(m)}}{H_n^{(m)}} \\ &< \frac{-2 \log H_n^{(m)}}{n(n+1)(n+2)} + \frac{2}{(n+1)^{m+1} H_n^{(m)}} \\ &\quad - \frac{2}{(n+2)^{m+1} H_n^{(m)}} + \frac{2}{(n+2)^{2m+1} (H_n^{(m)})^2}. \end{aligned}$$

Since $(n+2)^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} (n+1)^k$ by the binomial theorem, we obtain

$$\begin{aligned} \Delta_n(m) &\leq \frac{-2 \log H_n^{(m)}}{n(n+1)(n+2)} + \frac{2 \sum_{k=0}^m \binom{m+1}{k} (n+1)^k}{(n+1)^{m+1} (n+2)^{m+1} H_n^{(m)}} + \frac{2}{(n+2)^{m+2} H_n^{(m)}} \\ &< \frac{-2 \log H_n^{(m)}}{n(n+1)(n+2)} + \frac{2(n+1)^m \sum_{k=0}^m \binom{m+1}{k}}{(n+1)^{m+1} (n+2)^{m+1} H_n^{(m)}} + \frac{2}{(n+1)(n+2)^{m+1} H_n^{(m)}} \\ &= \frac{-2 \log H_n^{(m)}}{n(n+1)(n+2)} + \frac{2(2^{m+1} - 1) + 2}{(n+1)(n+2)^{m+1} H_n^{(m)}}. \end{aligned}$$

Thus

$$\begin{aligned} n(n+1)(n+2) \Delta_n(m) \frac{H_n^{(m)}}{2} &< -H_n^{(m)} \log H_n^{(m)} + \frac{2^{m+1} n}{(n+2)^m} \\ &< 4 \left(\frac{2}{n+2} \right)^{m-1} - H_n^{(m)} \log H_n^{(m)}. \end{aligned}$$

Applying (3) we find that $\Delta_n(m) < 0$. This completes the proof. $\square$

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References

- [1] R. P. STANLEY, Enumerative Combinatorics, Vol. 1, *Cambridge Univ. Press, Cambridge*, 1997.
- [2] Z.-W. SUN, Reduction of unknowns in diophantine representations, *Sci. China Ser. A* **35** (1992), 257–269.
- [3] Z.-W. SUN, On a sequence involving sums of primes, *Bull. Austral. Math. Soc.* **88** (2013), 197–205.
- [4] Z.-W. SUN, Conjectures involving arithmetical sequences, in: Number Theory: Arithmetic in Shangri-La, Proc. 6th China–Japan Seminar (Shanghai, August 15–17, 2011), (S. Kanemitsu, H. Li and J. Liu, eds.), *World Sci., Singapore*, 2013, 244–258.

QING-HU HOU
CENTER FOR APPLIED MATHEMATICS
TIANJIN UNIVERSITY
TIANJIN 300072
PEOPLE'S REPUBLIC OF CHINA
E-mail: qinghu.hou@gmail.com

ZHI-WEI SUN
DEPARTMENT OF MATHEMATICS
NANJING UNIVERSITY
NANJING 210093
PEOPLE'S REPUBLIC OF CHINA
E-mail: zwsun@nju.edu.cn

HAOMIN WEN
DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF PENNSYLVANIA
PHILADELPHIA, PA 19104
USA
E-mail: weh@math.upenn.edu

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