

## On a graph of a $p$ -solvable normal subgroup

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**Abstract.** Let  $N$  be a  $p$ -solvable normal subgroup of a group  $G$ . In this paper, we prove that  $N$  is solvable if  $\alpha > \beta > 1$  are the two maximal sizes in  $cs_G(N_{p'})$  such that  $(\alpha, \beta) = 1$  and  $\beta$  is a  $p'$ -number dividing  $|N/(N \cap Z(G))|$ . Moreover, the structure of  $N$  is given.

### 1. Introduction

All groups considered in this paper are finite. Let  $G$  be a group and  $x$  an element in  $G$ , we use  $x^G$  to denote the conjugacy class of  $G$  containing  $x$  and use  $|x^G|$  to denote the size of  $x^G$ . Also we write  $\text{Con}(G) = \{x^G \mid x \in G\}$  and  $cs(G) = \{|B| \mid B \in \text{Con}(G)\}$ . Furthermore, if  $H$  is a subset of  $G$ , we write  $\text{Con}_G(H) = \{x^G \mid x \in H\}$  and  $cs_G(H) = \{|B| \mid B \in \text{Con}_G(H)\}$ . If  $B$  is a non-empty subset of a group  $G$ , following [1], we set  $K_S = \{x \in G \mid xS = S\}$ . Clearly,  $K_S$  is a subgroup of  $G$ . Furthermore,  $K_S$  is a normal subgroup of  $G$  if  $S$  is a normal subset of  $G$ . Since  $S$  is the union of right cosets of  $K_S$ , we see that  $|K_S|$  divides  $|S|$ .

In 1904, BURNSIDE proved that a group  $G$  is not simple if some conjugacy class size of  $G$  is a prime power, see [4] for instance. Since then, many authors began to study how the set of conjugacy class sizes may determine the properties of a group, say solvability, non-simplicity and so on (see, e.g., [5], [6], [7], [9]).

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Let  $N$  be a normal subgroup of a group  $G$ . Then  $N$  is a union of several  $G$ -conjugacy classes contained in  $N$ . Therefore, the investigation into the relation between the structure of  $N$  and the  $G$ -conjugacy class sizes contained in it has attracted interests of many authors (see, e.g., [8], [11], [12]).

Recall that a group  $G$  is called a quasi-Frobenius group if  $G/Z(G)$  is a Frobenius group. The inverse images in  $G$  of the kernel and a complement of  $G/Z(G)$  are called the kernel and a complement of  $G$ . And a group  $G$  is said to be  $p$ -nilpotent or  $p$ -closed if it has a normal  $p$ -complement or a normal Sylow  $p$ -subgroup respectively.

If  $N$  is a  $p$ -solvable normal subgroup of a group  $G$ , we use  $N_{p'}$  to denote the set of all  $p'$ -elements in  $N$ . In this paper, we are interested in the relationship between the set  $cs_G(N_{p'})$  and the property of  $N$ . We have the following theorem.

**Theorem.** *Let  $N$  be a  $p$ -solvable normal subgroup of a group  $G$ . Suppose that  $\alpha > \beta > 1$  are the two maximal sizes in  $cs_G(N_{p'})$  with  $(\alpha, \beta) = 1$ . If  $\beta$  is a  $p'$ -number and  $\beta$  divides  $|N/(N \cap Z(G))|$ , then  $N$  is solvable. Furthermore, if a  $p$ -complement  $K$  of  $N$  is not nilpotent, then  $K$  is quasi-Frobenius with abelian kernel and complements and at least one of the following conditions is satisfied:*

- (1)  $N$  is  $p$ -nilpotent;
- (2)  $N$  is  $p$ -closed;
- (3)  $K = R \rtimes T$  with  $R$  an abelian normal Sylow  $r$ -subgroup of  $N$  and  $T$  an abelian  $\{p, r\}$ -complement of  $N$ .

In the proof of this theorem, we use the graphs  $\Gamma(G)$  of a group  $G$  and  $\Gamma_p(G)$  of a  $p$ -solvable group  $G$ , whose vertices are the non-central  $G$ -conjugacy classes of elements and  $p'$ -elements in  $G$  respectively and two vertices are joined by an edge if their cardinalities have a common primary divisor. Furthermore, we use  $n(\Gamma(G))$  and  $n(\Gamma_p(G))$  to denote the number of components of  $\Gamma(G)$  and  $\Gamma_p(G)$  respectively. Several authors have obtained interesting results about  $\Gamma(G)$  and  $\Gamma_p(G)$  (see, e.g., [1], [2], [3]).

## 2. Preliminaries

For a positive integer  $m$ , we write  $\pi(m) = \{p \mid p \text{ is a primary divisor of } m\}$ , and for a non-empty set  $K$ , we use  $\pi(K)$  to denote the set of primary divisors of  $|K|$ .

Inspired by the works of [3] and [10], we have the following generalized results. For completeness, we provide the corresponding proofs.

**Lemma 2.1.** *Let  $N$  be a  $p$ -solvable normal subgroup of a group  $G$ . Suppose that  $B = b^G$ ,  $C = c^G \in \text{Con}_G(N_{p'})$  such that  $(|B|, |C|) = 1$ . Then*

- (a)  $G = C_G(b)C_G(c)$ ;
- (b)  $BC \in \text{Con}_G(N_{p'})$  and  $|BC|$  divides  $|B||C|$ .

PROOF. (a) This is obvious since  $(|G : C_G(b)|, |G : C_G(c)|) = (|b^G|, |c^G|) = 1$ .

(b) We first claim that  $BC \in \text{Con}(G)$ . It suffices to prove that  $b^g c^h$  is conjugate to  $bc$  for any  $g, h \in G$ . Since  $gh^{-1} \in G = C_G(b)C_G(c)$ , there exists  $x \in C_G(b)$  and  $y \in C_G(c)$  such that  $gh^{-1} = x^{-1}y$ . Then  $xg = yh$  and  $b^g c^h = b^{xg} c^{yh} = (bc)^{xg}$ . Now, it suffices to prove that there is a  $p'$ -element in  $BC$ . In fact, let  $H$  be a  $p$ -complement of  $N$ . Then there exists  $g, h \in G$  such that  $b^g \in H$  and  $c^h \in H$ . Therefore,  $b^g c^h$  is a  $p'$ -element in  $N$  and  $b^g c^h \in BC$ .

Since  $C_G(b) \cap C_G(c) \leq C_G(bc)$ , we have that  $|BC| = |G : C_G(bc)|$  divides  $|G : C_G(b) \cap C_G(c)| = |G : C_G(b)| |G : C_G(c)| = |B| |C|$ .  $\square$

**Lemma 2.2.** *Let  $N$  be a  $p$ -solvable normal subgroup of a group  $G$ . Then the following two properties hold:*

- (a) *Suppose that  $B_0 \in \text{Con}_G(N_{p'})$  such that  $|B_0|$  is the maximal size in  $cs_G(N_{p'})$ . If  $C \in \text{Con}_G(N_{p'})$  such that  $(|B_0|, |C|) = 1$ , then  $C^{-1}CB_0 = B_0$  and  $|\langle C^{-1}C \rangle|$  divides  $|B_0|$ .*
- (b) *Suppose that  $m, n$  are the two maximal sizes in  $cs_G(N_{p'})$  such that  $m > n > 1$  and  $(m, n) = 1$ . If  $D$  is a non-central class in  $\text{Con}_G(N_{p'})$  with  $(|D|, n) = 1$ , then  $|D| = m$ .*

PROOF. (a) Lemma 2.1 (b) implies that  $CB_0 \in \text{Con}_G(N_{p'})$ , and it is obvious that  $|CB_0| \geq |B_0|$ . By the hypothesis, we see that  $|CB_0| = |B_0|$ . It follows that  $C^{-1}CB_0 = B_0$ , and thus  $\langle C^{-1}C \rangle B_0 = B_0$ . So,  $\langle C^{-1}C \rangle \leq K_B$ , which yields that  $|\langle C^{-1}C \rangle|$  divides  $|B_0|$ .

(b) Let  $A, B \in \text{Con}_G(N_{p'})$  such that  $|A| = n$  and  $|B| = m$ . Then  $DA \in \text{Con}_G(N_{p'})$  by Lemma 2.1 (b). Since  $|DA| \geq |A|$ ,  $|DA| = m$  or  $n$ . If  $|DA| = n$ , then  $D^{-1}DA \in \text{Con}_G(N_{p'})$  and thus  $D^{-1}DA = A$ , in which case we have  $\langle D^{-1}D \rangle A = A$ . Therefore,  $|\langle D^{-1}D \rangle|$  divides  $|A|$ . On the other hand, we see that  $\langle D^{-1}D \rangle \subseteq \langle AA^{-1} \rangle$ , so  $|\langle D^{-1}D \rangle|$  divides  $|\langle AA^{-1} \rangle|$ . By (a) of this lemma,  $|\langle AA^{-1} \rangle|$  divides  $|B|$ , so  $|\langle D^{-1}D \rangle|$  divides  $|B|$ . Therefore,  $|\langle D^{-1}D \rangle|$  divides  $(n, m) = 1$ , a contradiction. So  $|DA| = m$ . We conclude that  $|D| = m$  since  $|DA|$  divides  $|D||A|$ .  $\square$

**Lemma 2.3.** *Suppose that  $N$  is a  $p$ -solvable normal subgroup of a group  $G$  and  $B_0 \in \text{Con}_G(N_{p'})$  such that  $|B_0|$  is the maximal size in  $cs_G(N_{p'})$ . Let*

$$M = \langle D \mid D \in \text{Con}_G(N_{p'}) \text{ such that } (|D|, |B_0|) = 1 \rangle.$$

*Then  $M$  is an abelian  $p'$ -subgroup of  $N$ . Furthermore, if  $M \not\leq Z(G)$ , then  $\pi(M/(Z(G) \cap M)) \subseteq \pi(B_0)$  and in this case,  $|B_0|$  is not a  $p$ -number.*

PROOF. Let

$$K = \langle D^{-1}D \mid D \in \text{Con}_G(N_{p'}) \text{ such that } (|D|, |B_0|) = 1 \rangle.$$

It is easy to see that  $M$  and  $K$  are normal subgroups of  $G$  and  $K = [M, G]$ .

On the other hand, if  $C \in \text{Con}_G(N_{p'})$  such that  $(|C|, |B_0|) = 1$ , then  $C^{-1}CB_0 = B_0$  by Lemma 2.2 (a), and thus  $KB_0 = B_0$ . Therefore,  $|K|$  divides  $|B_0|$ , in particular,  $\pi(K) \subseteq \pi(B_0)$ . So  $(|K|, |C|) = 1$ . Suppose that  $C = c^G$ . Since  $|K : C_K(c)|$  divides  $(|K|, |C|)$ , we have  $K = C_K(c)$ , and thus  $K \leq Z(M)$ . Therefore,  $M$  is nilpotent since  $M/K \leq Z(G/K)$ . We conclude that  $M$  is a  $p'$ -group since all of its generators are  $p'$ -elements.

Suppose that  $M \not\leq Z(G)$ . Let  $r \in \pi(M/(Z(G) \cap M))$  and  $R \in \text{Syl}_r(M)$ . Then  $R \trianglelefteq G$  and  $1 \neq [R, G] \leq [M, G] = K$ . Therefore,  $r \in \pi(K) \subseteq \pi(B_0)$ , and thus  $\pi(M/(Z(G) \cap M)) \subseteq \pi(B_0)$ .

If  $R$  is a Sylow  $r$ -subgroup of  $M$ , we can assume that  $r \in \pi(M/(Z(G) \cap M))$ . For any generating class  $d^G$  of  $M$ , we have that  $|d^R|$  divides  $(|R|, |d^G|) = 1$ . Therefore,  $R = C_R(d)$ , which implies that  $R \leq Z(M)$ . Consequently, we conclude that  $M$  is abelian.  $\square$

### 3. Main Results

In this section, we give the proof of our main result.

**Theorem.** *Let  $N$  be a  $p$ -solvable normal subgroup of a group  $G$ . Suppose that  $\alpha > \beta > 1$  are the two maximal sizes in  $cs_G(N_{p'})$  with  $(\alpha, \beta) = 1$ . If  $\beta$  is a  $p'$ -number and  $\beta$  divides  $|N/(N \cap Z(G))|$ , then  $N$  is solvable. Furthermore, if a  $p$ -complement  $K$  of  $N$  is not nilpotent, then  $K$  is quasi-Frobenius with abelian kernel and complements and at least one of the following conditions is satisfied:*

- (1)  $N$  is  $p$ -nilpotent;
- (2)  $N$  is  $p$ -closed;
- (3)  $K = R \rtimes T$  with  $R$  an abelian normal Sylow  $r$ -subgroup of  $N$  and  $T$  an abelian  $\{p, r\}$ -complement of  $N$ .

PROOF. Let  $K$  and  $P$  be a  $p$ -complement and a Sylow  $p$ -subgroup of  $N$  respectively. If  $K$  is nilpotent, then  $N$  is solvable since it is a product of two nilpotent groups. We only need to consider the case that  $K$  is not nilpotent. Then  $N = PK$  and we can write  $K = U \times W$ , where  $W$  is the maximal Hall subgroup of  $K$  contained in  $Z(G)$ . Then  $N_1 = PU$  is a normal subgroup of  $G$  and no Sylow subgroup of  $U$  is contained in  $Z(G)$ . It is easy to see that  $U/Z(U) \cong K/Z(K)$  and that the theorem is true for  $N$  if and only if it is true for  $N_1$ . So, without loss of generality, we will assume that no Sylow subgroup of  $N$  with order prime to  $p$  is contained in  $Z(G)$ .

Since  $K$  is not nilpotent by the assumption, we have  $|\pi(K)| \geq 2$ .

Let

$$M = \langle D \mid D \in \text{Con}_G(N_{p'}) \text{ such that } (|D|, \alpha) = 1 \rangle.$$

Then  $M$  is an abelian  $p'$ -subgroup of  $N$  by Lemma 2.3, and  $M \leq G$ .

Let  $a$  and  $b$  be  $p'$ -elements in  $N$  such that  $|a^G| = \beta$  and  $|b^G| = \alpha$ . We will prove this theorem by the following steps.

*Step 1.*  $C_G(b)$  is maximal and minimal among centralizers of all non-central  $p'$ -elements in  $N$ . In particular,  $b$  can be assumed to be a  $q$ -element for some prime  $q \neq p$  and  $C_N(b) = P_1 Q \times L$ , with  $P_1 \in \text{Syl}_p(C_N(b))$ ,  $Q \in \text{Syl}_q(C_N(b))$  and  $L \leq Z(C_G(b))$ .

Suppose that  $x$  and  $y$  are non-central  $p'$ -elements in  $N$  such that  $C_G(b) \leq C_G(x)$  and  $C_G(y) \leq C_G(b)$ . Then  $|x^G|$  divides  $|b^G| = \alpha$ . Therefore,  $|x^G| = \alpha$  by Lemma 2.2 (b), and thus  $C_G(b) = C_G(x)$ . On the other hand,  $\alpha = |b^G|$  divides  $|y^G|$ , so  $|y^G| = \alpha$  and  $C_G(b) = C_G(y)$  by the hypothesis of this theorem.

Now, write  $b = b_1 b_2 \cdots b_s$  with each  $b_i$  an element of primary order and  $b_i b_j = b_j b_i$  for all  $i$  and  $j$ . We may assume that  $b_1 \notin Z(G)$  and  $b_1$  can be assumed to be a  $q$ -element for a prime  $q \neq p$ . It is obvious that  $C_G(b) \leq C_G(b_1)$ , so  $C_G(b) = C_G(b_1)$  by the above paragraph. By replacing  $b$  with  $b_1$  we can assume that  $b$  is a  $q$ -element.

For every  $\{p, q\}'$ -element  $x$  in  $C_N(b)$ , we have  $C_G(bx) = C_G(b) \cap C_G(x) \leq C_G(b)$ . Therefore,  $C_G(bx) = C_G(b)$  by the first paragraph. It follows that  $C_G(b) \leq C_G(x)$ , and thus  $x \in Z(C_G(b))$ .

*Step 2.*  $q \nmid \alpha$ .

Suppose that  $q \mid \alpha$ . Then  $q \nmid \beta$  and thus  $q \nmid |a^N| = |N : C_N(a)| = |C_N(b) : C_N(a) \cap C_N(b)|$ . Therefore, a Sylow  $q$ -subgroup of  $C_N(b)$  is contained in  $C_N(a) \cap C_N(b)$ . Without loss of generality, we may assume that  $Q \leq C_N(a)$ .

Since  $a \in C_N(b)$  and  $L \leq Z(C_G(b))$ , we have  $L \leq C_N(a)$ , and thus  $Q \times L \leq C_N(a)$ . Therefore,  $|a^N|_{p'}$  divides  $|b^N|_{p'}$ . Since  $\beta$  is a  $p'$ -number, we have  $|a^N| = 1$ ,

that is  $a \in Z(N)$ . Write  $a = a_q \cdot a_{q'}$  with  $a_q$  and  $a_{q'}$  the  $q$ -component and  $q'$ -component of  $a$  respectively. If  $a_{q'} \notin Z(G)$ , then  $C_G(ba_{q'}) = C_G(b) \cap C_G(a_{q'})$ . Therefore,  $C_G(b) = C_G(ba_{q'}) \leq C_G(a_{q'})$ . By Step 1, we have  $|a_{q'}^G| = \alpha$ , and thus  $|a^G| = \alpha$ , which is a contradiction. Therefore, we may assume  $a$  to be a  $q$ -element. For any  $\{p, q\}'$ -element  $x \in N = C_N(a)$ , we have  $C_G(ax) = C_G(a) \cap C_G(x) \leq C_G(a)$ , then  $C_G(ax) = C_G(a) \leq C_G(x)$  by the hypothesis, and thus  $x \in Z(C_G(a))$ . Since  $b \in C_G(a)$ , we have  $x \in C_N(b)$  and thus  $x \in L \leq Z(C_G(b))$ . If  $x \notin Z(G)$ , then  $C_G(x) = C_G(b)$  by Step 1. The fact that  $C_G(ax) = C_G(a) \cap C_G(x)$  gives the contradiction that  $\alpha\beta$  divides  $|(ax)^G|$ . So,  $x \in Z(G)$  for all  $\{p, q\}'$ -elements in  $N$ , which contradicts to the assumption of the beginning of the proof.

*Step 3.*  $K_B \cap C_G(b) = \{1\}$ , where  $B = b^G$ . In particular,  $a$  can be assumed to be a  $q'$ -element.

By Step 1 we can assume that  $|b| = q^k$  for some positive integer  $k$ .

Let  $x \in K_B \cap C_G(b)$ . Then  $xb \in B$ , and so  $xb = b^g$  for some  $g \in G$ . As  $x \in C_G(b)$ , we have

$$x^{q^k} = x^{q^k} b^{q^k} = (xb)^{q^k} = (b^g)^{q^k} = (b^{q^k})^g = 1.$$

On the other hand, since  $|K_B|$  divides  $|B| = \alpha$ , we see that  $q \nmid |K_B|$  by Step 2. Therefore,  $x = 1$ .

Now, let  $a_1 = a^s$  be the  $q$ -component of  $a$ . Then  $a_1 \in C_G(b)^w$  for some  $w \in G$ . For every  $g \in G$ . Since  $G = C_G(a)C_G(b)^w$ , we can write  $g = uv$  with  $u \in C_G(a)$  and  $v \in C_G(b)^w$ . By Lemma 2.2 (a) and Lemma 2.3,  $\langle a^G \rangle$  is a normal abelian subgroup of  $G$  and  $\langle A^{-1}A \rangle \leq K_B$ , where  $A = a^G$ . Therefore,

$$[a_1, g] = [a^s, g] = [a, g]^s = (a^{-1}a^g)^s \in K_B.$$

Also,

$$[a_1, g] = [a_1, uv] = [a_1, v] \in C_G(b)^w.$$

Therefore,

$$[a_1, g] \in K_B \cap C_G(b)^w = (K_B \cap C_G(b))^w = \{1\}.$$

So,  $a_1 \in Z(G)$  and by replacing  $a$  with  $aa_1^{-1}$  we can assume that  $a$  is a  $q'$ -element.

*Step 4.*  $(C_N(a) \cap C_N(b))_{p'} = Z(N)_{p'} = Z(G)_{p'} \cap N$ .

Let  $x$  be a  $p'$ -element in  $C_N(a) \cap C_N(b)$  and write  $x = x_q \cdot x_{q'}$  with  $x_q$  and  $x_{q'}$  the  $q$ -component and  $q'$ -component of  $x$  respectively. If  $x_{q'} \notin Z(G)$ , then  $C_G(b) \leq C_G(x_{q'})$  by Step 1, and thus  $C_G(b) = C_G(x_{q'})$  again by Step 1. It follows that  $a \in C_G(b)$ , and so  $C_G(b) \leq C_G(a)$ , which is a contradiction. Since  $x_q \in C_G(a)$ , we have  $C_G(ax_q) = C_G(a) \cap C_G(x_q) \leq C_G(a)$ . Then the hypothesis implies that  $C_G(ax_q) = C_G(a) \leq C_G(x_q)$ . If  $x_q \notin Z(G)$ , then  $x_q \in M$ , and thus

$q \in \pi(M/M \cap Z(G)) \subseteq \pi(\alpha)$ , which contradicts to Step 2. So  $(C_N(a) \cap C_N(b))_{p'} \leq Z(G)$ .

Now the equalities are obvious since  $Z(N)_{p'} \leq (C_N(a) \cap C_N(b))_{p'} \leq Z(G)_{p'} \cap N \leq Z(N)_{p'}$ .

*Step 5.*  $\pi(a^N) = \pi(\beta)$  and  $\pi(b^N) = \pi(\alpha)$ .

It is well known that  $\pi(a^N) \subseteq \pi(\beta)$ . On the other hand, if there exists a prime  $r \in \pi(\beta) \setminus \pi(a^N)$ , then  $r \notin \pi(\alpha)$ , and thus  $r \notin \pi(b^N)$ . By replacing  $b$  with a suitable conjugation, we can assume that a Sylow  $r$ -subgroup of  $N$ , say  $R$ , is contained in both  $C_N(a)$  and  $C_N(b)$ , and thus in  $C_N(a) \cap C_N(b)$ . Therefore,  $R \leq (C_N(a) \cap C_N(b))_{p'} \leq Z(G)$  by Step 4, which contradicts to the beginning of the proof. Hence,  $\pi(a^N) = \pi(\beta)$ . Similarly we have  $\pi(b^N) = \pi(\alpha)$ .

*Step 6.*  $n(\Gamma_p(N)) = 2$ .

For any non-central  $p'$ -element  $x$  in  $N$ , we can write  $x = x_q \cdot x_{q'}$ . Since  $q \nmid |b^N|$  by Step 2, we can assume that  $x \in Q$ . We will prove that  $|x^G| = \alpha$  when  $x_q \notin Z(G)$  and  $|x^G| = \beta$  when  $x_q \in Z(G)$ .

First suppose that  $x_q \notin Z(G)$ . If  $|x_q^G| = \alpha$ , then clearly  $|x^G| = \alpha$ . If  $|x_q^G| = \beta$ , then  $x_q \in M$  by the definition of  $M$  and thus  $q \in \pi(\alpha)$ , which is a contradiction. We next show that  $|x_q^G| < \beta$  may not happen. If  $L \not\leq Z(G)$ , then we choose  $z$  a non-central  $\{p, q\}'$ -element in  $C_N(b)$ . Then  $C_G(b) = C_G(z)$  by Step 1. Now  $x_q \in C_N(b) = C_N(z)$ , it follows that  $C_G(zx_q) = C_G(z) \cap C_G(x_q) \leq C_G(z)$ . Therefore,  $\alpha = |z^G|$  divides  $|(zx_q)^G|$ , which gives that  $C_G(zx_q) = C_G(z) = C_G(b) \leq C_G(x_q)$ . Again by Step 1, we have  $|x_q^G| = \alpha$ , which contradicts to our assumption. Therefore, in this case  $L \leq Z(G)$ . It follows that  $|a^N| = |N : C_N(a)| = |C_N(b) : C_N(a) \cap C_N(b)| = |PQL : (C_N(a) \cap C_N(b))_p Z(N)_{p'}|$  is a  $q$ -number. And so  $\beta$  is a  $q$ -number by Step 5. Since  $\langle x_q \rangle$  acts coprimely on the abelian group  $M_{q'}$ ,  $M_{q'} = [M_{q'}, \langle x_q \rangle] \times C_{M_{q'}}(x_q)$ . As  $a \in M_{q'}$ , we may write  $a = uw$  with  $u \in [M_{q'}, \langle x_q \rangle]$  and  $w \in C_{M_{q'}}(x_q)$ . Let  $g = wx_q$ . Then  $C_G(g) = C_G(w) \cap C_G(x_q) \leq C_G(x_q)$  and so  $|x_q^G|$  divides  $|g^G|$ . It is easy to see that  $|g^G| \neq \alpha, \beta$ . Hence  $|g^G| < \beta$ . Choose  $Q_0$  to be a Sylow  $q$ -subgroup of  $G$  such that  $Q \leq Q_0$ . It follows that  $M_{q'}Q_0 \leq G$  and

$$|M_{q'}Q_0 : C_{M_{q'}Q_0}(g)| \leq |g^G| < \beta = |Q_0| : |C_G(a)|_q.$$

Moreover, we have

$$\begin{aligned} C_{M_{q'}Q_0}(g) &= C_{M_{q'}Q_0}(w) \cap C_{M_{q'}Q_0}(x_q) = M_{q'}C_{Q_0}(w) \cap C_{M_{q'}}(x_q)C_{Q_0}(x_q) \\ &= C_{M_{q'}}(x_q)(C_{Q_0}(w) \cap C_{Q_0}(x_q)). \end{aligned}$$

Set  $D = C_{Q_0}(w) \cap C_{Q_0}(x_q)$ . Then

$$\frac{|Q_0|}{|C_G(a)|_q} > \frac{|M_{q'}||Q_0|}{|C_{M_{q'}}(x_q)||D|},$$

which implies that  $|D| : |C_G(a)|_q > |M_{q'}| : |C_{M_{q'}}(x_q)| = |[M_{q'}, \langle x_q \rangle]|$ . On the other hand, since  $D \leq C_G(x_q)$  and  $M_{q'} \leq G$ ,  $D$  acts on  $[M_{q'}, \langle x_q \rangle]$  by conjugation. Noticing that

$$C_D(u) = C_G(u) \cap D = C_{Q_0}(u) \cap C_{Q_0}(w) \cap C_{Q_0}(x_q) = C_{Q_0}(a) \cap C_{Q_0}(x_q),$$

we have

$$|C_D(u)| = |C_{Q_0}(a) \cap C_{Q_0}(x_q)| \leq |C_G(a)|_q.$$

Therefore,

$$|u^D| = |D| : |C_D(u)| \geq |D| : |C_G(a)|_q > |[M_{q'}, \langle x_q \rangle]|,$$

which is a contradiction. Therefore, if  $x_q \notin Z(G)$ , then  $|x^G| = \alpha$ .

Now, suppose that  $x_q \in Z(G)$ . Then  $x_{q'} \notin Z(G)$ . If  $|x_{q'}^G| = \alpha$ , then  $|x^G| = \alpha$ . Suppose that  $|x_{q'}^G| \neq \alpha$ . For any  $p'$ -element  $y \in C_N(x_{q'}) \cap C_N(b)$ , we will prove that  $y \in Z(G)$ . For otherwise, write  $y = y_q \cdot y_{q'}$  with  $y_q$  and  $y_{q'}$  the  $q$ -component and  $q'$ -component of  $y$  respectively. If  $y_{q'} \notin Z(G)$ , then  $C_G(b) \leq C_G(y_{q'})$  and thus  $C_G(y_{q'}) = C_G(b)$  by Step 1. Therefore,  $x_{q'} \in C_G(b)$  and thus we see that  $C_G(b) \leq C_G(x_{q'})$ . Again by Step 1 we have  $|x_{q'}^G| = \alpha$ , which is a contradiction. Now,  $y$  can be assumed to be a  $q$ -element. Then  $|y^G| = \alpha$  by the above paragraphs. Arguing similarly as above, we have the contradiction that  $|x_{q'}^G| = \alpha$ . Therefore, we have  $(C_N(x_{q'}) \cap C_N(b))_{p'} \leq Z(G)$ . Noticing that  $N = C_N(a)C_N(b)$ , we have  $|N| = |a^N| |b^N| |C_N(a) \cap C_N(b)|$ . Since  $\beta$  is a  $p'$ -number dividing  $|N/(N \cap Z(G))|$ , we see that  $\beta$  divides  $|N|_{p'} : |N \cap Z(G)|_{p'} = |N|_{p'} : |Z(N)|_{p'}$ , and thus  $\beta$  divides  $|a^N|$ . Therefore,  $|a^G| = \beta = |a^N| = |C_N(b) : C_N(a) \cap C_N(b)| = |C_N(b)|_{p'} : |C_N(a) \cap C_N(b)|_{p'} = |C_N(b)|_{p'} : |Z(N)|_{p'}$ . Let  $T$  be a  $p$ -complement of  $C_N(b)$ . Now, consider the factor group  $T/Z(N)_{p'}$  and the set  $x_{q'}^N$ . If  $zZ(N)_{p'}$  is an element in  $T/Z(N)_{p'}$ , we may assume that  $z \in T$ . For any  $y \in t^N$ , we define  $y^{zZ(N)_{p'}} = y^z$ . Then  $T/Z(N)_{p'}$  acts on the set  $x_{q'}^N$ . By the above paragraph we have  $x_{q'}^N \cap T = \emptyset$ . Therefore,  $T/Z(N)_{p'}$  acts on  $x_{q'}^N$  without fixed point, which implies that  $|T/Z(N)_{p'}|$  divides  $|x_{q'}^N|$ . Consequently,  $|x_{q'}^N| = |x^G| = \beta$ . Hence  $|x^G| = |x_{q'}^G| = \beta$ .

Consequently, if we denote

$$I = \{x \mid x \text{ is a } p'\text{-element in } N \text{ such that } |x^G| = \alpha\}$$

and

$$J = \{x \mid x \text{ is a } p'\text{-element in } N \text{ such that } |x^G| = \beta\}.$$

Then, from the above paragraphs and Step 4 we see that  $N_{p'} = Z(N)_{p'} \cup I \cup J$  and



that  $|x^N| \neq 1$  and  $|y^N| \neq 1$  for every  $x \in I$  and  $y \in J$ . Therefore,  $n(\Gamma_p(N)) = 2$  by [2, Theorem 1].

*Step 7. Final conclusion.*

Since  $\Gamma_p(N)$  has two components by Step 6, we use  $X_1$  and  $X_2$  to denote the two components and assume that  $x^N \in X_2$  where  $|x^N|$  is maximal in  $cs_N(N_{p'})$ . Furthermore, for  $i = 1$  and  $2$ , we write

$$\pi_i = \{r \mid r \text{ is a primary divisor of } |A|, \text{ where } A = x^N \in X_i\}.$$

If  $p$  does not divide  $\alpha$ , then all  $p'$ -elements in  $N$  have conjugacy class size in  $N$  coprime to  $p$ . Therefore,  $N = P \times K$ , where  $P$  is a Sylow  $p$ -subgroup in  $N$  and  $K$  is a  $p$ -complement of  $N$  by [2, Proposition 2]. In this case,  $cs(K) = cs_N(N_{p'})$  and thus  $n(\Gamma(K)) = 2$ . So  $K$  is quasi-Frobenius with abelian kernel and complements by [1, Theorem 2].

Now suppose that  $p$  divides  $\alpha$ . If the maximal size of conjugacy classes in  $\text{Con}_N(N_{p'})$  divides  $\beta$ , then  $p \in \pi(\alpha) = \pi_1$ . Therefore,  $N$  is  $p$ -nilpotent by [3, Theorem 8] and  $K$  is quasi-Frobenius with abelian kernel and complements. Otherwise, the maximal size of conjugacy classes in  $\text{Con}_N(N_{p'})$  divides  $\alpha$ , it follows that  $p \in \pi(\alpha) = \pi_2$ . If  $|\pi_2| \geq 3$ , then  $N$  has a normal Sylow  $p$ -subgroup and  $K$  is quasi-Frobenius with abelian kernel and complements by [3, Theorem 12]. It is easy to see that  $N$  is solvable in the above cases.

If  $|\pi_2| < 3$ , then  $|\pi_2| = 2$  by Lemma 2.3 and we may assume that  $\pi_2 = \{p, r\}$  for some prime  $r \neq p$ . By [3, Theorem 9], we see that  $N$  is  $\pi_2$ -separable and has abelian  $\pi_2$ -complements. Therefore, there exists a  $\pi_2$ -complement  $T$  and a Sylow  $r$ -subgroup  $R$  of  $K$  such that  $K = RT$ . Since  $r$  does not divide  $\beta$ , a Sylow  $r$ -subgroup of  $N$  is contained in  $C_N(a)$ . It is easy to see that  $M$  is the  $p$ -complement of  $C_N(a)$ , so  $R \leq M$ . It follows that  $R \trianglelefteq N$ , by Schur–Zassenhaus theorem, there is a complement  $V$  of  $R$  in  $N$ . Therefore,  $V$  is a product of a Sylow  $p$ -subgroup and an abelian  $p$ -complement, hence it is solvable. Consequently, we deduce that  $N$  is solvable since  $N/R \cong V$ . We will finally show that  $K$  is quasi-Frobenius with abelian kernel and complements.

If  $x \in Z(K)$ , then  $|x^N|$  is a  $p$ -number. Since  $\beta$  is a  $p'$ -number and  $|x^N|$  divides  $|x^G|$ ,  $|x^G| \neq \beta$ . If  $|x^G| = \alpha$ , then  $\alpha$  is a  $p$ -number since  $\pi(x^N) = \pi(\alpha)$  by Step 5, which contradicts to Lemma 2.3. Therefore,  $|x^G| = 1$ . It follows that  $x$  is a  $p'$ -element in  $Z(G) \cap N$  and thus  $x \in Z(N)_{p'}$ . Therefore,  $Z(K) \subseteq Z(N)_{p'}$ . The fact that  $Z(N)_{p'} \subseteq Z(K)$  is easy to see. So,  $Z(N)_{p'} = Z(K)$ .

Write  $\overline{K} = K/Z(K)$ ,  $\overline{R} = RZ(K)/Z(K)$  and  $\overline{T} = TZ(K)/Z(K)$ . Then  $\overline{K} = \overline{R} \rtimes \overline{T}$ . It suffices to show that  $C_{\overline{T}}(\overline{g}) = 1$  for any  $1 \neq \overline{g} \in \overline{R}$ . Suppose on contrary that there exists  $1 \neq \overline{t} \in C_{\overline{T}}(\overline{g})$  for some  $1 \neq \overline{g} \in \overline{R}$ . Since  $\overline{R} \cap \overline{T} = 1$

and  $t \notin Z(N)$ , we see  $t \notin M$  and thus  $|t^G| = \alpha$  by Step 6. On the other hand, since  $M \leq C_N(g)$ ,  $|g^N|$  divides  $|N : M| = |a^N| \cdot p^s$  for some integer  $s \geq 0$ . If  $|g^G| = \alpha$ , then  $|g^N|$  divides  $\alpha$ . Therefore,  $|g^N|$  is a  $p$ -number and so  $\alpha$  is a  $p$ -number too, which contradicts to Lemma 2.3. Therefore,  $|g^G| = \beta$  by Step 6 and so  $(C_N(t) \cap C_N(g))_{p'} \leq Z(G)_{p'}$ .

But on the other hand, since  $(|\bar{t}|, |\bar{g}|) = 1$ , we see that  $\overline{gt}^{|\bar{g}|} = \bar{t}^{|\bar{g}|} \neq 1$ . Hence,

$$\overline{gt}^{|\bar{g}|} = \bar{t}^{|\bar{g}|} \in \overline{C_N(t)} \cap \overline{C_N(gt)} = \overline{C_N(t) \cap C_N(gt)} = \overline{C_N(t) \cap C_N(g)},$$

which means that  $C_N(t) \cap C_N(g)$  contains a non-central  $p'$ -element, contradicting to the above paragraph.  $\square$

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