

## On conformally flat $(\alpha, \beta)$ -metrics with relatively isotropic mean Landsberg curvature

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**Abstract.** In this paper, we study conformally flat  $(\alpha, \beta)$ -metrics in the form of  $F = \alpha\phi(\beta/\alpha)$ , where  $\alpha$  is a Riemannian metric and  $\beta$  is a 1-form on the manifold. We prove that conformally flat weak Landsberg  $(\alpha, \beta)$ -metrics must be either Riemannian metrics or locally Minkowski metrics. Further, we prove that, if  $\phi(s)$  is a polynomial in  $s$ , then conformally flat  $(\alpha, \beta)$ -metrics with relatively isotropic mean Landsberg curvature must also be either Riemannian metrics or locally Minkowski metrics.

### 1. Introduction

The study on conformal geometry has a long and venerable history. From the beginning, conformal geometry has played an important role in physical theories. The conformal geometry of Riemannian metrics have been well studied by many geometers. There are many important local and global results in Riemannian conformal geometry, which in turn lead to a better understanding on Riemann manifolds. More generally, the conformal properties of a Finsler metric deserve extra attention. The Weyl theorem states that the projective and conformal properties of a Finsler space determine the metric properties uniquely ([15], [17]). Two Finsler metrics  $F$  and  $\tilde{F}$  on a manifold  $M$  are said to be *conformally related* if there is a scalar function  $\sigma(x)$  on  $M$  such that  $\tilde{F} = e^{\sigma(x)}F$ . In [2],

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S. BACSO and the first author studied the conformal transformations between two Finsler metrics which preserve Ricci curvature, Landsberg curvature, mean Landsberg curvature and  $S$ -curvature respectively. A Finsler metric is called *conformally Berwald* Finsler metric if it is conformally related to a Berwald metric. M. HASHIGUCHI and Y. ICHIJYŌ proved that a Finsler manifold is a conformally Berwald manifold if and only if it is a so-called Wagner manifold ([10]). C. VINCZE also characterized conformally Berwald Finsler manifolds in [19]. Particularly, a Finsler metric which is conformally related to a Minkowski metric is called *conformally flat Finsler metric*. In [12], Y. ICHIJYŌ and M. HASHIGUCHI defined a conformally invariant linear connection in a Finsler space with an  $(\alpha, \beta)$ -metric and gave a condition that a Randers metric is conformally flat based on their connection. Later, S. KIKUCHI found a conformally invariant Finsler connection and gave a necessary and sufficient condition for a Finsler metric to be conformally flat by a system of partial differential equations under an extra condition ([14]). By using Kikuchi's conformally invariant Finsler connection, S.-I. HOJO, M. MATSUMOTO and K. OKUBO studied conformally Berwald Finsler spaces and its applications to  $(\alpha, \beta)$ -metrics ([11]). On the other hand, L. KANG has proved that any conformally flat Randers metric of scalar flag curvature is projectively flat and classified completely conformally flat Randers metrics of scalar flag curvature ([13]). Further, he characterized conformally flat and projectively flat  $(\alpha, \beta)$ -metrics. Recently, the first author and G. CHEN proved that, if  $\phi = \phi(s)$  is a polynomial in  $s$ , the conformally flat weak Einstein  $(\alpha, \beta)$ -metric  $F = \alpha\phi(\beta/\alpha)$  must be either a locally Minkowski metric or a Riemannian metric. Moreover, they prove that conformally flat  $(\alpha, \beta)$ -metrics with isotropic  $S$ -curvature are also either locally Minkowski metrics or Riemannian metrics (See [4]).

In Finsler geometry, there are several very important non-Riemannian quantities. The Cartan torsion  $\mathbf{C}$  is a primary quantity. There is another quantity which is determined by the Busemann–Hausdorff volume form, that is the so-called distortion  $\tau$ . The vertical differential of  $\tau$  on each tangent space gives rise to the mean Cartan torsion  $\mathbf{I} := \tau_{y^k} dx^k$ .  $\mathbf{C}$ ,  $\tau$  and  $\mathbf{I}$  are the fundamental geometric quantities which characterize Riemannian metrics among Finsler metrics. Differentiating  $\mathbf{C}$  along geodesics gives rise to the *Landsberg curvature*  $\mathbf{L}$ . The horizontal derivative of  $\tau$  along geodesics is the so-called  *$S$ -curvature*  $\mathbf{S} := \tau_{;k} y^k$ . The horizontal derivative of  $\mathbf{I}$  along geodesics is called the *mean Landsberg curvature*  $\mathbf{J} := \mathbf{I}_{;k} y^k$ . The Riemann curvature measures the shape of the space while the non-Riemannian quantities describe the change of the “color” on the space. Hence Finsler spaces are “colorful” geometric spaces. It is found that the flag curvature is closely related to these non-Riemannian quantities (see [5], [18]).

By the definition,  $\mathbf{J}/\mathbf{I}$  can be regarded as the relative growth rate of the mean Cartan torsion along geodesic. We call a Finsler metric  $F$  is of *relatively isotropic mean Landsberg curvature* if  $F$  satisfies  $\mathbf{J} + cF\mathbf{I} = 0$ , where  $c = c(x)$  is a scalar function on the Finsler manifold. In particular, when  $c = 0$ , Finsler metrics with  $\mathbf{J} = 0$  are called *weak Landsberg metrics*. Many known Finsler metrics satisfy  $\mathbf{J} + cF\mathbf{I} = 0$  (see [5], [6], [18]). B. LI and Z. SHEN characterized weak Landsberg metrics in  $(\alpha, \beta)$ -metrics and showed that there exist weak Landsberg metrics which are not Landsberg metrics in dimension greater than two ([16]). Further, the first author and H. Wang and M. Wang studied and characterized  $(\alpha, \beta)$ -metrics with relatively isotropic mean Landsberg curvature ([8]).

In this paper, we first study and characterize conformally flat weak Landsberg  $(\alpha, \beta)$ -metrics and obtain the following theorem.

**Theorem 1.1.** *Any conformally flat weak Landsberg  $(\alpha, \beta)$ -metric  $F = \alpha\phi(\frac{\beta}{\alpha})$  on a manifold  $M$  must be either a Riemannian metric or a locally Minkowski metric.*

Further, we study conformally flat  $(\alpha, \beta)$ -metrics with relatively isotropic mean Landsberg curvature. We get the following theorem.

**Theorem 1.2.** *Let  $F = \alpha\phi(s)$ ,  $s = \frac{\beta}{\alpha}$  be a conformally flat  $(\alpha, \beta)$ -metric on a manifold  $M$ , where  $\phi(s)$  is a polynomial in  $s$ . If  $F$  is of relatively isotropic mean Landsberg curvature, then it is either a Riemannian metric or a locally Minkowski metric.*

## 2. Preliminary

Let  $M$  be an  $n$ -dimensional  $C^\infty$  manifold and  $TM$  denote the tangent bundle of  $M$ . A Finsler metric on  $M$  is a function  $F : TM \rightarrow [0, \infty)$  with the following properties:

- (1)  $F$  is  $C^\infty$  on  $TM \setminus \{0\}$ ;
- (2)  $F(x, \lambda y) = \lambda F(x, y)$ ,  $\forall \lambda > 0$ ;
- (3) the fundamental tensor  $(g_{ij}(x, y))$  is positive definite, where

$$g_{ij}(x, y) := \frac{1}{2} [F^2]_{y^i y^j}(x, y).$$

We call the pair  $(M, F)$  an  $n$ -dimensional *Finsler manifold*.

Let  $(M, F)$  be a Finsler manifold. For any non-zero vector  $y = y^i \frac{\partial}{\partial x^i} \in T_x M$ ,  $F$  induces an inner product  $g_y$  on  $T_x M$  as follows

$$g_y(u, v) := g_{ij}(x, y) u^i v^j,$$

where  $u = u^i \frac{\partial}{\partial x^i} \in T_x M$ ,  $v = v^i \frac{\partial}{\partial x^i} \in T_x M$ .

For a Finsler metric  $F$ , the geodesics are characterized by the following system of 2nd order ordinary differential equations:

$$\frac{d^2 x^i}{dt^2} + 2G^i \left( x, \frac{dx}{dt} \right) = 0,$$

where

$$G^i := \frac{1}{4} g^{il} \{ [F^2]_{x^k y^l y^k} - [F^2]_{x^l} \} \quad (1)$$

and  $(g^{ij}) = (g_{ij})^{-1}$ .  $G^i$  are called the *geodesic coefficients* of  $F$ .

Let

$$C_{ijk} := \frac{1}{4} [F^2]_{y^i y^j y^k} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}.$$

Define symmetric trilinear form  $\mathbf{C} := C_{ijk}(x, y) dx^i \otimes dx^j \otimes dx^k$  on  $TM \setminus \{0\}$ . We call  $\mathbf{C}$  the *Cartan torsion*. The *mean Cartan torsion*  $\mathbf{I} = I_i dx^i$  is defined by

$$I_i := g^{jk} C_{ijk}.$$

Further, we have ([5], [18])

$$I_i = g^{jk} C_{ijk} = \frac{\partial}{\partial y^i} \left[ \ln \sqrt{\det(g_{jk})} \right]. \quad (2)$$

On the slit tangent bundle  $TM \setminus \{0\}$ , the *Landsberg curvature*  $\mathbf{L} := L_{ijk} dx^i \otimes dx^j \otimes dx^k$  is defined by  $L_{ijk} := C_{ijk; m} y^m$ , where “;” denotes the horizontal covariant derivative with respect to  $F$ . Further,  $\mathbf{L}$  can be expressed as

$$L_{ijk} = -\frac{1}{2} F F_{y^m} [G^m]_{y^i y^j y^k}. \quad (3)$$

A Finsler metric is called the *Landsberg metric* if  $L_{ijk} = 0$ . The *mean Landsberg curvature*  $\mathbf{J} := J_i dx^i$  is defined by

$$J_i := g^{jk} L_{ijk}. \quad (4)$$

It is easy to see that  $J_i = I_{i; m} y^m$ . A Finsler metric is called the *weak Landsberg metric* if  $J = 0$ . More generally, if  $F$  satisfies  $\mathbf{J} + cF\mathbf{I} = 0$ , where  $c = c(x)$  is a

scalar function on the manifold, then it is said to be of *relatively isotropic mean Landsberg curvature*.

$(\alpha, \beta)$ -metrics form a rich class of computable Finsler metrics. They play an important role in Finsler geometry (see [3]). The important applications of  $(\alpha, \beta)$ -metrics in physics and biology(ecology) have also been found. The study for  $(\alpha, \beta)$ -metrics can help us to understand better geometric properties of Finsler metrics in general case. Hence, it is worthy of doing study for such metrics deeply. Let  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  be a Riemannian metric and  $\beta = b_i(x)y^i$  be a 1-form on a manifold  $M$ . An  $(\alpha, \beta)$ -metric is a scalar function on  $TM$  defined by

$$F := \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where  $\phi = \phi(s)$  is a  $C^\infty$  function on an open interval  $(-b_0, b_0)$ . By a direct computation and a lemma in linear algebra, one gets

$$\det(g_{ij}) = \phi^{n+1}(\phi - s\phi')^{n-2} [(\phi - s\phi') + (b^2 - s^2)\phi''] \det(a_{ij}),$$

where  $b := \|\beta_x\|_\alpha$ . Using the above formula, one can easily prove that for Riemannian metric  $\alpha$  and 1-form  $\beta$  with  $\|\beta_x\|_\alpha < b_0$ ,  $x \in M$ , the function  $F = \alpha\phi(\beta/\alpha)$  is a positive definite Finsler metric if and only if the function  $\phi$  satisfies

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + (\rho^2 - s^2)\phi''(s) > 0, \quad |s| \leq \rho < b_0. \quad (5)$$

Such  $(\alpha, \beta)$ -metrics are said to be *regular*. For any  $s$  with  $|s| < b_0$ , taking  $\rho = |s|$  in (5), we obtain

$$\phi(s) - s\phi'(s) > 0, \quad |s| < b_0. \quad (6)$$

See [9]. In particular, when  $\phi = 1 + s$ , the metric  $F = \alpha\phi(\beta/\alpha)$  is just the Randers metric  $F = \alpha + \beta$ . In this case,  $b_0 = 1$ .

In this paper, we will focus on studying regular  $(\alpha, \beta)$ -metrics. Let “|” denote the covariant derivative with respect to the Levi-Civita connection of  $\alpha$ . Denote

$$\begin{aligned} r_{ij} &:= \frac{1}{2}(b_{i|j} + b_{j|i}), & s_{ij} &:= \frac{1}{2}(b_{i|j} - b_{j|i}), \\ s^i{}_j &:= a^{il}s_{lj}, & r^i{}_j &:= a^{il}r_{lj}, & r_i &:= b^j r_{ji}, & s_i &:= b^j s_{ji}, \end{aligned}$$

where  $(a^{ij}) := (a_{ij})^{-1}$  and  $b^j := a^{jk}b_k$ . We put  $r_0 := r_i y^i$ ,  $s_0 := s_i y^i$ ,  $r_{00} := r_{ij}y^i y^j$ ,  $s^i{}_0 := s^i{}_j y^j$ , etc. Let  $G^i$  and  $G_\alpha^i$  denote the geodesic coefficients of  $F$  and  $\alpha$  respectively in the same coordinate system. Then we have ([9])

$$G^i = G_\alpha^i + \alpha Q s^i{}_0 + \{-2Q\alpha s_0 + r_{00}\}\{\Psi b^i + \Theta \alpha^{-1}y^i\}, \quad (7)$$

where

$$\begin{aligned} Q &:= \frac{\phi'}{\phi - s\phi'}, \\ \Theta &:= \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi[(\phi - s\phi') + (b^2 - s^2)\phi'']}, \\ \Psi &:= \frac{\phi''}{2[(\phi - s\phi') + (b^2 - s^2)\phi'']}. \end{aligned}$$

Let

$$\begin{aligned} \Delta &:= 1 + sQ + (b^2 - s^2)Q', \\ \Phi &:= -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q'', \\ \Psi_1 &:= \sqrt{b^2 - s^2}\Delta^{\frac{1}{2}} \left[ \frac{\sqrt{b^2 - s^2}\Phi}{\Delta^{\frac{3}{2}}} \right]', \\ h_j &:= b_j - \alpha^{-1}sy_j. \end{aligned}$$

By (3), (4), (7), the mean Landsberg curvature of the  $(\alpha, \beta)$ -metric  $F = \alpha\phi(\beta/\alpha)$  is given by (see [8], [16])

$$\begin{aligned} J_j &= -\frac{1}{2\alpha^4\Delta} \left\{ \frac{2\alpha^3}{b^2 - s^2} \left[ \frac{\Phi}{\Delta} + (n+1)(Q - sQ') \right] (s_0 + r_0)h_j \right. \\ &\quad + \frac{\alpha^2}{b^2 - s^2} \left[ \Psi_1 + s\frac{\Phi}{\Delta} \right] (r_{00} - 2\alpha Qs_0)h_j \\ &\quad + \alpha \left[ -\alpha^2 Q' s_0 h_j + \alpha Q(\alpha^2 s_j - y_j s_0) + \alpha^2 \Delta s_{j0} \right. \\ &\quad \left. \left. + \alpha^2 (r_{j0} - 2\alpha Qs_j) - (r_{00} - 2\alpha Qs_0)y_j \right] \frac{\Phi}{\Delta} \right\}. \end{aligned} \quad (8)$$

Here,  $y_j = a_{ij}y^i$ .

### 3. The Proof of Theorem 1.1

Now we are in the position to prove the theorems. For our aim, we need the following formula for the mean Cartan torsion of  $(\alpha, \beta)$ -metrics.

**Lemma 3.1** ([8]). *For an  $(\alpha, \beta)$ -metric  $F = \alpha\phi(\frac{\beta}{\alpha})$ , the mean Cartan torsion is given by*

$$I_i = -\frac{1}{2F} \frac{\Phi}{\Delta} (\phi - s\phi')h_i. \quad (9)$$

By Deicke's theorem, a Finsler metric is Riemannian if and only if  $I = 0$ . By (6) and Lemma 3.1, we have the following

**Lemma 3.2** ([8]). *An  $(\alpha, \beta)$ -metric  $F$  is a Riemannian metric if and only if  $\Phi = 0$ .*

Let

$$J := J_j b^j.$$

By (8), we get

**Lemma 3.3** ([16]). *For an  $(\alpha, \beta)$ -metric  $F = \alpha\phi(\frac{\beta}{\alpha})$ , the quantity  $J$  is given by*

$$J = -\frac{1}{2\alpha^2\Delta}\{\Psi_1(r_{00} - 2\alpha Qs_0) + \alpha\Psi_2(r_0 + s_0)\}, \quad (10)$$

where  $\Psi_2 := 2(n+1)(Q - sQ') + 3\frac{\Phi}{\Delta}$ .

By (8) and (9), we have the following ([8])

$$\begin{aligned} J_j + c(x)FI_j = & \frac{(-1)}{2\alpha^4\Delta}\left\{\frac{2\alpha^3}{b^2 - s^2}\left[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')\right](s_0 + r_0)h_j \right. \\ & + \frac{\alpha^2}{b^2 - s^2}\left[\Psi_1 + s\frac{\Phi}{\Delta}\right](r_{00} - 2\alpha Qs_0)h_j \\ & + \alpha\left[-\alpha^2Q's_0h_j + \alpha Q(\alpha^2s_j - y_js_0) + \alpha^2\Delta s_{j0} \right. \\ & \left. + \alpha^2(r_{j0} - 2\alpha Qs_j) - (r_{00} - 2\alpha Qs_0)y_j\right]\frac{\Phi}{\Delta} \\ & \left. + c(x)\alpha^4\Phi(\phi - s\phi')h_j\right\}. \end{aligned} \quad (11)$$

Further, we need the following lemmas.

**Lemma 3.4** ([1]). *Let  $F$  be an  $(\alpha, \beta)$ -metric. Then  $F$  is locally Minkowskian if and only if  $\alpha$  is flat and  $b_{i|j} = 0$  (that is,  $\beta$  is parallel with respect to  $\alpha$ ).*

**Lemma 3.5.** *If  $\phi = \phi(s)$  satisfies  $\Psi_1 = 0$ , then  $F$  is Riemannian.*

PROOF.  $\Psi_1 = 0$  means that  $\left[\frac{\sqrt{b^2 - s^2}\Phi}{\Delta^{\frac{3}{2}}}\right]' = 0$ . Then  $\Lambda(s) := \frac{\sqrt{b^2 - s^2}\Phi}{\Delta^{\frac{3}{2}}}$  is a constant for  $|s| \leq b < b_0$ . Letting  $s = b$  yields  $\Lambda(s) = 0$ . Thus  $\Lambda(s) \equiv 0$ , which implies that  $\Phi = 0$ . By Lemma 3.2,  $F$  is Riemannian.  $\square$

Now, assume that  $F = \alpha\phi(\beta/\alpha)$  is conformally flat, that is,  $F$  is conformally related to a Minkowski metric  $\tilde{F}$ . Then there exists a scalar function  $\sigma = \sigma(x)$  on the manifold, so that  $\tilde{F} = e^{\sigma(x)}F$ . It is easy to see that  $\tilde{F} = \tilde{\alpha}\phi(\tilde{\beta}/\tilde{\alpha})$ . We have

$$\tilde{\alpha} = e^{\sigma(x)}\alpha, \quad \tilde{\beta} = e^{\sigma(x)}\beta,$$

which are equivalent to

$$\tilde{a}_{ij} = e^{2\sigma(x)}a_{ij}, \quad \tilde{b}_i = e^{\sigma(x)}b_i.$$

Let “ $\parallel$ ” denote the covariant derivative with respect to the Levi-Civita connection of  $\tilde{\alpha}$ . Putting  $\sigma_i := \frac{\partial\sigma}{\partial x^i}$  and  $\sigma^i := a^{ij}\sigma_j$ . The Christoffel symbols  $\Gamma_{jk}^i$  of

$\alpha$  and the Christoffel symbols  $\tilde{\Gamma}_{jk}^i$  of  $\tilde{\alpha}$  are related by

$$\tilde{\Gamma}_{jk}^i = \Gamma_{jk}^i + \delta_j^i \sigma_k + \delta_k^i \sigma_j - \sigma^i a_{jk}.$$

Hence we can obtain

$$\tilde{b}_{i||j} = \frac{\partial \tilde{b}_i}{\partial x^j} - \tilde{b}_s \tilde{\Gamma}_{jk}^i = e^\sigma (b_{i||j} - b_j \sigma_i + b_r \sigma^r a_{ij}). \quad (12)$$

By Lemma 3.4, for Minkowski metric  $\tilde{F}$ ,  $\tilde{b}_{i||j} = 0$ . Thus we have

$$b_{i||j} = b_j \sigma_i - b_r \sigma^r a_{ij}, \quad (13)$$

$$r_{ij} = \frac{1}{2}(\sigma_i b_j + \sigma_j b_i) - b_r \sigma^r a_{ij}, \quad s_{ij} = \frac{1}{2}(\sigma_i b_j - \sigma_j b_i), \quad (14)$$

$$r_j = -\frac{1}{2}(b_r \sigma^r) b_j + \frac{1}{2} \sigma_j b^2, \quad s_j = \frac{1}{2}(b_r \sigma^r) b_j - \frac{1}{2} \sigma_j b^2, \quad (15)$$

$$r_{i0} = \frac{1}{2}[\sigma_i \beta + (\sigma_r y^r) b_i] - \sigma_r b^r y_i, \quad s_{i0} = \frac{1}{2}[\sigma_i \beta - (\sigma_r y^r) b_i]. \quad (16)$$

Further, we have

$$r_{00} = (\sigma_r y^r) \beta - (\sigma_r y^r) \alpha^2, \quad (17)$$

$$r_0 = \frac{1}{2}(\sigma_r y^r) b^2 - \frac{1}{2}(\sigma_r b^r) \beta, \quad s_0 = \frac{1}{2}(\sigma_r y^r) \beta - \frac{1}{2}(\sigma_r y^r) b^2. \quad (18)$$

By (18), it is easy to see that, for conformally flat  $(\alpha, \beta)$ -metrics,

$$r_0 + s_0 = 0, \quad (19)$$

which is equivalent to the length of  $\beta$  with respect to  $\alpha$  being a constant.

In order to overcome the difficulty in computation, we take an orthonormal basis at any point  $x$  with respect to  $\alpha$  such that

$$\alpha = \sqrt{\sum_{i=1}^n (y^i)^2}, \quad \beta = b y^1,$$

where  $b := \|\beta_x\|_\alpha$ . Then we take the following coordinate transformation in  $T_x M$  (see [7]),  $\psi : (s, u^A) \rightarrow (y^i)$ :

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad y^A = u^A, \quad 2 \leq A \leq n, \quad (20)$$

where  $\bar{\alpha} = \sqrt{\sum_{i=2}^n (u^A)^2}$ . We have

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}} \bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}} \bar{\alpha}. \quad (21)$$

By (14)–(18) and (20), (21), we have

$$r_{00} = -b\sigma_1 \bar{\alpha}^2 + \frac{bs\bar{\sigma}_0 \bar{\alpha}}{\sqrt{b^2 - s^2}}, \quad (22)$$

$$r_0 = \frac{1}{2} b^2 \bar{\sigma}_0 = -s_0, \quad (23)$$

$$r_{10} = \frac{1}{2} b \bar{\sigma}_0, \quad r_{A0} = \frac{1}{2} \frac{\sigma_A b s \bar{\alpha}}{\sqrt{b^2 - s^2}} - (b\sigma_1) u_A, \quad (24)$$

$$s_1 = 0, \quad s_A = -\frac{1}{2} \sigma_A b^2, \quad (25)$$

$$s_{10} = -\frac{1}{2} b \bar{\sigma}_0, \quad s_{A0} = \frac{1}{2} \frac{\sigma_A b s \bar{\alpha}}{\sqrt{b^2 - s^2}}, \quad (26)$$

$$h_1 = b - \frac{s^2}{b}, \quad h_A = -\frac{\sqrt{b^2 - s^2} s u_A}{b \bar{\alpha}}, \quad (27)$$

here,  $\bar{\sigma}_0 := \sigma_A u^A$ .

PROOF OF THEOREM 1.1. Assume that  $F$  is a weak Landsberg metric, then it satisfies  $J = 0$ . Putting  $r_0 + s_0 = 0$  into (10), we get  $\Psi_1 = 0$  or  $r_{00} - 2\alpha Q s_0 = 0$ .

If  $\Psi_1 = 0$ , by Lemma 3.5,  $F$  is Riemannian.

If  $r_{00} - 2\alpha Q s_0 = 0$ , by (21), (22) and (23), we get

$$-b\sigma_1 \bar{\alpha}^2 + \frac{bs\bar{\sigma}_0}{\sqrt{b^2 - s^2}} (b^2 Q + s) \bar{\alpha} = 0. \quad (28)$$

Note that  $\bar{\alpha}^2$  is a quadratic form in  $(u^A)$  and  $\bar{\alpha}$  is an irrational expression of  $(u^A)$ . We have the following (also see Lemma 6.1 in [18])

$$\sigma_1 = 0, \quad (b^2 Q + s) \sigma_A = 0.$$

If  $\phi = \phi(s) > 0$  satisfies  $b^2 Q + s = 0$ , we can get  $\phi = k\sqrt{b^2 - s^2}$  (see [7]). In this case,  $F$  is Riemannian. If  $b^2 Q + s \neq 0$ , then  $\sigma_A = 0$ . Together with  $\sigma_1 = 0$ , we know that  $\sigma = \text{constant}$ . Hence  $F$  is Minkowskian.  $\square$

#### 4. The Proof of Theorem 1.2

Assume that  $F$  is a conformally flat  $(\alpha, \beta)$ -metric with relatively isotropic mean Landsberg curvature. By (11) and  $r_0 + s_0 = 0$ , we get

$$\begin{aligned} & \frac{\alpha^2}{b^2 - s^2} \left[ \Psi_1 + s \frac{\Phi}{\Delta} \right] (r_{00} - 2\alpha Q s_0) h_j + \alpha \left[ -\alpha^2 Q' s_0 h_j + \alpha Q (\alpha^2 s_j - y_j s_0) \right. \\ & \quad \left. + \alpha^2 \Delta s_{j0} + \alpha^2 (r_{j0} - 2\alpha Q s_j) - (r_{00} - 2\alpha Q s_0) y_j \right] \frac{\Phi}{\Delta} \\ & \quad + c(x) \alpha^4 \Phi (\phi - s\phi') h_j = 0. \end{aligned} \quad (29)$$

Letting  $j = 1$  in (29), we have

$$\begin{aligned} & \frac{\alpha^2}{b^2 - s^2} \left[ \Psi_1 + s \frac{\Phi}{\Delta} \right] (r_{00} - 2\alpha Q s_0) h_1 + \alpha \left[ -\alpha^2 Q' s_0 h_1 + \alpha Q (\alpha^2 s_1 - y_1 s_0) \right. \\ & \quad \left. + \alpha^2 \Delta s_{10} + \alpha^2 (r_{10} - 2\alpha Q s_1) - (r_{00} - 2\alpha Q s_0) y_1 \right] \frac{\Phi}{\Delta} \\ & \quad + c(x) \alpha^4 \Phi (\phi - s\phi') h_1 = 0. \end{aligned} \quad (30)$$

Putting (21)-(27) into (30), and multiplying the resulting equation by  $2\Delta(b^2 - s^2)^{5/2}$ , we get

$$\begin{aligned} & 2b^2(b^2 - s^2)^{3/2} \Delta (b\Phi\phi c - b\Phi s\phi' c - \Psi_1 \sigma_1) \bar{\alpha}^4 \\ & \quad + b^2(b^2 - s^2) \bar{\sigma}_0 (b^4 \Phi Q' - b^2 \Phi \Delta - b^2 \Phi Q' s^2 \\ & \quad + 2b^2 \Psi_1 \Delta Q + b^2 \Phi + b^2 \Phi Q s + 2\Psi_1 \Delta s) \bar{\alpha}^3 = 0. \end{aligned} \quad (31)$$

From (31), we get

$$\Delta [b\Phi(\phi - s\phi')c - \Psi_1 \sigma_1] = 0, \quad (32)$$

$$\bar{\sigma}_0 (b^4 \Phi Q' - b^2 \Phi \Delta - b^2 \Phi Q' s^2 + 2b^2 \Psi_1 \Delta Q + b^2 \Phi + b^2 \Phi Q s + 2\Psi_1 \Delta s) = 0. \quad (33)$$

Note that  $\Delta = Q'(b^2 - s^2) + sQ + 1$ . Simplifying (33) yields,

$$(b^2 \Psi_1 \Delta Q + \Psi_1 \Delta s) \bar{\sigma}_0 = 0,$$

that is,

$$\Psi_1 \Delta (b^2 Q + s) \bar{\sigma}_0 = 0. \quad (34)$$

Letting  $j = A$  in (29), we have

$$\begin{aligned} & \frac{\alpha^2}{b^2 - s^2} \left[ \Psi_1 + s \frac{\Phi}{\Delta} \right] (r_{00} - 2\alpha Q s_0) h_A + \alpha \left[ -\alpha^2 Q' s_0 h_A \right. \\ & \quad \left. + \alpha Q (\alpha^2 s_A - y_A s_0) + \alpha^2 \Delta s_{A0} + \alpha^2 (r_{A0} - 2\alpha Q s_A) - (r_{00} - 2\alpha Q s_0) y_A \right] \frac{\Phi}{\Delta} \\ & \quad + c(x) \alpha^4 \Phi (\phi - s\phi') h_A = 0. \end{aligned} \quad (35)$$

Putting (21)–(27) into (35) and by using the similar method in the case of  $j = 1$ , we can get

$$-(s\Delta + s + b^2Q)b^2\Phi\sigma_A\bar{\alpha}^2 + [(s\Delta + s + b^2Q)b^2\Phi + 2s(b^2Q + s)\Psi_1\Delta]\bar{\sigma}_0u_A = 0, \quad (36)$$

$$s(b^2 - s^2)[b(\phi - s\phi')\Phi c - \Psi_1\sigma_1]\Delta u_A = 0. \quad (37)$$

It is easy to see that (37) is equivalent to (32). Further, Multiplying (36) by  $u^A$ , we get

$$s(b^2Q + s)\Psi_1\Delta\bar{\sigma}_0\bar{\alpha}^2 = 0. \quad (38)$$

It is easy to see that (38) is equivalent to (34).

In summary, conformally flat  $(\alpha, \beta)$ -metrics with relatively isotropic mean Landsberg curvature satisfy (32) and (34).

If  $b^2Q + s = 0$ , we have  $\phi = k\sqrt{b^2 - s^2}$  (see proof of Theorem 1.1), which is a irrational expression of  $s$  and a contradiction with the assumption that  $\phi = \phi(s)$  is a polynomial in  $s$ . Then we have  $b^2Q + s \neq 0$ . From (34), we have  $\Psi_1 = 0$  or  $\sigma_A = 0$ .

If  $\Psi_1 = 0$ , by Lemma 3.5,  $F$  is Riemannian. In this case,  $\Phi = 0$  by Lemma 3.2. Hence (32) holds.

If  $\Psi_1 \neq 0$ , then  $\sigma_A = 0$ . In the following, we prove that  $\sigma_1 = 0$  from (32) when  $\Psi_1 \neq 0$ .

By the assumption,  $\phi(s)$  is a polynomial in  $s$ . Assume that

$$\phi = 1 + a_1s + a_2s^2 + \cdots + a_ms^m, \quad (39)$$

here  $a_1, a_2, \dots, a_m$  are numbers independent of  $s$  and  $a_m \neq 0$ .

Firstly we consider the situation of  $m > 1$  in (39).

Simplifying (32) and multiplying it by  $\Delta^2$ , we get

$$\left\{ [-s\Phi + (b^2 - s^2)\Phi']\Delta - \frac{3}{2}(b^2 - s^2)\Phi\Delta' \right\} \sigma_1 - b\Delta^2\Phi(\phi - s\phi')c = 0. \quad (40)$$

Putting (39) into (40) and multiplying it by

$$2[-1 + a_2s^2 + 2a_3s^3 + \cdots + (m-1)a_ms^m],$$

by using maple program, we can obtain the following

$$k_1nbca_m^8s^{8m-1} + (k_2n + k_3)bca_{m-1}a_m^7s^{8m-2} + \cdots + \sum_{i=1}^{i=7} \eta_i b^i = 0, \quad (41)$$

where  $k_1$ ,  $k_2$  and  $k_3$  are non-zero constants depending on  $m$ . For example,

$$\begin{aligned} \text{when } m = 2, \quad k_1 &= -108, & k_2 &= -405, & k_3 &= 27; \\ \text{when } m = 3, \quad k_1 &= -6144, & k_2 &= -33024, & k_3 &= 384; \\ \text{when } m = 4, \quad k_1 &= -81000, & k_2 &= -490725, & k_3 &= 2025; \\ \text{when } m = 10, \quad k_1 &= -174653820, & k_2 &= -1260771237, & k_3 &= 264627 \end{aligned}$$

and when  $m = 100$ ,  $k_1 = -19794060593980200$ ,  $k_2 = -156773780240111397$ ,  $k_3 = 2969406029$ , etc. Further,  $\eta_i$  ( $1 \leq i \leq 7$ ) are polynomials of  $a_1$ ,  $a_2$ ,  $a_3$  and  $a_4$  independent of  $s$  and  $m$ ,

$$\begin{aligned} \eta_1 &= -2ca_1 - 2cna_1; \\ \eta_2 &= -(n+1)a_1^2; \\ \eta_3 &= -12(n+1)ca_1a_2 - 12ca_3; \\ \eta_4 &= -4(n+1)a_1^2a_2 + 24a_2^2 - 6na_1a_3 + 48a_4; \\ \eta_5 &= -24c(n+1)a_1a_2^2 - 48ca_2a_3; \\ \eta_6 &= -4(n+1)a_1^2a_2^2 + 48a_2^2 - 12na_1a_2a_3 - 108a_3^2 + 96a_2a_4; \\ \eta_7 &= -16c(n+1)a_1a_2^2 - 48ca_2^2a_3. \end{aligned} \tag{42}$$

We must point out that, when  $m = 2$ ,  $a_3 = a_4 = 0$  in (42) while when  $m = 3$ ,  $a_4 = 0$  in (42).

From (41), we can see that  $k_1nbca_m^8 = 0$ . Thus we have  $c = 0$ . By (32), we have  $\sigma_1 = 0$ . Together with  $\sigma_A = 0$ , we know that  $\sigma$  is a constant, which means that  $F$  is a locally Minkowski metric.

Secondly, we consider the situation of  $m = 1$  in (39). In this case,  $F$  is a Randers metric. In [6], the first author and Z. SHEN have proved that a Randers metric  $F = \alpha + \beta$  is of relatively isotropic mean Landsberg curvature if and only if it is of isotropic  $S$ -curvature ( $S = (n+1)cF$ ) and  $\beta$  is closed. Further, in [13], L. KANG has proved that conformally flat Randers metric with almost isotropic  $S$ -curvature must be either a Riemannian metric or a Minkowski metric. This completes the proof of Theorem 1.2.

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