

## Quasi Ahlfors–David regularity of Moran sets

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**Abstract.** For Moran fractals in Euclidean spaces, under suitable assumption, we obtain their quasi Ahlfors–David regularity and quasi-Lipschitz equivalence.

### 1. Introduction

**1.1. Moran set.** Recall some notions in fractals. A map  $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called a similitude with ratio  $r$ , if  $|S(x) - S(y)| = r|x - y|$  for all  $x, y \in \mathbb{R}^n$ . For two subsets  $A$  and  $B$  of  $\mathbb{R}^n$ , we say that  $A$  is geometrically similar to  $B$  with ratio  $r$ , if there is a similitude  $S$  of ratio  $r$  such that  $S(A) = B$ . For subset  $A$ , let  $|A|$  be the diameter of the set, and  $\bar{A}$  the closure of  $A$ .

The notion of Moran set is introduced by WEN in [16].

Given integers  $\{n_k\}_{k \geq 1}$  with  $n_k \geq 2$  and ratios  $\{c_{k,i}\}_{k \geq 1, 1 \leq i \leq n_k} \subset (0, 1)$ , let  $\tilde{\Sigma}^* = \bigcup_{k=0}^{\infty} \prod_{i=1}^k \{1, \dots, n_i\}$  be the collection of finite words with  $k$ -th letter in  $\{1, \dots, n_k\}$  for each  $k$ , suppose  $\{V_{i_1 \dots i_k}\}_{i_1 \dots i_k \in \tilde{\Sigma}^*}$  are non-empty open sets satisfying

$$V_{i_1 \dots i_{k-1} i_k} \subset V_{i_1 \dots i_{k-1}} \text{ and } V_{i_1 \dots i_{k-1} i_k} \cap V_{i_1 \dots i_{k-1} j_k} = \emptyset \text{ if } i_k \neq j_k, \quad (1.1)$$

$V_{i_1 \dots i_{k-1} i_k}$  is geometrically similar to  $V_{i_1 \dots i_{k-1}}$  with ratio

$$|V_{i_1 \dots i_{k-1} i_k}| / |V_{i_1 \dots i_{k-1}}| = c_{k, i_k}. \quad (1.2)$$

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A Moran set  $F$  is defined by

$$F = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \dots i_k \in \tilde{\Sigma}^*} \bar{V}_{i_1 \dots i_k}. \quad (1.3)$$

For convenience, we write  $d_k = \min_{1 \leq i \leq n_k} c_{k,i}$ ,  $D_k = \max_{1 \leq i \leq n_k} c_{k,i}$  and suppose  $s_k$  satisfies  $\prod_{j=1}^k (\sum_{i=1}^{n_j} (c_{j,i})^{s_k}) = 1$ . Let  $s_* = \underline{\lim}_{k \rightarrow \infty} s_k$  and  $s^* = \overline{\lim}_{k \rightarrow \infty} s_k$ .

*Remark 1.* The self-similar set satisfying the open set condition is a Moran set. In fact, suppose  $E = \cup_{i=1}^m S_i(E)$  is a self-similar set (see [7]), where  $\{S_i\}_{i=1}^m$  are contracting similitudes with ratios  $\{r_i\}_{i=1}^m$  such that the open set condition holds, i.e., there is a non-empty open set  $U$  such that

$$\cup_i S_i(U) \subset U \text{ and } S_i(U) \cap S_j(U) = \emptyset \text{ for any } i \neq j. \quad (1.4)$$

Let  $n_k \equiv m$  and  $c_{k,i_k} = r_{i_k}$ ,  $\Sigma^* = \cup_{k=0}^{\infty} \{1, \dots, m\}^k$  and  $\emptyset$  the empty word. We write  $U_{\emptyset} = U$  and  $U_{i_1 \dots i_k} = S_{i_1} \circ \dots \circ S_{i_k}(U)$  for word  $i_1 \dots i_k \in \Sigma^*$ , then (1.4) implies  $U_{i_1 \dots i_{k-1} i_k} \subset U_{i_1 \dots i_{k-1}}$  and  $U_{i_1 \dots i_{k-1} i_k} \cap U_{i_1 \dots i_{k-1} j_k} = \emptyset$  for any  $i_1 \dots i_k \in \Sigma^*$  and  $i_k \neq j_k$  and  $U_{i_1 \dots i_{k-1} i_k}$  is geometrically similar to  $U_{i_1 \dots i_{k-1}}$  with ratio  $|U_{i_1 \dots i_{k-1} i_k}|/|U_{i_1 \dots i_{k-1}}| = r_{i_k}$ . It follows from (1.4) that  $\cup_i S_i(\bar{U}) \subset \bar{U}$ , hence the self-similar set

$$E = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \dots i_k \in \Sigma^*} \bar{U}_{i_1 \dots i_k}. \quad (1.5)$$

**1.2. Ahlfors–David regularity.** A compact set  $E$  is said to be Ahlfors–David  $s$ -regular [11], [12], if there is a Borel measure  $\nu$  supported on  $E$  such that

$$C^{-1}r^s \leq \nu(B(x, r)) \leq Cr^s \quad (1.6)$$

for any  $x \in E$  and  $r \leq |E|$ . For self-similar set  $E$  satisfying the open set condition as above, as shown in [7], the self-similar measure  $\nu$  satisfies (1.6) where  $s = \dim_H E = \dim_B E$  is the solution of the equation  $(r_1)^s + \dots + (r_m)^s = 1$ .

For Ahlfors–David  $s$ -regular set  $E$ , by Theorem 5.7 of [11], we have

$$\overline{\dim}_B E = \underline{\dim}_B E = \dim_H E = s.$$

It was shown in [16] that, if  $c_* = \inf d_k > 0$  for Moran set  $F$  as above, then

$$\dim_H F = s_* \text{ and } \overline{\dim}_B F = s^*.$$

Therefore, if  $s_* < s^*$ , then  $F$  can not be Ahlfors–David regular.

*Example 1.* Let  $n_k \equiv 2$  and  $c_k \in \{1/3, 1/5\}$ . Then  $c_* > 0$ . Take a sequence  $\{c_k\}_k$  such that  $a = \underline{\lim}_{k \rightarrow \infty} q_k < \overline{\lim}_{k \rightarrow \infty} q_k = b$ , where  $q_k = \frac{\#\{i \leq k: c_i = 1/3\}}{k}$ . Then  $\underline{\lim}_{k \rightarrow \infty} s_k = \underline{\lim}_{k \rightarrow \infty} \frac{k \log 2}{-(\log c_1 \cdots c_k)} = \underline{\lim}_{k \rightarrow \infty} \frac{\log 2}{q_k \log 3 + (1-q_k) \log 5} = \frac{\log 2}{a \log 3 + (1-a) \log 5}$  and  $\overline{\lim}_{k \rightarrow \infty} s_k = \frac{\log 2}{b \log 3 + (1-b) \log 5}$ , which means  $\dim_H F < \dim_B F$ .

In the following example,  $\dim_H F = \dim_B F$  but  $F$  is not Ahlfors–David regular.

*Example 2.* Let  $s = \log 3 / \log 5$  and  $c_k = \frac{1}{5}(1 + \frac{1}{2^k})$  for all  $k$ , then  $\lim_{k \rightarrow \infty} c_k = 1/5$  and  $\lim_{k \rightarrow \infty} 3^k(c_1 \cdots c_k)^s = \lim_{k \rightarrow \infty} [\prod_{i=1}^k (1 + \frac{1}{2^i})]^s = \infty$ . Set  $n_k \equiv 3$ . Fix  $I_\phi = [0, 1]$  for empty word  $\phi$ . Given an interval  $I_{i_1 \cdots i_{k-1}} = [c, d]$  with  $i_1 \cdots i_{k-1} \in \{1, 2, 3\}^{k-1}$ , we put  $I_{i_1 \cdots i_{k-1}1} = [c, c + c_k(d - c)]$ ,  $I_{i_1 \cdots i_{k-1}2} = [\frac{c+d}{2} - \frac{c_k(d-c)}{2}, \frac{c+d}{2} + \frac{c_k(d-c)}{2}]$  and  $I_{i_1 \cdots i_{k-1}3} = [d - (d - c)c_k, d]$ . Let

$$F = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \cdots i_k \in \{1, 2, 3\}^k} I_{i_1 \cdots i_k}.$$

Using result in [16], we have  $\dim_H F = \dim_B F = s$ . We will show that  $F$  is not Ahlfors–David  $s$ -regular. On the contrary, there is a constant  $\alpha > 0$  such that for the corresponding Ahlfors–David  $s$ -regular measure  $\nu$ , we have  $(c_1 \cdots c_k)^s \leq \alpha \nu(I_{i_1 \cdots i_k})$  for all  $i_1 \cdots i_k \in \{1, 2, 3\}^k$ . Therefore,

$$3^k(c_1 \cdots c_k)^s \leq \alpha \sum_{i_1 \cdots i_k \in \{1, 2, 3\}^k} \nu(I_{i_1 \cdots i_k}) = \alpha \nu([0, 1]) < \infty.$$

Letting  $k \rightarrow \infty$ , we obtain a contradiction.

However, the Moran set in Example 2 is quasi Ahlfors–David  $s$ -regular.

**1.3. Quasi Ahlfors–David regularity.** We say that  $F$  is quasi Ahlfors–David  $s$ -regular [15], if there is a Borel measure  $\mu$  supported on  $F$  such that

$$\frac{\log \mu(B(x, r))}{\log r} \rightarrow s \text{ uniformly for } x \in F \text{ as } r \rightarrow 0. \quad (1.7)$$

Notice that Ahlfors–David regularity implies quasi Ahlfors–David regularity.

Can we find the conditions for a Moran set to be quasi Ahlfors–David regular? Now, we shall pose some assumptions upon the Moran sets:

- (A1):  $\lim_{k \rightarrow \infty} \frac{\log d_k}{\log(D_1 D_2 \cdots D_{k-1})} = 0$ ;
- (A2):  $\sup_k \frac{\log(n_1 n_2 \cdots n_k)}{\log(D_1 D_2 \cdots D_k)} < \infty$ ;
- (A3):  $\lim_{k \rightarrow \infty} s_k = s \in (0, \infty)$ .

*Remark 2.* (A1) means  $\frac{\log |\tilde{V}_{i_1 \dots i_{k-1} i_k}|}{\log |V_{i_1 \dots i_{k-1}}|} \rightarrow 1$  uniformly as  $k \rightarrow \infty$ , this is a natural assumption. We pose (A2) as a technical need.

*Remark 3.* If  $c_* = \inf_k d_k > 0$ , then (A1)–(A2) hold since  $\sup_k n_k \leq 1/(c_*)^n$  and  $0 < c_* \leq d_k \leq D_k \leq \sqrt[n]{1 - (c_*)^n}$  by (1.1)–(1.2).

*Remark 4.* (A3) means  $\dim_H F = \dim_B F = s$  in some sense as in [16]. If  $c_* = \inf_k d_k > 0$ , it follows from [16] that  $\dim_H F = s_*$  and  $s^* = \bar{\dim}_B F$ . Using (A3), we have  $\dim_H F = \dim_B F = s$ .

*Remark 5.* For self-similar set  $E$  satisfying the open set condition as above, we have  $D_k \equiv \max_i r_i$ ,  $d_k \equiv \min_i r_i$ ,  $n_k \equiv m$ ,  $c_{k,i} = r_i$  and  $s_k \equiv s$  with

$$(r_1)^s + \dots + (r_m)^s = 1.$$

It is obvious that assumptions (A1)–(A3) hold. The Ahlfors–David regularity of  $E$  supports the rationality of these assumptions.

Under these assumption, we get the quasi Ahlfors–David regularity.

**Theorem 1.** *Suppose  $F$  is a Moran set satisfying (A1)–(A3). Then  $F$  is quasi Ahlfors–David  $s$ -regular.*

Besides Example 2, we give the following planar Moran set satisfying (A1)–(A3).

*Example 3* ([3]). Consider a plane fractal defined as follows. Given two sequences  $\{l_k\}_{k \geq 1}$  and  $\{n_k\}_{k \geq 1}$  with  $n_k \leq l_k^2$  and

$$\lim_{k \rightarrow \infty} \frac{\log(n_1 \dots n_k)}{\log(l_1 \dots l_k)} = s.$$

Take a unit square  $[0, 1]^2$  as the initial set. In the first step, we divide  $[0, 1]^2$  into  $(l_1)^2$  equal squares with side length  $1/l_1$ , then we select  $n_1$  ones. By induction, assume that we get a square  $Q_{i_1 \dots i_{k-1}}$  ( $i_1 \dots i_{k-1} \in \tilde{\Sigma}^*$ ) of side length  $(l_1 \dots l_{k-1})^{-1}$ , we divide it into  $(l_k)^2$  equal squares with side length  $(l_1 \dots l_{k-1} l_k)^{-1}$ , and then we select  $n_k$  ones from them, denoted by  $\{Q_{i_1 \dots i_{k-1} i_k}\}_{i_k=1}^{n_k}$ . Again and again, we get a limit set which is a Moran set, where  $c_{k,i} \equiv (l_k)^{-1}$  and  $V_{i_1 \dots i_k}$  is the interior of  $Q_{i_1 \dots i_k}$ . It is easy to check that assumptions (A1)–(A3) hold.

**1.4. Quasi Lipschitz equivalence of Moran sets.** We say that two subsets  $A$  and  $B$  of Euclidean spaces are Lipschitz equivalent, if there is a bijection from  $A$  to  $B$  such that for all  $x, y \in A$ ,

$$C^{-1}|x - y| \leq |f(x) - f(y)| \leq C|x - y|$$

for some constant  $C > 0$ .

Even for self-similar sets, their Lipschitz equivalences are very difficult to study, for example see COOPER and PIGNATARO [1], DAVID and SEMMES [3], DENG and HE [4], DENG, WEN, XIONG and XI [5], FALCONER and MARSH [6], LAU and LUO [10], LLORENTE and MATTILA [9], RAO, RUAN and WANG [13], XI and XIONG [17], [19], [20], [21]. As we known, two dust-like self-similar sets with the same dimension may not be Lipschitz equivalent. But they are quasi Lipschitz equivalent.

The notion of quasi-Lipschitz equivalence is introduced by XI [18].

We say that two subsets  $A$  and  $B$  of Euclidean spaces are quasi-Lipschitz equivalent, if there is a bijection from  $A$  to  $B$  such that for all  $x, y \in A$ ,

$$\frac{\log |f(x) - f(y)|}{\log |x - y|} \rightarrow 1 \text{ uniformly as } |x - y| \rightarrow 0.$$

It is proved in [18] that two self-conformal sets are quasi-Lipschitz equivalent if and only if they have the same dimension. This result was developed by WANG and XI [14], [15], XIONG and XI [22].

LI *et al.* [8] studied the quasi-Lipschitz equivalence for some homogeneous Moran sets in line, a special class of Moran sets. For any Moran set in this class, we have  $c_{k,1} = \cdots = c_{k,n_k} = c_k$  and an additional assumption  $\lim_{k \rightarrow \infty} \frac{\log n_1 \cdots n_k}{\log c_1 \cdots c_k} = s \in (0, 1)$ . We see that the condition  $\lim_{k \rightarrow \infty} \frac{\log n_1 \cdots n_k}{\log c_1 \cdots c_k} = s$  is like (A3). But the condition  $s < 1$  plays an important role which is like the separation condition for self-similar sets.

In this paper, we study the general Moran sets under the separation assumption:

(A4): For any  $i_1 \cdots i_{k-1} i_k \in \tilde{\Sigma}^*$ ,

$$\frac{\log d(\bar{V}_{i_1 \cdots i_{k-1} i_k}, F \setminus \bar{V}_{i_1 \cdots i_{k-1} i_k})}{\log |\bar{V}_{i_1 \cdots i_{k-1}}|} \rightarrow 1 \text{ uniformly as } k \rightarrow \infty.$$

Here  $d(A, B)$  is the least distance between compact sets  $A$  and  $B$ .

For  $s > 0$ , a class  $\mathcal{A}_s$  of Moran sets is defined by

$$\mathcal{A}_s = \begin{cases} \{F : F \text{ is a Moran set satisfying (A1)–(A4)}\} & \text{if } s \geq 1, \\ \{F : F \text{ is a Moran set satisfying (A1)–(A3)}\} & \text{if } s < 1. \end{cases}$$

**Theorem 2.** *For every  $s > 0$ , any two Moran sets in  $\mathcal{A}_s$  are quasi-Lipschitz equivalent.*

We pose the last assumption:

(A5): There exists a constant  $c > 0$  such that for any  $i_1 \cdots i_{k-1} i_k \in \tilde{\Sigma}^*$ ,

$$\frac{|\bar{V}_{i_1 \cdots i_{k-1} i_k}|}{|\bar{V}_{i_1 \cdots i_{k-1}}|} = c_{k, i_k} \geq c \quad \text{and} \quad \frac{\min_{j_k \neq i_k} d(\bar{V}_{i_1 \cdots i_{k-1} i_k}, \bar{V}_{i_1 \cdots i_{k-1} j_k})}{|\bar{V}_{i_1 \cdots i_{k-1}}|} \geq c.$$

Then (A5) implies (A1)–(A2) (see Remark 3) and (A4).

Using Theorem 2, we have

**Theorem 3.** *Fix  $s > 0$ . Suppose  $F_1$  and  $F_2$  are Moran sets satisfying (A3) (with the same parameter  $s$ ) and (A5), then they are quasi-Lipschitz equivalent.*

*Remark 6.* In fact, (A3) and (A5) hold for self-similar sets satisfying the strong separation condition. Then we obtain the result in [18]: two dust-like self-similar sets are quasi-Lipschitz equivalent if and only if they have the same dimension.

The paper is organized as follows. Section 2 is the preliminaries, including the construction of the measure which plays an important role in this paper. Section 3 is the proof of Theorem 1. In Section 4, we get Theorem 2 based on the main result in [15]. In fact, [15] proved that if two compact sets are quasi Ahlfors–David  $s$ -regular and quasi uniformly disconnected, then they are quasi-Lipschitz equivalent.

## 2. Preliminaries

For  $\sigma = \sigma_1 \cdots \sigma_k \in \tilde{\Sigma}_*$ , let  $|\sigma| (= k)$  denote its length. Write

$$(\sigma_1 \cdots \sigma_k) * \sigma_{k+1} = \sigma_1 \cdots \sigma_k \sigma_{k+1}.$$

We say that  $\bar{V}_\sigma$  is of rank  $k$ , if  $|\sigma| = k$ . Without loss of generality, we assume the diameter

$$|\bar{V}_\emptyset| = 1.$$

If  $\sigma = \sigma_1 \cdots \sigma_k$ , then every  $\bar{V}_\sigma$  is similar to  $|\bar{V}_\emptyset|$  with ratio  $c_{1, \sigma_1} c_{2, \sigma_2} \cdots c_{k, \sigma_k}$  and

$$|\bar{V}_\sigma| = c_{1, \sigma_1} c_{2, \sigma_2} \cdots c_{k, \sigma_k}. \quad (2.1)$$

**2.1. Construction of measure.** Fix  $s > 0$ . As in [2] by DAI *et al.*, we can define a probability measure supported on  $F$  depending on  $s$ .

Let  $\mu(\bar{V}_\emptyset) = 1$ , where  $\emptyset$  is the empty word.

By induction, for every  $k \geq 1$  and  $\bar{V}_\sigma$  of rank  $(k - 1)$ , we define

$$\mu(\bar{V}_{\sigma * i}) = \frac{c_{k,i}^s}{\sum_{j=1}^{n_k} c_{k,j}^s} \mu(\bar{V}_\sigma) \quad \text{for } 1 \leq i \leq n_k.$$

In fact, if  $\sigma = \sigma_1 \cdots \sigma_k$ , then

$$\mu(\bar{V}_\sigma) = \frac{|\bar{V}_\sigma|^s}{\prod_{i=1}^k \left( \sum_{j=1}^{n_i} c_{i,j}^s \right)}. \quad (2.2)$$

More and more, we get a probability measure supported on  $F$ . In fact, we use the condition (1.1) during the above construction of measure, as in the way by HUTCHINSON [7] for the open set condition.

**2.2. Disconnectedness and quasi-Lipschitz equivalence.** As above, we introduce the Moran set on the analogy of the self-similar sets satisfying the open set condition. Now, we will give some property of disconnectedness on the analogy of the self-similar sets satisfying the strong separation condition (SSC in short), here SSC holds for  $E = \cup_{i=1}^m S_i(E)$ , if

$$\min_{i \neq j} d(S_i(E), S_j(E)) > 0,$$

where  $d(A, B)$  is the distance between  $A$  and  $B$ .

*Definition 1* ([15]). We say that a subset  $K$  of Euclidean space is *quasi uniformly disconnected* if there is a function  $\eta: (0, +\infty) \rightarrow (0, +\infty)$  with  $\lim_{t \rightarrow 0} \frac{\log \eta(t)}{\log t} = 1$  such that for any  $x \in K$  and  $r > 0$ , there is a subset  $B \subset K$  such that

$$K \cap B(x, \eta(r)) \subset B \subset B(x, r) \quad \text{and} \quad d(B, K \setminus B) > \eta(r). \quad (2.3)$$

We notice that if  $E$  is a self-similar set satisfying SSC, then  $E$  is quasi uniformly disconnected. In fact, we can take

$$\eta(r) = (\min_i r_i) \frac{\min_{i \neq j} d(S_i(E), S_j(E))}{|E|} \cdot r.$$

The following two lemmas come from [15] by WANG and XI.

**Lemma 1.** *Suppose that compact and quasi uniformly disconnected subsets  $E_1$  and  $E_2$  of Euclidean spaces are quasi Ahlfors–David  $s$ -regular. Then  $E_1$  and  $E_2$  are quasi Lipschitz equivalent.*

**Lemma 2.** *If  $E$  is quasi Ahlfors–David  $s$ -regular with  $s \in (0, 1)$ , then  $E$  is quasi uniformly disconnected.*

In fact, MATTILA and SAARANEN [12] proved that if  $E$  is Ahlfors–David  $s$ -regular with  $s \in (0, 1)$ , then  $E$  is uniformly disconnected.

### 3. Quasi Ahlfors–David Regularity

In this section, we will prove Theorem 1. Now, fix

$$s = \lim_{k \rightarrow \infty} s_k,$$

where  $\prod_{i=1}^k \left( \sum_{j=1}^{n_i} c_{i,j}^{s_k} \right) = 1$ .

**3.1. Measure of  $\bar{V}_\sigma$ .** We will estimate the measure of  $\bar{V}_\sigma$ .

**Lemma 3.** *There is a non-increasing function  $\delta : \mathbb{N} \rightarrow \mathbb{R}^+$  such that  $\lim_{k \rightarrow \infty} \delta(k) = 0$  and for any basic interval  $I_\sigma$ ,*

$$|\bar{V}_\sigma|^{s+\delta(|\sigma|)} \leq \mu(\bar{V}_\sigma) \leq |\bar{V}_\sigma|^{s-\delta(|\sigma|)}.$$

PROOF. Suppose  $|\sigma| = k$ . By (2.2), we only need to show

$$\frac{\log \prod_{i=1}^k \left( \sum_{j=1}^{n_i} c_{i,j}^s \right)}{\log |\bar{V}_\sigma|} \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (3.1)$$

In fact, using (2.1) and  $D_k = \max_i c_{k,i}$ , we have

$$|\log |\bar{V}_\sigma|| \geq |\log(D_1 \cdots D_k)|. \quad (3.2)$$

On the other hand, since  $\prod_{i=1}^k \left( \sum_{j=1}^{n_i} c_{i,j}^{s_k} \right) = 1$ , we have

$$\begin{aligned} & \left| \log \prod_{i=1}^k \left( \sum_{j=1}^{n_i} c_{i,j}^s \right) \right| \\ &= \left| \log \prod_{i=1}^k \left( \sum_{j=1}^{n_i} c_{i,j}^s \right) - \log \prod_{i=1}^k \left( \sum_{j=1}^{n_i} c_{i,j}^{s_k} \right) \right| = |g(s) - g(s_k)|, \end{aligned}$$

where

$$g(x) = \sum_{i=1}^k \log \left( \sum_{j=1}^{n_i} c_{i,j}^x \right).$$

By Mean value theorem, there exists  $\xi_k$  lying in the interval with endpoints  $s$  and  $s_k$  such that

$$|g(s) - g(s_k)| = (s_k - s) \sum_{i=1}^k \frac{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k} \log c_{i,j}}{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k}}. \quad (3.3)$$

Write  $c_{i,j}^{\sim} = \frac{c_{i,j}^{\xi_k}}{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k}}$ , then  $\sum_{j=1}^{n_i} c_{i,j}^{\sim} = 1$  and

$$\begin{aligned} \sum_{i=1}^k \frac{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k} \log c_{i,j}}{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k}} &= \frac{1}{\xi_k} \cdot \sum_{i=1}^k \frac{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k} \log c_{i,j}^{\xi_k}}{\sum_{j=1}^{n_i} c_{i,j}^{\xi_k}} \\ &= \frac{1}{\xi_k} \cdot \left( \sum_{i=1}^k \sum_{j=1}^{n_i} c_{i,j}^{\sim} \log c_{i,j}^{\sim} + \sum_{i=1}^k \log \sum_{j=1}^{n_i} c_{i,j}^{\xi_k} \right). \end{aligned} \quad (3.4)$$

Using the convexity of function  $f(x) = -x \log x$  on  $[0, 1]$ , we have

$$\left| \sum_{j=1}^{n_i} c_{i,j}^{\sim} \log c_{i,j}^{\sim} \right| \leq \log n_i \quad \text{and} \quad \prod_{i=1}^k D_i^{\xi_k} \leq \prod_{i=1}^k \sum_{j=1}^{n_i} c_{i,j}^{\xi_k} \leq \prod_{i=1}^k n_i. \quad (3.5)$$

Then it follows from assumption (A2) and (3.3)–(3.5) that

$$\lim_{k \rightarrow \infty} \frac{|\log \prod_{i=1}^k (\sum_{j=1}^{n_i} c_{i,j}^s)|}{\log(D_1 \cdots D_k)} = 0. \quad (3.6)$$

Therefore, we get (3.1) by (3.2) and (3.6).  $\square$

**3.2. Proof of Theorem 1.** For  $x \in F$  and  $r$  small enough, now we shall estimate

$$\frac{\log \mu(B(x, r))}{\log r}.$$

Let  $N_r = \min\{k : d_1 \cdots d_k \leq r\} \rightarrow \infty$  as  $r \rightarrow 0$ . Let  $\sigma^-$  be the father of  $\sigma$ , that is  $\sigma = \sigma^- * i_{|\sigma|}$  for some letter  $i_{|\sigma|}$ .

(1) *Lower bound:*

Using assumption (A1)  $\lim_{k \rightarrow \infty} \frac{\log d_k}{\log(D_1 D_2 \cdots D_{k-1})} = 0$ , we have

$$\lim_{|\sigma| \rightarrow \infty} \frac{\log |\bar{V}_\sigma|}{\log |\bar{V}_{\sigma^-}|} = 1, \quad (3.7)$$

then there exists a non-increasing function  $\alpha : \mathbb{N} \rightarrow (0, \infty)$  with  $\lim_{k \rightarrow \infty} \alpha(k) = 0$  such that

$$|\bar{V}_{\sigma-}| \geq |\bar{V}_{\sigma}| \geq |\bar{V}_{\sigma-}|^{1+\alpha(|\sigma|)}. \quad (3.8)$$

Take a word  $\sigma$  such that

$$|\bar{V}_{\sigma}| \leq r \quad \text{and} \quad |\bar{V}_{\sigma-}| > r, \quad (3.9)$$

Then by Lemma 3, we have

$$\mu(B(x, r)) \geq \mu(\bar{V}_{\sigma}) \geq |\bar{V}_{\sigma}|^{s+\delta(|\sigma|)}. \quad (3.10)$$

It follows from (3.8), (3.9) and (3.10) that

$$\mu(B(x, r)) \geq |\bar{V}_{\sigma-}|^{(s+\delta(|\sigma|))(1+\alpha(|\sigma|))} > r^{(s+\delta(N_r))(1+\alpha(N_r))}, \quad (3.11)$$

where

$$(s + \delta(N_r))(1 + \alpha(N_r)) \rightarrow s \text{ as } r \rightarrow 0.$$

(2) *Upper bound:*

Let

$$\Omega_{x,r} = \{\sigma : B(x, r) \cap \bar{V}_{\sigma} \neq \emptyset \quad \text{and} \quad |\bar{V}_{\sigma}| \leq r, |\bar{V}_{\sigma-}| > r\},$$

and  $l_{x,r} := \min_{\sigma \in \Omega_{x,r}} |\bar{V}_{\sigma}| = \min_{\sigma \in \Omega_{x,r}} |V_{\sigma}|$ .

It follows from Lemma 3 that for  $\sigma \in \Omega_{x,r}$ ,

$$\mu(\bar{V}_{\sigma}) \leq r^{s-\delta(N_r)}. \quad (3.12)$$

By (3.8), we have

$$l_{x,r} \geq r^{1+\alpha(N_r)} \text{ with } \alpha(N_r) \rightarrow 0 \text{ as } r \rightarrow 0. \quad (3.13)$$

Denote by  $\mathcal{L}^n$  the Lebesgue measure on  $\mathbb{R}^n$ . Since  $\bigcup_{\sigma \in \Omega_{x,r}} V_{\sigma}$  is a disjoint union contained in  $B(x, 2r)$ , we have

$$\begin{aligned} \mathcal{L}^n B(x, 2r) &\geq \mathcal{L}^n \left( \bigcup_{\sigma \in \Omega_{x,r}} V_{\sigma} \right) = \sum_{\sigma \in \Omega_{x,r}} \mathcal{L}^n(V_{\sigma}) \\ &= \sum_{\sigma \in \Omega_{x,r}} \mathcal{L}^n(V_{\emptyset}) \cdot \frac{|V_{\sigma}|^n}{|V_{\emptyset}|^n} \geq \frac{\mathcal{L}^n(V_{\emptyset})}{|V_{\emptyset}|^n} (\#\Omega_{x,r}) (l_{x,r})^n. \end{aligned}$$

This means the cardinality

$$\#\Omega_{x,r} \leq C_1 \frac{r^n}{(l_{x,r})^n} \quad (3.14)$$

for some constant  $C_1 > 0$ .

For any  $y \in F \cap B(x, r)$ , we can find a word  $\sigma$  with  $y \in \bar{V}_\sigma$  such that  $|\bar{V}_\sigma| \leq r$  and  $|\bar{V}_{\sigma-}| > r$ . That means

$$F \cap B(x, r) \subset \bigcup_{\sigma \in \Omega_{x,r}} \bar{V}_\sigma. \quad (3.15)$$

By (3.12)–(3.15), we have

$$\begin{aligned} \mu(B(x, r)) &= \mu(F \cap B(x, r)) \leq \mu\left(\bigcup_{\sigma \in \Omega_{x,r}} \bar{V}_\sigma\right) \leq \#\Omega_{x,r} \cdot \max_{\sigma \in \Omega_{x,r}} \mu(\bar{V}_\sigma) \\ &\leq C_1 \frac{r^n}{(l_{x,r})^n} \max_{\sigma \in \Omega_{x,r}} \mu(\bar{V}_\sigma) \leq C_1 \cdot r^{s-\delta(N_r)-n\alpha(N_r)}, \end{aligned}$$

where

$$s - \delta(N_r) - n\alpha(N_r) \rightarrow s \quad \text{as } r \rightarrow 0.$$

Then Theorem 1 follows from the estimates of lower and upper bounds.

#### 4. Quasi-Lipschitz equivalence

In this section, we will prove Theorem 2.

For  $E_1, E_2 \in \mathcal{A}_s$ , Theorem 1 implies that they are quasi Ahlfors–David  $s$ -regular.

By Lemmas 1 and 2, we only need to show that for  $s \geq 1$  any set in  $F \in \mathcal{A}_s$  is quasi uniformly disconnected by using assumption (A4).

In fact, given  $F \in \mathcal{A}_s$  with  $|\bar{V}_\emptyset| = 1$ , the assumption (A4) shows that for any  $\bar{V}_\sigma$ ,

$$\lim_{|\sigma| \rightarrow \infty} \frac{\log d(\bar{V}_\sigma, F \setminus \bar{V}_\sigma)}{\log |\bar{V}_{\sigma-}|} = 1. \quad (4.1)$$

Using assumption (A1)  $\lim_{k \rightarrow \infty} \frac{\log d_k}{\log(D_1 D_2 \cdots D_{k-1})} = 0$ , we have

$$\lim_{|\sigma| \rightarrow \infty} \frac{\log |\bar{V}_\sigma|}{\log |\bar{V}_{\sigma-}|} = 1. \quad (4.2)$$

By (4.1) and (4.2), we can take a non-increasing function  $\varepsilon : \mathbb{N} \rightarrow (0, \infty)$  with  $\lim_{k \rightarrow \infty} \varepsilon(k) = 0$  such that

$$|\bar{V}_{\sigma-}| \geq d(\bar{V}_\sigma, F \setminus \bar{V}_\sigma) \geq |\bar{V}_{\sigma-}|^{1+\varepsilon(|\sigma|)} \quad (4.3)$$

and

$$|\bar{V}_{\sigma-}| \geq |\bar{V}_\sigma| \geq |\bar{V}_{\sigma-}|^{1+\varepsilon(|\sigma|)}. \quad (4.4)$$

Take

$$\eta(r) = r^{1+\varepsilon(N_r)},$$

where  $N_r = \min\{k : d_1 \cdots d_k \leq r\} \rightarrow \infty$  as  $r \rightarrow 0$  which implies

$$\lim_{r \rightarrow 0} \frac{\log \eta(r)}{\log r} = 1.$$

For  $B(x, r)$  with  $x \in F$  and  $r$  small enough, take a word  $\sigma$  such that

$$|\bar{V}_\sigma| \leq r \quad \text{and} \quad |\bar{V}_{\sigma^-}| > r.$$

Then by (4.3)–(4.4), we have

$$F \cap B(x, \eta(r)) \subset \bar{V}_\sigma \subset B(x, r)$$

and

$$d(\bar{V}_\sigma, F \setminus \bar{V}_\sigma) \geq \eta(r).$$

Hence  $F$  is quasi uniformly disconnected.

Applying Lemma 1, we obtain Theorem 2.

## References

- [1] D. COOPER and T. PIGNATARO, On the shape of Cantor sets, *J. Diff. Geom.* **28** (1988), 203–221.
- [2] Y. X. DAI, Z. X. WEN, L. F. XI and Y. XIONG, *Ann. Acad. Sci. Fenn. Math.* **36** (2011), 139–151.
- [3] G. DAVID and S. SEMMES, Fractured Fractals and Broken Dreams: Self-Similar Geometry Through Metric and Measure, *Oxford University Press, Oxford*, 1997.
- [4] G. T. DENG and X. G. H, Lipschitz equivalence of fractal sets, *R. Sci. China Math.* **55** (2012), 2095–2107.
- [5] J. DENG, Z. Y. WEN, Y. XIONG, and L. F. XI, Bilipschitz embedding of self-similar sets, *J. Anal. Math.* **114** (2011), 63–97.
- [6] K. J. FALCONER and D. T. MARSH, On the Lipschitz equivalence of Cantor sets, *Mathematika* **39** (1992), 223–233.
- [7] J. E. HUTCHINSON, Fractals and self-similarity, *Indiana Univ. Math. J.* **30** (1981), 713–747.
- [8] H. LI, Q. WANG and L. F. XI, Classification of Moran fractals, *J. Math. Anal. Appl.* **378** (2011), 230–237.
- [9] M. LLORENTE and P. MATTILA, Lipschitz equivalence of subsets of self-conformal sets, *Nonlinearity* **23** (2010), 875–882.
- [10] J. J. LUO and K. S. LAU, Lipschitz equivalence of self-similar sets and hyperbolic boundaries, *Adv. Math.* **235** (2013), 555–579.
- [11] P. MATTILA, Geometry of sets and measures in Euclidean spaces, Fractals and Rectifiability, *Cambridge University Press, Cambridge*, 1995.

- [12] P. MATTILA and P. SAARANEN, *Ann. Acad. Sci. Fenn. Math.* **34** (2009), 487–502.
- [13] H. RAO, H. J. RUAN and Y. WANG, Lipschitz equivalence of Cantor sets and algebraic properties of contraction ratios, *Trans. Amer. Math. Soc.* **364** (2012), 1109–1126.
- [14] Q. WANG and L. F. XI, Quasi-Lipschitz equivalence of Ahlfors–David regular sets, *Nonlinearity* **24** (2011), 941–950.
- [15] Q. WANG and L. F. XI, Quasi-Lipschitz equivalence of quasi Ahlfors–David regular sets, *Sci. China Math.* **54** (2011), 2573–2582.
- [16] Z. Y. WEN, Moran sets and Moran classes, *Chinese Sci. Bull.* **46** (2001), 1849–1856.
- [17] L. F. XI, Lipschitz equivalence of self-conformal sets, *J. London Math. Soc. (2)* **70** (2004), 369–382.
- [18] L. F. XI, Quasi-Lipschitz equivalence of fractals, *Israel J. Math.* **160** (2007), 1–21.
- [19] L. F. XI, Lipschitz equivalence of dust-like self-similar sets, *Math. Z.* **266** (2010), 683–691.
- [20] L. F. XI and Y. XIONG, Self-similar sets with initial cubic patterns, *C. R. Math. Acad. Sci. Paris* **348** (2010), 15–20.
- [21] L. F. XI and Y. XIONG, Lipschitz equivalence of fractals generated by nested cubes, *Math. Z.* **271** (2012), 1287–1308.
- [22] L. F. XI and Y. XIONG, *Studia Math.*, Vol. 194, 2009, 197–205.

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