

Corrigendum to “Chern connection of a pseudo-Finsler metric as a family of affine connections”

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Abstract. In this note we give the correct statements of [2, Proposition 3.3 and Theorem 3.4] and a formula of the Chern curvature in terms of the curvature tensor R^V of the affine connection ∇^V and the Chern tensor P .

1. Curvature of two parameter maps

Throughout this note, we will use the same notation and conventions as in [2] with a small exception: in local calculation $(\Omega, (u^i)_{i=1}^n)$ denotes a chart on the base manifold M , and $(\pi^{-1}(\Omega), (x^i)_{i=1}^n, (y^i)_{i=1}^n)$ is the induced chart on TM . Then $x^i = u^i \circ \pi$, $y^i(v) = v(u^i)$ for every $v \in T_p M$, $p \in \Omega$. We agree to abbreviate composite mappings like $f \circ g$ as $f(g)$. Vector fields on Ω can naturally be regarded as local sections of the pull-back bundle $\pi^*(TM)$; we use such harmless identifications all the time.

Proposition 3.3 and Theorem 3.4 in [2] are not correct. In particular, the problem in Proposition 3.3 is that $R^V(V, U)W$ depends also on $D_\gamma^\gamma U(\gamma(a))$, so it is not independent of the extension of u . The corrected versions of such results

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are given in Theorems 2.2 and 1.1, respectively. In order to formulate the correct results we need to associate a curvature operator to every two-parameter map.

Let us begin with some definitions. Given a pseudo-Finsler manifold (M, L) and an L -admissible vector field V on $\Omega \subset M$, we define a $(1, 3)$ tensor P_V by

$$P_V(X, Y, Z) = \frac{\partial}{\partial t} (\nabla_X^{V+tZ} Y) |_{t=0},$$

where X, Y, Z are arbitrary smooth vector fields on Ω . Let us observe that the tensor P_V is symmetric in its first two arguments because ∇^V is torsion-free. Moreover, in coordinates P_V is given by

$$P_V(X, Y, Z) = X^j Y^k Z^l P_{jkl}^i(V) \frac{\partial}{\partial u^k},$$

where $P_{jkl}^i(V) = \frac{\partial \Gamma_{jk}^i}{\partial y^l}(V)$. In particular, it is clear that the value of $P_V(X, Y, Z)$ at a point $p \in \Omega$ depends only on $V(p)$ and not on the extension V used to compute it. From the homogeneity of ∇^V , namely, from the property $\nabla^{\lambda V} = \nabla^V$ for every $\lambda > 0$, it follows easily that

$$P_V(X, Y, V) = 0. \quad (1)$$

In [4, equation (7.23)], this tensor is called the *Chern curvature*. To avoid confusion with the Chern curvature in [2, equation (15)] we will refer to it as *Chern tensor*. Observe that there is a misprint in [2, equation (15)]. The right formula for Chern curvature is

$$R_v(V(q), U(q))W(q) = V^k U^l W^j(q) R_j^i{}_{kl}(v) \frac{\partial}{\partial u^i}(q), \quad q := \pi(v).$$

Let us also define the curvature of any L -admissible two parameter map

$$\Lambda : [a, b] \times (-\varepsilon, \varepsilon) \rightarrow M, \quad (t, s) \rightarrow \Lambda(t, s).$$

(Here ‘ L -admissible’ means that $\Lambda_t(t, s) \in A$ for every $(t, s) \in [a, b] \times (-\varepsilon, \varepsilon)$). To avoid problems with differentiability, we assume that this map can be extended to a smooth map defined in an open subset $(\bar{a}, \bar{b}) \times (-\varepsilon, \varepsilon) \subset \mathbb{R}^2$, with $[a, b] \subset (\bar{a}, \bar{b})$. Then for every smooth vector field W along Λ we define the curvature operator

$$R^\Lambda(W) := D_{\gamma_s}^{\Lambda_t} D_{\beta_t}^{\Lambda_t} W - D_{\beta_t}^{\Lambda_t} D_{\gamma_s}^{\Lambda_t} W,$$

(see the notation in [2, Section 3.1]). In the following theorem we will relate the curvature of a two parameter map with the Chern curvature and the tensor P_V . Then we obtain the correct version of [2, Theorem 3.4].

Theorem 1.1. *Let (M, L) be a pseudo-Finsler manifold. Consider an L -admissible smooth curve $\gamma : [a, b] \rightarrow M$, an L -admissible two parameter map $\Lambda : [a, b] \times (-\varepsilon, \varepsilon) \rightarrow M$ such that $\Lambda(\cdot, 0) = \gamma$ and a smooth vector field W along Λ . With the above notation,*

$$R^\Lambda(W) = R_\gamma(\dot{\gamma}, U)W + P_\gamma(U, W, D_\gamma^\dot{\gamma}\dot{\gamma}) - P_\gamma(\dot{\gamma}, W, D_\gamma^\dot{\gamma}U), \quad (2)$$

where U is the variational vector field of Λ along γ , namely, $U(t) = \Lambda_s(t, 0)$.

PROOF. Observe that using the formula for $D_{\gamma_s}^{\Lambda_t} D_{\beta_t}^{\Lambda_t} W - D_{\beta_t}^{\Lambda_t} D_{\gamma_s}^{\Lambda_t} W$ in the proof of [2, Proposition 3.3] and formulas (13) and (14) in [2], we get

$$\begin{aligned} R^\Lambda(W) = & \left[W^i U^j \dot{\gamma}^p \frac{\partial \Gamma_{ij}^k}{\partial x^p}(\dot{\gamma}) + W^i U^j \Lambda_{tt}^p \frac{\partial \Gamma_{ij}^k}{\partial y^p}(\dot{\gamma}) - W^i \dot{\gamma}^j U^p \frac{\partial \Gamma_{ij}^k}{\partial x^p}(\dot{\gamma}) \right. \\ & \left. - W^i \dot{\gamma}^j \Lambda_{ts}^p \frac{\partial \Gamma_{ij}^k}{\partial y^p}(\dot{\gamma}) + W^i U^j \dot{\gamma}^m (\Gamma_{ij}^l(\dot{\gamma}) \Gamma_{lm}^k(\dot{\gamma}) - \Gamma_{im}^l(\dot{\gamma}) \Gamma_{lj}^k(\dot{\gamma})) \right] \frac{\partial}{\partial u^k}(\gamma). \end{aligned}$$

Since

$$\Lambda_{tt}^p = (D_\gamma^\dot{\gamma}\dot{\gamma})^p - \dot{\gamma}^i \dot{\gamma}^j \Gamma_{ij}^p(\dot{\gamma}), \quad \Lambda_{ts}^p = (D_\gamma^\dot{\gamma}U)^p - \dot{\gamma}^i U^j \Gamma_{ij}^p(\dot{\gamma}),$$

we find that

$$\begin{aligned} R^\Lambda(W) = & \left[W^i U^j \dot{\gamma}^p \frac{\partial \Gamma_{ij}^k}{\partial x^p}(\dot{\gamma}) - W^i U^j \dot{\gamma}^m \dot{\gamma}^n \Gamma_{mn}^p(\dot{\gamma}) \frac{\partial \Gamma_{ij}^k}{\partial y^p}(\dot{\gamma}) \right. \\ & - W^i \dot{\gamma}^j U^p \frac{\partial \Gamma_{ij}^k}{\partial x^p}(\dot{\gamma}) + W^i \dot{\gamma}^j \dot{\gamma}^m U^n \Gamma_{mn}^p(\dot{\gamma}) \frac{\partial \Gamma_{ij}^k}{\partial x^p}(\dot{\gamma}) \\ & + W^i U^j (D_\gamma^\dot{\gamma}\dot{\gamma})^p \frac{\partial \Gamma_{ij}^k}{\partial y^p} - W^i \dot{\gamma}^j (D_\gamma^\dot{\gamma}U)^p \frac{\partial \Gamma_{ij}^k}{\partial y^p}(\dot{\gamma}) \\ & \left. + W^i U^j \dot{\gamma}^m (\Gamma_{ij}^l(\dot{\gamma}) \Gamma_{lm}^k(\dot{\gamma}) - \Gamma_{im}^l(\dot{\gamma}) \Gamma_{lj}^k(\dot{\gamma})) \right] \frac{\partial}{\partial u^k}(\gamma). \quad (3) \end{aligned}$$

Finally, (7) follows easily from the last relation and definitions taking into account that $\dot{\gamma}^i \Gamma_{ij}^k(\dot{\gamma}) = N_j^k(\dot{\gamma})$. $\square$

As a consequence of Theorem 1.1, we can define

$$R^\gamma(\dot{\gamma}, U)W := R^\Lambda(\tilde{W})$$

for any vector fields U and W along γ , where Λ is any L -admissible two parameter map such that $\Lambda(\cdot, 0) = \gamma$ and $\Lambda_s(t, 0) = U(t)$, and $\tilde{W}$ is any extension of W to Λ . The last theorem ensures that R^γ does not depend on the choice of two parameter map, neither on the extension of W .

Lemma 1.2. *Given a pseudo-Finsler manifold (M, L) and its Chern tensor P , we have that $P_v(v, v, u) = 0$ for any $v \in A$ and $u \in T_{\pi(v)}M$.*

PROOF. It is enough to show that $y^i y^j \frac{\partial \Gamma_{ij}^k}{\partial y^p} = 0$ for any $k = 1, \dots, n$. Using that $y^i \Gamma_{ij}^k = N_j^k$ we get

$$y^i \frac{\partial \Gamma_{ij}^k}{\partial y^l} = \frac{\partial N_j^k}{\partial y^l} - \Gamma_{lj}^k. \quad (4)$$

Since the functions $N_j^i(v)$ are positive homogeneous of degree one, we have

$$y^l \frac{\partial N_j^k}{\partial y^l} = N_j^k. \quad (5)$$

Using (4) and $y^j \Gamma_{lj}^k = N_l^k$, it follows that

$$y^j y^i \frac{\partial \Gamma_{ij}^k}{\partial y^l} = y^j \frac{\partial N_j^k}{\partial y^l} - y^j \Gamma_{lj}^k = y^j \frac{\partial N_j^k}{\partial y^l} - N_l^k. \quad (6)$$

Introduce the spray coefficients $G^i = \gamma_{jk}^i y^j y^k$ and observe that $\frac{1}{2} \frac{\partial G^i}{\partial y^j} = N_j^i$ (see [1, equation (3.8.2)]). Using the last relation and (5), we get

$$y^j \frac{\partial N_j^k}{\partial y^l} = y^j \frac{1}{2} \frac{\partial^2 G^k}{\partial y^l \partial y^j} = y^j \frac{\partial N_l^k}{\partial y^j} = N_l^k.$$

Substituting this in (6) we finally conclude that $y^j y^i \frac{\partial \Gamma_{ij}^k}{\partial y^l} = 0$. $\square$

Corollary 1.3. *For any L -admissible curve $\gamma : [a, b] \rightarrow M$ we have*

$$R^\gamma(\dot{\gamma}, U)\dot{\gamma} = R_{\dot{\gamma}(a)}(\dot{\gamma}(a), U)\dot{\gamma} + P_{\dot{\gamma}}(U, \dot{\gamma}, D_{\dot{\gamma}}\dot{\gamma}). \quad (7)$$

Therefore, the value of $R^\gamma(\dot{\gamma}, U)\dot{\gamma}$ at $t = s_0 \in [a, b]$ depends only on $U(s_0)$ and γ , and not on the particular extension of U used to compute it.

PROOF. A straightforward consequence from (2) and Lemma 1.2. $\square$

2. Relation with the curvature tensor R^V

Let us see how the curvature tensor R^V relates to the Chern curvature R_V and the Chern tensor P_V .

Theorem 2.1. *Let (M, L) be a pseudo-Finsler manifold, V an L -admissible vector field on an open subset $\Omega \subset M$, and let X, Y, Z be arbitrary smooth vector fields on Ω . Then*

$$R^V(X, Y)Z = R_V(X, Y)Z + P_V(Y, Z, \nabla_X^V V) - P_V(X, Z, \nabla_Y^V V). \quad (8)$$

PROOF. By definition, using the same notation as in the proof of [2, Proposition 3.3], over Ω we have

$$R^V(X, Y)Z = \left[Z^i Y^j X^p \frac{\partial \tilde{\Gamma}_{ij}^k}{\partial u^p} - Z^i X^j Y^p \frac{\partial \tilde{\Gamma}_{ij}^k}{\partial u^p} + Z^i Y^j X^m \left(\tilde{\Gamma}_{ij}^l \tilde{\Gamma}_{lm}^k - \tilde{\Gamma}_{im}^l \tilde{\Gamma}_{lj}^k \right) \right] \frac{\partial}{\partial u^k}. \quad (9)$$

Now observe that $\frac{\partial \tilde{\Gamma}_{ij}^k}{\partial u^p} = \frac{\partial \Gamma_{ij}^k}{\partial x^p}(V) + \frac{\partial V^l}{\partial u^p} \frac{\partial \Gamma_{ij}^k}{\partial y^l}(V)$ and $(\nabla_X^V V)^k = X^p \frac{\partial V^k}{\partial u^p} + X^p V^l \Gamma_{pl}^k(V)$. Using this, we conclude that

$$\begin{aligned} X^p \frac{\partial \tilde{\Gamma}_{ij}^k}{\partial u^p} &= X^p \frac{\partial \Gamma_{ij}^k}{\partial x^p}(V) + X^p \frac{\partial V^l}{\partial u^p} \frac{\partial \Gamma_{ij}^k}{\partial y^l}(V) \\ &= X^p \frac{\partial \Gamma_{ij}^k}{\partial x^p}(V) + (\nabla_X^V V)^l \frac{\partial \Gamma_{ij}^k}{\partial y^l}(V) - X^m V^n \Gamma_{mn}^l(V) \frac{\partial \Gamma_{ij}^k}{\partial y^l}(V). \end{aligned}$$

Taking into account that $V^n \Gamma_{mn}^l(V) = N_m^l(V)$, we get

$$X^p \frac{\partial \tilde{\Gamma}_{ij}^k}{\partial u^p} = X^p \frac{\partial \Gamma_{ij}^k}{\partial x^p}(V) + (\nabla_X^V V)^l \frac{\partial \Gamma_{ij}^k}{\partial y^l}(V) - X^m N_m^l(V) \frac{\partial \Gamma_{ij}^k}{\partial y^l}(V). \quad (10)$$

In the same way,

$$Y^p \frac{\partial \tilde{\Gamma}_{ij}^k}{\partial u^p} = Y^p \frac{\partial \Gamma_{ij}^k}{\partial x^p}(V) + (\nabla_Y^V V)^l \frac{\partial \Gamma_{ij}^k}{\partial y^l}(V) - Y^m N_m^l(V) \frac{\partial \Gamma_{ij}^k}{\partial y^l}(V). \quad (11)$$

Substituting (10) and (11) in (9), we obtain (8) in coordinates. $\square$

Finally we can give the correct version of [2, Proposition 3.3].

Theorem 2.2. *Let $\gamma : [a, b] \rightarrow M$ be a smooth embedded L -admissible curve and V an L -admissible smooth vector field defined on an open subset $\Omega \subset M$. Assume that $\gamma([a, b]) \subset \Omega$ and V coincides with $\dot{\gamma}$ along γ . Then*

$$R^\gamma(\dot{\gamma}, U)W = \left(R^V(V, \tilde{U})\tilde{W} + P_V(V, \tilde{W}, [\tilde{U}, V]) \right)(\gamma), \quad (12)$$

where U and W are smooth vector fields along γ , and $\tilde{U}, \tilde{W}$ are extensions of U, W to Ω , resp.

PROOF. This follows easily from (2) and (8) since ∇^V is torsion-free. $\square$

Remark 2.3. Observe that the expression in [2, Corollary 3.5] is valid. Indeed, more generally, for every $v \in A$ and $u, w \in T_{\pi(v)}M$, it holds that

$$K_v(u, w) = \frac{g_v((R^{\gamma_v}(\dot{\gamma}_v, U)W)(t_0), \dot{\gamma}_v(t_0))}{L(v)g_v(u, w) - g_v(v, u)g_v(v, w)}, \quad (13)$$

where γ_v is the geodesic such that $\dot{\gamma}_v(t_0) = v$ and U, W are arbitrary extensions of u, w along γ_v . Recall the $K_v(u, w)$, the predecessor of the flag curvature, is defined as

$$K_v(u, w) = \frac{g_v(R_v(v, u)w, v)}{L(v)g_v(u, w) - g_v(v, u)g_v(v, w)}.$$

To prove (13), we show that if γ is a geodesic, then

$$g_{\dot{\gamma}}(R^{\gamma}(\dot{\gamma}, U)W, \dot{\gamma}) = -g_{\dot{\gamma}}(R^{\gamma}(\dot{\gamma}, U)\dot{\gamma}, W) \quad (14)$$

where $g_{\dot{\gamma}}$ is given by the rule $g_{\dot{\gamma}}(X, Y) := g_{\dot{\gamma}(t)}(X(t), Y(t))$ for any two vector fields X, Y along γ . This holds trivially in the interior of the set where U is proportional to $\dot{\gamma}$, because in this case $[\tilde{U}, V]$ is proportional to V , and then applying (1) and the antisymmetry of R^V in its first two arguments to (12), we get $R^{\gamma}(\dot{\gamma}, U)W = R^{\gamma}(\dot{\gamma}, U)\dot{\gamma} = 0$. If $\dot{\gamma}$ and U are linearly independent, then we can choose extensions V and $\tilde{U}$ with $[\tilde{U}, V] = 0$, and (12) together with [2, Proposition 3.1] and [3, Lemma 3.10] conclude (14). By continuity we can extend (14) to the interval of definition of γ . As the right hand side of (14) does not depend on the extension U of u along γ , we can compute the left hand side assuming that $D_{\dot{\gamma}}U = 0$, and using (2), we get (13).

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