Publ. Math. Debrecen
86/1-2 (2015), 19–30
DOI: 10.5486/PMD.2015.5916

Characterization of finite simple group A_{p+3} by its order and degree pattern

By ALI MAHMOUDIFAR (Tehran) and BEHROOZ KHOSRAVI (Tehran)

Abstract. It is proved that some finite groups are OD-characterizable, i.e. they are uniquely determined by order and degree pattern. In [R. Kogani-Moghadam and A. R. Moghaddamfar, Groups with the same order and degree pattern, *Science China Mathematics*, 2012], the authors posed the following conjecture:
Conjecture. *All alternating groups A_m with $m \neq 10$ are* OD*-characterizable.*

Up to now it has been proved that this conjecture is correct for $m = p$, $p+1$, $p+2$, where p is a prime number. Also it has been proved that the conjecture is true for A_{106} and A_{112}. In this paper, by an example we show that this conjecture is not true in general and so we reformulate this conjecture as follows:
Conjecture. *If $m \neq 10$ is even, then all alternating groups A_m are* OD*-characterizable.*

Recently, the OD-characterization of A_{p+3}, where $p \neq 7$ and $p < 100$ has been proved. In this paper we continue this work and we prove that if $p \neq 7$ is a prime number, then the alternating group A_{p+3} is OD-characterizable. We note that this is the first work that verify an infinite family of alternating groups with connected prime graphs.

1. Introduction

In this paper every group is finite. If n is a natural number, then we denote by $\pi(n)$, the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. Also for a prime number p, n_p denotes the p-part of n, i.e. $n_p = p^k$ if $p^k \mid n$ but $p^{k+1} \nmid n$. If $p \in \pi(G)$, then the set of all Sylow p-subgroups

---

Mathematics Subject Classification: 20D05, 20D60, 20D08.
Key words and phrases: alternating group, prime graph, degree pattern, order, characterization.
The second author was supported in part by a grant from Institute for Research in Fundamental sciences, Tehran, Iran (IPM) (No. 92050120).

of G is denoted by $\mathrm{Syl}_p(G)$. The prime graph $GK(G)$ of a group G is defined as a graph with vertex set $\pi(G)$ in which two distinct primes $p, q \in \pi(G)$ are adjacent (and we write $p \sim q$) if G contains an element of order pq. A subset ρ of vertices of $GK(G)$ is called an independent subset of $GK(G)$, if there does not exist any edge between any two elements of ρ. The maximal number of vertices in the independent subsets of $GK(G)$ is denoted by $t(G)$ and we denote by $\rho(G)$ an independent subset of $GK(G)$ with $t(G)$ elements. Also the maximal number of vertices in the independent subsets of $GK(G)$ containing 2 is denoted by $t(2, G)$. If n is a natural number and p is a prime divisor of n, then we denote the p-part of n by n_p.

Definition 1.1 ([4]). Let G be a group and $\pi(G) = \{p_1, p_2, \ldots, p_k\}$, where $p_1 < p_2 < \cdots < p_k$. Then the degree pattern of G is defined as follows:

$$D(G) := (\deg(p_1), \deg(p_2), \ldots, \deg(p_k)),$$

where $\deg(p_i)$, $1 \leq i \leq k$, is the degree of vertex p_i in $GK(G)$.

Definition 1.2. ([4]) Let G be a finite group. We say that G is k-fold OD-characterizable if there exist exactly k non-isomorphic groups $H_1, H_2, \ldots, H_k$ such that $|H_i| = |G|$ and $D(H_i) = D(G)$, for $1 \leq i \leq k$. When $k = 1$, the finite group G is called OD-characterizable.

The k-fold OD-characterization of some finite groups are considered by some authors (see [3], [4], [9], [6], [10], [11]). If G is a finite group such that $D(G) = D(A_{10})$ and $|G| = |A_{10}|$, then $G \cong A_{10}$ or $G \cong J_2 \times \mathbb{Z}_3$ (see [11]). Therefore A_{10} is not OD-characterizable. In [4], the authors put the following conjecture:

Conjecture 1.3. *All the alternating groups A_m with $m \neq 10$ are OD-characterizable.*

Until now, the above conjecture is valid for $n = p$, $p + 1$, $p + 2$, where p is an odd prime number. We note that in these cases the prime graph of A_n is disconnected and p is an isolated vertex. In this paper, by an example, we show that the above conjecture is not true in general and so we reformulate this conjecture as follows:

Conjecture 1.4. *If $m \neq 10$ is even, then all alternating groups A_m are OD-characterizable.*

In [1], the OD-characterization of alternating group A_{p+3} is considered, where $p \neq 7$ and $p < 100$ is an odd prime number. In this paper, we continue their work and we prove the validity of Conjecture 1.4 for A_{p+3}, where p is an odd prime number. In fact we prove the following result:

Main Theorem 1.5. *If $p \neq 7$ is an odd prime, then A_{p+3} is* OD*-characterizable.*

We note that in this case the prime graph of A_{p+3} is a connected graph, unless $p+2$ is a prime number. This paper is the first paper giving the OD-characterization of an infinite family of alternating groups A_n, where $GK(A_n)$ is connected.

In [5], the following problem about OD-characterization of finite simple groups is possed:

Problem 1.6. *Is there a simple group which is k-fold OD-characterizable for $k \geq 3$?*

Finally we introduce some finite simple groups where these groups are k-fold OD-characterizable, where $k \geq 3$, as an answer to the above problem.

2. Preliminary results

Lemma 2.1 ([8, Proposition 1.1])**.** *Let $G = A_n$ be an alternating group of degree n.*

(1) *Let $r, s \in \pi(G)$ be odd primes. Then r, s are nonadjacent if and only if $r + s > n$.*

(2) *Let $r \in \pi(G)$ be an odd prime. Then 2, r are nonadjacent if and only if $r + 4 > n$.*

Lemma 2.2 ([4, Lemma 2.15])**.** *Let S be a symmetric (resp., an alternating) group and let G be a finite group with $\pi(G) = \pi(S)$ and $D(G) = D(S)$. Then the prime graph $GK(G)$ coincides with $GK(S)$.*

Lemma 2.3. ([12]) *Let q and $s \geq 2$ be some natural numbers. Then one of the following holds:*

(a) *there exists a prime p such that p divides $q^s - 1$ and p does not divide $q^t - 1$ for all natural numbers $t < s$;*

(b) *$s = 6$ and $q = 2$;*

(c) *$s = 2$ and $q = 2^t - 1$, for some t.*

A prime p satisfying condition (a) of Lemma 2.3, is said to be a primitive prime divisor of $q^s - 1$. Also a primitive prime divisor of $q^n - 1$ is denoted by $r_n(q)$ or briefly r_n.

Lemma 2.4 ([2, Lemma 2.12]). *Let n, s and q be positive integers and x be an odd prime. If q is odd, then $\left(\frac{q^{2n}-1}{q^2-1}\right)_2 = n_2$. If $x \mid (q^s - 1)$ and $s \mid n$, then $\left(\frac{q^n-1}{q^s-1}\right)_x$ divides $(n/s)_x$.*

Lemma 2.5. *Let $q > 1$ be an integer.*

(i) *If $n \geq 5$ is a prime, then $3 \nmid (q^n - 1)/(q - 1)$ and $3 \nmid (q^n + 1)/(q + 1)$.*

(ii) *If $n = 2^\beta$ for some $\beta > 0$, then $3 \nmid (q^n + 1)$.*

Proof. (i) If $3 \mid q$, obviously we get the result. If $3 \mid (q - 1)$, then by Lemma 2.4, $((q^n - 1)/(q - 1))_3 = n_3 = 1$. If $3 \mid (q + 1)$, then $3 \nmid (q^n - 1)$, since n is odd. Also $(q^n + 1) \mid (q^{2n} - 1)$ and by the above discussion $3 \nmid (q^{2n} - 1)/(q^2 - 1)$.

(ii) Obviously the result holds if $3 \mid q$. Otherwise, $q \equiv \pm 1 \pmod 3$. Hence we have $q^n \equiv 1 \pmod 3$ and so $3 \nmid (q^n + 1)$. □

Lemma 2.6 ([8, Lemma 1.2]). *With the notations in* [8], *we have:*

(1) *Every maximal torus T of $G = A^{\varepsilon}_{n-1}(q)$ $(\varepsilon \in \{+, -\})$ has the order*

$$\frac{1}{(n, q - \varepsilon 1)(q - \varepsilon 1)}(q^{n_1} - (\varepsilon 1)^{n_1})(q^{n_2} - (\varepsilon 1)^{n_2}) \dots (q^{n_k} - (\varepsilon 1)^{n_k})$$

for an appropriate partition $n_1 + n_2 + \cdots + n_k = n$ of n. Moreover, for every partition, there exists a torus of corresponding order.

(2) *Every maximal torus T of G, where $G = B_n(q)$ or $G = C_n(q)$, has the order*

$$\frac{1}{(2, q - 1)}(q^{n_1} - 1)(q^{n_2} - 1) \dots (q^{n_k} - 1)(q^{l_1} + 1)(q^{l_2} + 1) \dots (q^{l_m} + 1)$$

for an appropriate partition $n_1 + n_2 + \cdots + n_k + l_1 + l_2 + \cdots + l_m = n$ of n. Moreover, for every partition, there exists a torus of corresponding order.

(3) *Every maximal torus T of $G = D^{\varepsilon}_n(q)$ has the order*

$$\frac{1}{(4, q^n - \varepsilon 1)}(q^{n_1} - 1)(q^{n_2} - 1) \dots (q^{n_k} - 1)(q^{l_1} + 1)(q^{l_2} + 1) \dots (q^{l_m} + 1)$$

for an appropriate partition $n_1 + n_2 + \cdots + n_k + l_1 + l_2 + \cdots + l_m = n$ of n, where m is even if $\varepsilon = +$ and m is odd if $\varepsilon = -$. Moreover, for every partition, there exists a torus of corresponding order.

3. Proof of the Main Theorem

First we prove some result about the OD-characterization of an alternating group with degree m, where m is a natural number greater that 106. Then we consider the special case $m = p+3$, where p is an odd prime number. So we have the following lemma.

Lemma 3.1. *Let G be a finite group such that $|G| = |A_m|$ and $D(G) = D(A_m)$, where $m \geq 106$. Also let:*

$$T := \Big\{t \text{ is a prime number} \,\Big|\, \frac{m}{2} < t \leq m\Big\}.$$

Then the following assertions hold:

(1) *T is an independent subset of $GK(G)$. If $t \in T$, then $|G|_t = t$ and also $|T| \geq 9$.*

(2) *There exists a nonabelian simple group S such that:*

$$S \leq \bar{G} := \frac{G}{N} \leq \mathrm{Aut}(S),$$

where N is the largest normal subgroup of G with $\pi(N) \cap T = \emptyset$. Also we have $T \subseteq \pi(S)$ and $\pi(\bar{G}/S) \cap T = \emptyset$.

(3) *The simple group S is isomorphic to a classical simple group of Lie type or an alternating group.*

Proof. (1) Since $|G| = |A_m|$ and $D(G) = D(A_m)$, by Lemma 2.2, we get that $GK(G) = GK(A_m)$. Also by the definition of T and Lemma 2.1, it is obvious that T is an independent subset of $GK(G)$. Since $|G| = m!/2$, if $t \in T$, then $|G|_t = t$.

Now since $m \geq 106$, by [7, Corollary 3], the number of prime numbers t such that $m/2 < t \leq m$, is bigger than $(3m/2)/(5\ln(m/2))$. So by the definition of T, we get that

$$|T| > \frac{3m/2}{5\ln(m/2)} \geq \frac{3(53)}{5\ln(53)} > 8,$$

which implies that $|T| \geq 9$.

(2) Let N be the largest normal subgroup of G such that $\pi(N) \cap T = \emptyset$. Also let S be a minimal normal subgroup of $\bar{G} := G/N$.

By the properties of N, we get that $\pi(S) \cap T \neq \emptyset$. Also we know that $S = S_1 \times \cdots \times S_k$, where S_i's are isomorphic to a simple group. So since $\pi(S) \cap T \neq \emptyset$, we get that for each $1 \leq i \leq k$, $\pi(S_i) \cap T \neq \emptyset$. On the other hand by (1), we conclude that if $t \in \pi(S) \cap T$, then $|S|_t = t$. It follows that $k = 1$ and so S is a simple group.

By (1), we have $|T| \geq 9$. Also by the above discussion, $\pi(S) \cap T \neq \emptyset$. We claim that $T \subseteq \pi(S)$. On the contrary, let $t_1 \in \pi(S) \cap T$ and $t_2 \in T \setminus \pi(S)$. By the Frattini argument, if $\bar{Q} \in \mathrm{Syl}_{t_1}(S)$, then $\bar{G} = SN_{\bar{G}}(\bar{Q})$. Thus since $t_2 \in T \setminus \pi(S)$, we get that $t_2 \in \pi(N_{\bar{G}}(\bar{Q}))$. So we may assume that $\bar{G}$ contains a subgroup $\bar{H}$ such that $|\bar{H}| = t_2$ and $\bar{Q} \rtimes \bar{H}$ is a subgroup of $\bar{G}$. Hence since t_1 and t_2 are nonadjacent in $GK(G)$, we conclude that $\bar{Q} \rtimes \bar{H}$ is a Frobenius subgroup of $\bar{G}$. This implies that $|\bar{H}| \mid (|\bar{Q}| - 1)$ and so $t_2 \mid (t_1 - 1)$. On the other hand, we have $m/2 < t_2 < t_1 \leq m$, which implies that $t_2 \nmid (t_1 - 1)$. So we get a contradiction and so $T \subseteq \pi(S)$. So since $|T| \geq 9$, we get that S is a nonabelian simple group. Also since for every $t \in T$, $|G|_t = t$, we deduce that $\pi(\bar{G}/S) \cap T = \emptyset$.

Let $C_{\bar{G}}(S) \neq 1$. Similarly to the above, we get that $\pi(C_{\bar{G}}(S)) \cap T \neq \emptyset$. On the other hand S is a nonabelian simple group, which implies that $S \cap C_{\bar{G}}(S) = 1$. Also by the above argument, $\pi(\bar{G}/S) \cap T = \emptyset$, which contradicts to $\pi(C_{\bar{G}}(S)) \cap T \neq \emptyset$. Therefore $C_{\bar{G}}(S) = 1$ and so we have:

$$S \leq \bar{G} = \frac{G}{N} \leq \mathrm{Aut}(S).$$

(3) We note that by the classification of finite simple groups, the nonableian simple group S is isomorphic to a sporadic simple group, an exceptional simple group of Lie type, a classical simple group of Lie type or an alternating group.

Let S be isomorphic to a sporadic simple group. By (2), we know that $\pi(S)$ contains every prime number t such that $m/2 < t \leq m$, where $m \geq 106$. This implies that there is at least a prime number $t > 100$ such that $t \in \pi(S)$. So by the orders of the sporadic simple groups we get a contradiction.

Now let S be isomorphic to an exceptional simple group of Lie type. By (2), we have $t(S) \geq |T| \geq 9$. Therefore by Table 9 in [8], it follows that $S \cong E_8(q)$. Also by this table, we get that

$$\rho(S) = \{r_5, r_7, r_8, r_9, r_{10}, r_{12}, r_{14}, r_{15}, r_{18}, r_{20}, r_{24}, r_{30}\}.$$

By (2), there exist at least 9 prime numbers $t_1, t_2, \ldots, t_9$ in $\pi(S)$ which are nonadjacent to each other in $GK(S)$ and $|S|_{t_i} = t_i$, for $1 \leq i \leq 9$. The order of $E_8(q)$ shows that if r is a prime divisor of $|E_8(q)|$ and $r \notin A := \{r_7, r_9, r_{14}, r_{15}, r_{18}, r_{20}, r_{24}, r_{30}\}$, then $r^2 \mid |E_8(q)|$. Now since $|A| = 8$, we get a contradiction.

Therefore S is isomorphic to a classical simple group of Lie type or an alternating group. $\square$

Now by the previous lemma, we consider the special case $m = p + 3$, where p is a prime number.

Proof of the Main Theorem. Let G be a finite group such that $|G| = |A_{p+3}| = (p + 3)!/2$ and $D(G) = D(A_{p+3})$, where p is a prime number. We note that if $p < 100$ and $p \neq 7$, the conjecture is proved in [1]. Also if $p' = p + 2$ is a prime number, then $A_{p+3} = A_{p'+1}$, which is considered in [4]. Therefore we suppose that $p \geq 101$ is a prime number such that $p+2$ is not a prime so $p \geq 103$. Let

$$T := \{t \text{ is a prime number} \mid (p + 3)/2 < t \leq p + 3\}.$$

By Lemma 3.1, there exists a nonabelian simple group S such that:

$$S \leq \bar{G} := \frac{G}{N} \leq \mathrm{Aut}(S),$$

where N is the largest normal subgroup of G with $\pi(N) \cap T = \emptyset$. Also by Lemma 3.1, $T \subseteq \pi(S)$, which implies that $p \in \pi(S)$ and S is isomorphic to a classical simple group of Lie type or an alternating group.

In the sequel we consider each possibility for S, separately:

(1) Let $S \cong A_{n-1}(q)$. By Lemma 3.2 and Table 8 in [8], $t(S) = [(n + 1)/2] \geq 9$ and so $n \geq 17$. Also $p \nsim 2$ in $GK(S)$. Then by Table 6 in [8], p is a primitive prime divisor of $q^n - 1$ or $q^{n-1} - 1$.

(1.a) Let p be a primitive prime divisor of $q^n - 1$. By Lemma 2.6, S contains an abelian subgroup of order $(q^n - 1)/((q - 1)(n, q - 1))$. Therefore every prime divisor of $(q^n - 1)/((q - 1)(n, q - 1))$ is adjacent to p in $GK(G)$. On the other hand p is only adjacent to 3 in $GK(G)$. This implies that $\pi((q^n - 1)/((q - 1)(n, q - 1))) \subseteq \{p, 3\}$.

Suppose that n is not a prime number. Also let m be the largest divisor of n such that $m < n$. Since $n \geq 17$, easily we get that $m > 4$ and $m \neq 6$. Hence by Lemma 2.3, there exists a primitive prime divisor u of $q^m - 1$. Since $m < n$, we get that $u \neq p$. Also $m > 4$ implies that $u \neq 3$. So we get a contradiction since $\pi(q^m - 1) \subseteq \pi(q^n - 1)$. Therefore n is an odd prime number.

By Lemma 2.5, we know that $3 \nmid (q^n - 1)/((q - 1)(n, q - 1))$. Therefore we have $\pi((q^n - 1)/((q - 1)(n, q - 1))) = \{p\}$. Since $|G|_p = |S|_p = p$, we conclude that $p = (q^n - 1)/((q - 1)(n, q - 1))$.

(1.b) Let p be a primitive prime divisor of $q^{n-1} - 1$. By Lemma 2.6, S contains an abelian subgroup of order $(q^{n-1} - 1)/(n, q - 1)(q - 1)$. Similarly to the above, this shows that $\pi((q^{n-1}-1)/(n,q-1)(q-1)) \subseteq \{p,3\}$ which implies that $n-1 \geq 16$ is an odd prime number and $p = (q^{n-1}-1)/(n,q-1)(q-1)$.

Now let $t \in T \subseteq \pi(S)$. Then in each case, we get that $t > (p+3)/2 > q^{n-4}$. On the other hand, by the order of S, we know that t is a primitive prime divisor of $q^m - 1$, where $1 \leq m \leq n$. Therefore $q^m \geq t > q^{n-4}$, which implies that t is a primitive prime divisor of $q^n - 1$, $q^{n-1} - 1$, $q^{n-2} - 1$ or $q^{n-3} - 1$. This shows that $|T| \leq 4$, which is a contradiction since $|T| \geq 9$.

(2) Let $S \cong {}^2A_{n-1}(q)$. By Table 8 in [8], we have $t(S) = [(n+1)/2] \geq 9$ and so $n \geq 17$. Also by Table 6 in [8], we get that p is a primitive prime divisor of $q^{2n}-1$, $q^{2n-2}-1$, q^n-1, $q^{(n-1)/2}-1$ or $q^{n/2}-1$.

(2.a) Let p be a primitive prime divisor of $q^{2n}-1$. By Tables in [8], n is odd. Also by Lemma 2.6, S contains an abelian subgroup of order $(q^n+1)/((q+1)(n,q+1))$. Similarly to the above, we get that n is an odd prime number and $p = (q^n+1)/((q+1)(n,q+1))$.

(2.b) Let p be a primitive prime divisor of $q^{2n-2}-1$. By Tables 4 and 6 in [8], we get that n is an even number. In this case since S contains an abelian subgroup of order $(q^{n-1}+1)/(n,q+1)$, similarly to the above we conclude that $p = (q^{n-1}+1)/(n,q+1)$.

Now let $t \in T$. Then in Cases (2.a) and (2.b), we have $t > (p+3)/2 > q^{n-4}$. Similarly to the above, we get that $|T| \leq 7$, which is a contradiction, since $|T| \geq 9$.

(2.c) Let p be a primitive prime divisor of q^n-1, $q^{n/2}-1$ or $q^{(n-1)/2}-1$. By Lemma 2.6, S contains an abelian subgroup of order $(q^n-1)/((n,q+1)(q+1))$ if n is even and $(q^{n-1}-1)/(n,q+1)$ if n is odd. This implies that p is adjacent to some prime number different from 3 in $GK(S)$, which is a contradiction.

(3) Let $S \cong B_n(q)$ or $S \cong C_n(q)$. By Table 8 in [8], we have $t(S) = [(3n+5)/4] \geq 9$ and so $n \geq 11$. Also by Table 6 in [8], we get that p is a primitive prime divisor of q^n-1 or $q^{2n}-1$.

(3.a) Let p be a primitive prime divisor of q^n-1. Using the maximal torus of order $(q^n-1)/(2,q-1)(q-1)$, we conclude that n is an odd prime number and $p = (q^n-1)/((2,q-1)(q-1))$.

(3.b) Let p be a primitive prime divisor of $q^{2n}-1$. Using the maximal torus of order $(q^n+1)/(2,q-1)$, we get that n is an odd prime number or a power of 2.

Hence $p = (q^n+1)/(2,q-1)$ if $n = 2^\beta$ and $p = (q^n+1)/((q+1)(2,q-1))$ if n is an odd prime, by Lemma 2.5.

Now let $t \in T$. By the above discussion, we get that $t > (p+3)/2 > q^{n-3}$. This implies that $|T| \leq 6$, which is a contradiction.

(4) Let $S \cong D_n(q)$. By Table 8 in [8], we get that $(3n+3)/4 \geq t(S) \geq 9$ and so $n \geq 11$. Also since $p \nsim 2$ in $GK(S)$, by Tables 4 and 6 in [8], we get that p is a primitive prime divisor of $q^n - 1$, $q^{n-1} - 1$ or $q^{2(n-1)} - 1$.

(4.a) Let p be a primitive prime divisor of $q^n - 1$. Then by Tables 4 and 6 in [8], n is odd. Using the maximal torus of order $(q^n - 1)/(4, q^n - 1)(q - 1)$, we get that $n \geq 11$ is an odd prime number and $p = (q^n - 1)/(4, q^n - 1)(q - 1)$.

(4.b) Let p be a primitive prime divisor of $q^{2(n-1)} - 1$. Using the maximal torus of order $(q^{n-1}+1)(q+1)/(4, q^n - 1)$, we conclude that $n-1$ is an odd prime number or a power of 2.

Hence we get that $\pi(q^{n-1}+1) \subseteq \{2, 3, p\}$. Now we consider the following cases:

(i) Let q be a power of 2 and $n-1$ is a power of 2. Then $\pi(q^{n-1}+1) \subseteq \{3, p\}$ and so by Lemma 2.5, we conclude that $\pi(q^{n-1}+1) \subseteq \{p\}$. Hence similarly to the above we have $p = q^{n-1}+1$.

(ii) Let q be a power of 2 and $n-1$ is an odd prime number. Then $\pi(q^{n-1}+1) \subseteq \{3, p\}$ and so by Lemma 2.5, we get that $\pi((q^{n-1}+1)/(q+1)) \subseteq \{p\}$. Hence similarly to the above we get that $p = (q^{n-1}+1)/(q+1)$.

(iii) Let q be an odd number and $n-1$ is a power of 2. Then by Lemma 2.5, we get that $\pi(q^{n-1}+1) \subseteq \{2, p\}$. Since $n-1$ is a power of 2, we have $4 \mid (q^{n-1}-1)$ and so $(q^{n-1}+1)_2 = 2$. Therefore $\pi((q^{n-1}+1)/2) \subseteq \{p\}$ and so $p = (q^{n-1}+1)/2$.

(iv) Let q be an odd number and $n-1$ is an odd prime number. Then by Lemma 2.4, we get that $2 \nmid (q^{n-1}+1)/(q+1)$. This implies that $\pi((q^{n-1}+1)/(q+1)) \subseteq \{3, p\}$ and so by Lemma 2.5, we conclude that $\pi((q^{n-1}+1)/(q+1)) \subseteq \{p\}$. Hence similarly to the above we get that $p = (q^{n-1}+1)/(q+1)$.

Therefore in each case we get that $p \geq (q^{n-1}+1)/(q+1)$.

(4.c) Let p be a primitive prime divisor of $q^{n-1} - 1$. Using the maximal torus of order $(q^{n-1}-1)(q-1)/(4, q^n - 1)$, we conclude that $n-1$ is an odd prime number and $p = (q^{n-1}-1)/(q-1)$.

Now let $t \in T$. By the above discussion we have $t > (p+3)/2 > q^{n-4}$. This implies that $|T| \leq 8$, which is a contradiction.

If $S \cong {}^2D_n(q)$, then similarly we get a contradiction and for convenience we omit the proof.

By the above discussion, we get that S is isomorphic to an alternating group A_m, where $m \geq 5$.

Hence we have $A_m \leq G/N \leq \mathrm{Aut}(A_m)$. Since $p \in \pi(S)$, we get that $m \geq p \geq 103$. This implies that $A_m \leq G/N \leq S_m$ and so $|G| = |A_m| \cdot |N|$ or $|G| = 2|A_m| \cdot |N|$. Also since $|A_m| \mid |G| = (p+3)!/2$, we conclude that $p \leq m \leq p+3$. Therefore $A_m \cong A_p$, A_{p+1}, A_{p+2} or A_{p+3}.

We claim that $S \cong A_{p+3}$. Otherwise let $S \cong A_m$, where $p \leq m \leq p+2$. In each case we can easily see that $(p+3)/2 \mid |N|$. Thus we have $|N| = k(p+3)/2$, where $k \mid 2(p+1)(p+2)$.

First we show that $(p+3)/2$ is a power of 2. Let $r \in \pi((p+3)/2)$ be an odd prime number and $((p+3)/2)_r = r^\alpha$, for some natural number α. We note that $r \neq 3$. Since $|N| = k(p+3)/2$, where $k \mid 2(p+1)(p+2)$, we get that $|N|_r = ((p+3)/2)_r = r^\alpha$.

So let $R \in \mathrm{Syl}_r(N)$. Then by the Frattini argument we have $G = N_G(R)N$. This implies that $p \in \pi(N_G(R))$ and so there exists a subgroup of G, like $R \rtimes P$, where $P \in \mathrm{Syl}_p(N_G(R))$. Since $r \neq 3$ and $r \not\sim p$ in $GK(G)$ we get that $R \rtimes P$ is a Frobenius subgroup of G.

Therefore we conclude that $|P| \mid (|R| - 1)$. Since $R \in \mathrm{Syl}_r(N)$ and $P \in \mathrm{Syl}_p(N_G(R))$, we have $|R| = |N|_r = ((p+3)/2)_r = r^\alpha$ and $|P| = |N_G(R)|_p = |G|_p = p$. Therefore $p \mid (r^\alpha - 1)$, which is a contradiction, since $r^\alpha \leq (p+3)/2 < p$.

Hence we conclude that $(p+3)/2$ is a power of 2. So let $(p+3)/2 = 2^\beta$, where $\beta > 1$ is a natural number. Since $|N| = k(p+3)/2$, where $k \mid 2(p+1)(p+2)$, we get that $|N|_2 = 2^\beta$, $2^{\beta+1}$ or $2^{\beta+2}$. Similarly to the above if $Q \in \mathrm{Syl}_2(N)$, then we conclude that G contains a Frobenius subgroup $Q \rtimes P$, where $P \in \mathrm{Syl}_p(N_G(Q))$. So we have the following cases:

(1) If $|N|_2 = 2^\beta$, then $p \mid (2^\beta - 1)$, which is impossible since $2^\beta = (p+3)/2 < p$.

(2) If $|N|_2 = 2^{\beta+1}$, then $p \mid (2^{\beta+1} - 1)$. Since $2^\beta = (p+3)/2$, we get that $p \mid (p+2)$, which is a contradiction.

(3) If $|N|_2 = 2^{\beta+2}$, then $p \mid (2^{\beta+2} - 1)$. Since $2^\beta = (p+3)/2$, we get that $p \mid (2p+5)$, which is a contradiction.

Therefore $S \cong A_{p+3}$ and so $A_{p+3} \leq G/N \leq S_{p+3}$. Since $|G| = |A_{p+3}|$, we conclude that $|N| = 1$ and so $G \cong A_{p+3}$. Therefore A_{p+3} is OD-characterizable. $\square$

Proposition 3.2. *Let* $m = 3^a \cdot 5^b$ *(or* $3^a \cdot 5^b \cdot 7^c$, *if* $a > 0$*) with* $m > 3$. *Suppose that neither* $m-2$ *nor* $m-4$ *is a prime. Let* $G = A_m$. *Let* R *be any finite group with* $|R| = m$ *and let* $H = A_{m-1} \times R$. *Then* G *and* H *have the same order and degree pattern. In particular,* G *is not OD-characterizable.*

Proof. Clearly, $|G| = |H|$. Let $\pi(G) = \{p_1, p_2, \ldots, p_k\}$ with $p_1 < p_2 < \ldots < p_k$. By the hypothesis, $p_k \leq m-6$ and if $a > 0$, $p_k \leq m-8$. Thus, each vertex in the prime graph of G and H is equal or connected to 2, 3 and 5 and also to 7 if $a > 0$. Now if $7 < p_i < p_j$, then p_i is connected to p_j in the prime graph of G if and only if $p_i + p_j \leq m$. Likewise, p_i and p_j are connected in the prime graph of H if and only if $p_i + p_j \leq m - 1$. Thus, connectivity differs if and only if $p_i + p_j = m$. But $p_i + p_j$ is even and m is odd. Hence G and H have the same degree pattern. □

Remark 3.3. We note that if $m = 5^3$ or $m = 3^{14}$, then the assumptions of the above proposition satisfy and so Conjecture 1.1 is not true in general. Also this implies that there are some finite simple groups, like A_{125} and $A_{3^{14}}$, which are k-fold OD-characterizable, where $k \geq 3$, which is an answer to Problem 1.1.

Acknowledgements. The authors would like to thank the referees for their constructive comments and valuable suggestions specially in Proposition 3.1, which improved the manuscript seriously.

The work of B. Khosravi was supported in part by a grant from IPM (School of Mathematics, Institute for Research in Fundamental Sciences, Tehran, Iran) (grant no. 92050120).

References

[1] A. A. Hoseini and A. R. Moghaddamfar, Recognizing alternating groups A_{p+3} for certain primes p by their orders and degree patterns, *Front. Math. China* **5** (2010), 541–553.

[2] B. Khosravi, B. Khosraviand B. Khosravi, Characterizability of $\mathrm{PSL}(p+1, q)$ by its order components, *Houston J. Math.* **32** (2006), 683–700.

[3] B. Khosravi, Some characterizations of $L_9(2)$ related to its prime graph, *Publ. Math. Debrecen* **75** (2009), 375–385.

[4] R. Kogani-Moghadam and A. R. Moghaddamfar, Groups with the same order and degree pattern, *Sci. China Math.* **55** (2012), 701–720.

[5] A. R. Moghaddamfar and A. R. Zokayi, Recognizing finite groups through order and degree pattern, *Algebra Colloq.* **15** (2008), 449–456.

[6] A. R. Moghaddamfar and A. R. Zokayi, OD-Characterization of certain finite groups having connected prime graphs, *Algebra Colloq.* **17** (2010), 121–130.

[7] J. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime number, *Illinois J. Math.* **6** (1962), 64–94.

[8] A. V. Vasil'ev and E. P. Vdovin, An adjacency criterion for the prime graph of a finite simple group, *Algebra Logic* **44** (2005), 381–406.

[9] Y. X. Yan and G. Y. Chen, OD-characterization of alternating and symmetric groups of degree 106 and 112, *Proceedings of the International Conference on Algebra* (2010), 690–696.

[10] L. C. Zhang, W. J. Shi, L. L. Wang and C. G. Shao, OD-characterization of A_{16}, *Journal of Suzhou University (Natural Science Edition)* **24** (2008), 7–10.

[11] L. C. Zhang and W. J. Shi, OD-characterization of simple K_4-groups, *Algebra Colloq.* **6** (2009), 275–282.

[12] K. Zsigmondy, Zur Theorie der Potenzreste, *Monatsh. Math. Phys.* **3** (1892), 265–284.

ALI MAHMOUDIFAR
DEPARTMENT OF PURE MATHEMATIC
FACULTY OF MATHEMATIC
AND COMPUTER SCIENCE
AMIRKABIR UNIVERSITY OF TECHNOLOGY
(TEHRAN POLYTECHNIC)
424, HAFEZ AVE., TEHRAN 15914
IRAN

E-mail: alimahmoudifar@gmail.com

BEHROOZ KHOSRAVI
DEPARTMENT OF PURE MATHEMATIC
FACULTY OF MATHEMATIC
AND COMPUTER SCIENCE
AMIRKABIR UNIVERSITY OF TECHNOLOGY
(TEHRAN POLYTECHNIC)
424, HAFEZ AVE., TEHRAN 15914
IRAN

E-mail: khosravibbb@yahoo.com
URL: http://www.aut.ac.ir/bkhosravi

(Received July 4, 2013; revised May 13, 2014)