

On group algebras with unit groups of derived length at most four

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Abstract. In this paper, we examine the relation between a finite group G and the units U in the group algebra of G over a field K of positive characteristic. By imposing certain natural conditions on the derived subgroups of U so that it has solvable length at most four, we show that the group G must be commutative.

1. Introduction

Let KG be the group algebra of a group G over a field K of positive characteristic p and let $U(KG) = U$ denote its multiplicative group of units. For subsets X, Y of a group G , we denote by (X, Y) the subgroup of G generated by all commutators $(x, y) = x^{-1}y^{-1}xy$ with $x \in X$ and $y \in Y$. The derived subgroups of G are defined as $G^{(0)} = G$, $G^{(1)} = G' = (G, G)$, and $G^{(i)} = (G^{(i-1)}, G^{(i-1)})$ for all $i > 0$. The descending normal series $G^{(0)} \triangleright G^{(1)} \triangleright \dots G^{(i)}$ is called the derived series of G . If the series terminates, i.e., if $G^{(n)} = 1$ for some integer n , then G is said to be solvable and the smallest such integer is called the derived length of G .

The investigation of necessary and sufficient conditions for the solvability of $U(KG)$ dates back to the 1970s with the works of BATEMAN and PASSMAN [1], [2]. A lot of work has been done on this context with a complete solution of the problem being given by BOVDI [3]. However, computation of the derived length of U and the converse problem of finding the nature of G or its commutator subgroup G' for a fixed derived length of U still remain open. Various results on the derived length of U have been obtained (for example, [4]–[6]). SHALEV [7] has classified

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group algebras of finite groups over fields of odd characteristic whose unit group is metabelian, and KURDICS [8] has done the same for even characteristic. Further, a necessary and sufficient condition for U to be centrally metabelian, that is, $(U^{(2)}, U) = 1$, is given by SAHAI [9]. Also, characterisation of group algebras over fields of odd characteristic such that $(U^{(2)}, U') = 1$ has been given by SAHAI [10] and the same with U satisfying $(U^{(2)}, U^{(2)}) = 1$ has been investigated by CHANDRA and SAHAI [11], [12].

In this article we consider group algebras with unit group U which satisfies $(U^{(3)}, U') = 1$. Such U is obviously of derived length at most four, i.e., $U^{(4)} = 1$. Lie algebraic properties of KG play an important role in our investigation. For $X, Y \subseteq KG$, we denote by $[X, Y]$ the additive subgroup generated by all Lie commutators $[x, y] = xy - yx$, where $x \in X$ and $y \in Y$. Also, $O_p(G)$ stands for the maximal normal p -subgroup of G , and $\Delta(G)$ denotes the augmentation ideal of the group algebra KG . For any two elements $x, h \in G$, x^h denotes the conjugation of x by h , that is, $h^{-1}xh$. We denote the Frattini subgroup of a group G by $\Phi(G)$, which is the intersection of all maximal subgroups of G . It is well-known that $\Phi(G)$ is a characteristic subgroup of G . By a p' -element or a p' -automorphism of a group G we mean an element or an automorphism of G whose order is not divisible by p . All groups considered are finite. Our main result is as follows:

Theorem 1.1. *Let K be a field of characteristic $p \geq 17$ and let G be a group of odd order. Then G is abelian if and only if U satisfies $(U^{(3)}, U') = 1$.*

2. Background

In this section we discuss a few important known results which provide useful tools for the proof of our theorem. We first state the complete set of necessary and sufficient conditions for U to be solvable, given by BOVDI [3].

Theorem 2.1. *Let K be a field of finite characteristic p , and $O_p(G)$ a maximal normal p -subgroup of the finite group G . Then the group $U(KG)$ is solvable if and only if one of the following statements holds:*

- (i) G is abelian.
- (ii) $G/O_p(G)$ is abelian and F is a field of characteristic p .
- (iii) $|K| = 2$ and $G/O_p(G)$ is an extension of an elementary abelian 3-group A by a group $\langle b \rangle$ of order 2, and $bab = a^{-1}$ for all $a \in A$.

- (iv) $|K| = 3$ and $G/O_p(G)$ is an extension of an elementary abelian 2-group A by a group $\langle b \rangle$ of order 2.
- (v) $|K| = 3$ and G is an extension of an abelian group A of exponent 4 by a group $\langle b \rangle$ of order 2 and $bab^{-1} = a^{-1}$ for all $a \in A$.
- (vi) $|K| = 3$ and $G/O_p(G)$ is an extension of an abelian group A of exponent 8 by a group $\langle b \rangle$ of order 2 and $bab = a^3$ for all $a \in A$.

The next result can be found in ([13], Theorem 3.5)

Theorem 2.2. *Let G be a group of order $p^a b$ and $(p, b) = 1$ and let K be a field of characteristic p . Assume that G has a normal Sylow p -subgroup P . Then the Jacobson radical $J = J(KG)$ of KG is $J = \Delta(P)KG$.*

The following result can be found in ([14], Chapter 9.6). Recall that the nilpotency index of a nilpotent ideal I is the least positive integer n such that $I^n = 0$.

Theorem 2.3. *Let G be a finite p -group of order p^m . Then the nilpotency index of $\Delta(G)$ over a field of characteristic p is p^m if and only if G is cyclic. Further, if $G = P_1 \times P_2 \times \cdots \times P_k$, where each P_i is a cyclic subgroup of order p^{t_i} , $t_i \in \mathbb{Z}$, for $i = 1, 2, \dots, k$, then the nilpotency index of $\Delta(G)$ is $(p^{t_1} + p^{t_2} + \cdots + p^{t_k} - k + 1)$.*

Next, we state a theorem by Burnside which can be found in ([15], Theorem 5.1.4).

Theorem 2.4. *Let ψ be a p' -automorphism of a p -group P , which induces the identity on $P/\Phi(P)$. Then ψ is the identity automorphism on P .*

The following can be found in ([15], Theorem 5.3.6).

Theorem 2.5. *If A is a p' -group of automorphisms of the p -group P , then $(P, A, A) = (P, A)$. In particular, if $(P, A, A) = 1$, then $A = 1$.*

If r is a real number, then by $\lceil r \rceil$ we denote the minimal integer not smaller than r . The next result can be found in [5].

Result 2.6. *If KG is a non-commutative group algebra of a torsion nilpotent group G over a field K of positive characteristic p such that U is solvable, then the derived length of U is at least $\lceil \log_2(p+1) \rceil$.*

Let $\mathfrak{J}(G')$ denote the ideal $KG\Delta(G')$. Let $x, y \in U$ be such that $x-1 \in \mathfrak{J}(G')^i$ and $y-1 \in \mathfrak{J}(G')^j$ for some $i, j > 0$. Then we have

$$(x, y) \equiv 1 + [x, y] \pmod{\mathfrak{J}(G')^{i+j+1}}. \quad (2.1)$$

Another important identity is, for any two elements x, y in KG , we have:

$$xy - 1 = (x - 1)(y - 1) + (x - 1) + (y - 1) \quad (2.2)$$

3. Proof of Theorem 1.1

- **Necessary conditions:**

Let G be a group of odd order. When G is abelian, the result follows trivially.

- **Sufficient conditions:**

Let KG be the group algebra of a finite group G over a field K of characteristic $p \geq 5$, such that, the unit group $U = U(KG)$ satisfies the condition $(U^{(3)}, U') = 1$. Then U is solvable and according to Theorem 2.1, $G/O_p(G)$ is abelian.

If $G/O_p(G)$ is abelian then $O_p(G)$ is a Sylow p -subgroup of G . Let $P = O_p(G)$. Now, $|P|$ and $[G : P]$ are relatively prime, hence by Schur–Zassenhaus Theorem ([15], Theorem 6.2.1), we have $G = P \rtimes H$, where H is a p' -prime subgroup of G . Also, by the above conditions, H is abelian.

Lemma 3.1. *Let $\text{Char } K = p \geq 11$. Let G be a group of odd order. Suppose that U satisfies $(U^{(3)}, U') = 1$. Then $G = P \times H$, where P is a p -group and H is an abelian p' -group, where p' is odd.*

PROOF. We know from above that $G = P \rtimes H$, where P is a p -group and H is an abelian p' -group. Also $P \trianglelefteq G$. Since G is of odd order, we have p' is odd. We need to show that $(P, H) = 1$. We will show that if $(P, H) \neq 1$, then we can construct nontrivial element in $(U^{(3)}, U')$. $\square$

We first assume that P is elementary abelian. Suppose, $(P, h) \neq 1$ for some $h \in H$. Then, $h^2 \neq 1$, and $(P, h) \leq P$ (as $P \trianglelefteq G$) and hence (P, h) is a p -group. Since h induces a p' -automorphism on P , by Theorem 2.5, $(P, h, h) = (P, h)$. Let $L = \langle (P, h), h \rangle$. Then $L' = (P, h, h) = (P, h)$ and $(P, h) \trianglelefteq L$. So on replacing G with L if necessary, we may assume that $P = G' = (P, h)$. By Theorem 2.2, the Jacobson radical $J = J(KG) = \Delta(P)KG$. Now, since $(P, h) \neq 1$, we can find $x \in P$ such that $(x, h) \neq 1$. Put $\alpha = x - 1$. Then $u = 1 + h\alpha$ is a unit in KG . Also, $(x, h), (x, h)^h, (x, h)^2 \in P$. By forming commutators of suitable elements in U , we obtain elements in $U', U^{(2)}$ and then in $U^{(3)}$. Now consider $u_1 = (u, h) \in U'$ and $v_1 = (u, x) \in U'$. We have

$$\begin{aligned} u_1 &= (u, h) = 1 + u^{-1}h^{-1}[u, h] \\ &\equiv 1 + (1 - h\alpha)(\alpha h - h\alpha) \pmod{J^2} \end{aligned}$$

$$\begin{aligned}
&\equiv 1 + \alpha h - h\alpha \pmod{J^2} \\
&= 1 + hx((x, h) - 1) \pmod{J^2} \\
&\equiv 1 + h((x, h) - 1) \pmod{J^2} \text{ (as } x \equiv 1 \pmod{J}). \tag{3.1}
\end{aligned}$$

As $G' = P$, we use identity (2.1) to obtain the following:

$$\begin{aligned}
v_1 &= (u, x) \equiv 1 + [u, x] \pmod{J^3} \\
&= 1 + (x + h\alpha x - x - xh\alpha) \pmod{J^3} \\
&= 1 + \{h(x-1)x - xh(x-1)\} \pmod{J^3} \\
&= 1 + (hx - xh)(x-1) \pmod{J^3} \\
&= 1 + hx(1 - (x, h))(x-1) \pmod{J^3} \\
&\equiv 1 - h((x, h) - 1)(x-1) \pmod{J^3} \text{ (as } x \equiv 1 \pmod{J}). \tag{3.2}
\end{aligned}$$

Next we consider $u_2 = (u_1, x)$ and $v_2 = (v_1, x)$. As $x \in P = G' \subset U'$, u_2 and v_2 are in $U^{(2)}$.

$$\begin{aligned}
u_2 &= (u_1, x) \equiv 1 + [u_1, x] \pmod{J^3} \\
&= 1 + \{x + h((x, h) - 1)x - x - xh((x, h) - 1)\} \pmod{J^3} \\
&= 1 + (hx - xh)((x, h) - 1) \pmod{J^3} \\
&= 1 + hx(1 - (x, h))((x, h) - 1) \pmod{J^3} \\
&\equiv 1 - h((x, h) - 1)^2 \pmod{J^3} \text{ (as } x \equiv 1 \pmod{J}). \tag{3.3}
\end{aligned}$$

$$\begin{aligned}
v_2 &= (v_1, x) \equiv 1 + [v_1, x] \pmod{J^4} \\
&= 1 + \{x - h((x, h) - 1)(x-1)x - x + xh((x, h) - 1)(x-1)\} \pmod{J^4} \\
&= 1 + (xh - hx)((x, h) - 1)(x-1) \pmod{J^4} \\
&= 1 + hx((x, h) - 1)^2(x-1) \pmod{J^4} \\
&\equiv 1 + h((x, h) - 1)^2(x-1) \pmod{J^4} \text{ (as } x \equiv 1 \pmod{J}). \tag{3.4}
\end{aligned}$$

Finally we obtain an element $w = (u_2, v_2)$ in $U^{(3)}$. We have

$$\begin{aligned}
w &= (u_2, v_2) \equiv 1 + [u_2, v_2] \pmod{J^6} \\
&= 1 + [u_2 - 1, v_2 - 1] \pmod{J^6} \\
&= 1 + \{-h((x, h) - 1)^2 h((x, h) - 1)^2 (x-1) \\
&\quad + h((x, h) - 1)^2 (x-1) h((x, h) - 1)^2\} \pmod{J^6}
\end{aligned}$$

$$\begin{aligned}
&= 1 + h((x, h) - 1)^2 \{(x - 1)h - h(x - 1)\}((x, h) - 1)^2 \pmod{J^6} \\
&= 1 + h((x, h) - 1)^2 hx((x, h) - 1)^3 \pmod{J^6} \\
&\equiv 1 + h((x, h) - 1)^2 h((x, h) - 1)^3 \pmod{J^6} \text{ (as } x \equiv 1 \pmod{J}) \\
&= 1 + h^2((x, h)^h - 1)^2((x, h) - 1)^3 \pmod{J^6}. \tag{3.5}
\end{aligned}$$

We now show that the element (w, x) in $(U^{(3)}, U')$ is nontrivial. It suffices to show that $V_1 = (w, x)$ is nontrivial modulo J^7 . Now,

$$\begin{aligned}
V_1 &= (w, x) \equiv 1 + [w, x] \pmod{J^7} \\
&= 1 + [w - 1, x] \pmod{J^7} \\
&= 1 + \{h^2((x, h)^h - 1)^2((x, h) - 1)^3 x - xh^2((x, h)^h - 1)^2((x, h) - 1)^3\} \pmod{J^7} \\
&= 1 + (h^2x - xh^2)((x, h)^h - 1)^2((x, h) - 1)^3 \pmod{J^7} \\
&= 1 + h^2x(1 - (x, h^2))((x, h)^h - 1)^2((x, h) - 1)^3 \pmod{J^7} \\
&\equiv 1 - h^2((x, h^2) - 1)((x, h)^h - 1)^2((x, h) - 1)^3 \pmod{J^7} \\
&\quad \text{(as } x \equiv 1 \pmod{J}). \tag{3.6}
\end{aligned}$$

Since P is elementary abelian, let $P = P_1 \times P_2 \times \cdots \times P_k$, where $k \geq 1$, each $P_i \cong C_p$, $i = 1, 2, \dots, k$ and C_p is a cyclic group of order p . Let x_i be the generator of P_i , for $i = 1, 2, \dots, k$. Let

$$\begin{aligned}
(x, h^2) &= x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} \\
(x, h)^h &= x_1^{b_1} x_2^{b_2} \cdots x_k^{b_k} \\
(x, h) &= x_1^{c_1} x_2^{c_2} \cdots x_k^{c_k}
\end{aligned}$$

where a_i 's, b_i 's and c_i 's are integers such that $1 \leq a_i, b_i, c_i \leq p$ for every $i = 1, 2, \dots, k$, with at least one element in every set of a_i 's, b_i 's and c_i 's being greater than or equal to one but strictly less than p . Then equation (3.6) can be written as:

$$\begin{aligned}
V_1 &= (w, x) \equiv 1 - h^2(x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} - 1)(x_1^{b_1} x_2^{b_2} \cdots x_k^{b_k} - 1)^2 \\
&\quad (x_1^{c_1} x_2^{c_2} \cdots x_k^{c_k} - 1)^3 \pmod{J^7}. \tag{3.7}
\end{aligned}$$

Now by repeated use of identity (2.2) and working modulo J^7 , equation (3.7) becomes

$$\begin{aligned}
V_1 &\equiv 1 - h^2\{(x_1^{a_1} - 1) + (x_2^{a_2} - 1) + \cdots + (x_k^{a_k} - 1)\} \\
&\quad \times \{(x_1^{b_1} - 1) + (x_2^{b_2} - 1) + \cdots + (x_k^{b_k} - 1)\}^2 \\
&\quad \times \{(x_1^{c_1} - 1) + (x_2^{c_2} - 1) + \cdots + (x_k^{c_k} - 1)\}^3. \tag{3.8}
\end{aligned}$$

Now, whenever we have $d \in \mathbb{N}$, we can write:

$$\begin{aligned} \therefore x^d - 1 &= (x - 1)(1 + x + \cdots + x^{d-1}) \\ &= (x - 1)\{1 + ((x - 1) + 1) + \cdots + ((x^{d-1} - 1) + 1)\} \\ &= (x - 1)(d + D_1) \text{ where } D_1 \in \Delta(P) \subseteq J = (x - 1)D. \end{aligned} \quad (3.9)$$

If $p \nmid d$, then $D = (d + D_1)$ is a unit in KG . With the help of this technique, the second term in RHS of equation (3.8) can be written as:

$$\begin{aligned} M &= \{(x_1 - 1)A_1 + (x_2 - 1)A_2 + \cdots + (x_k - 1)A_k\} \\ &\quad \times \{(x_1 - 1)B_1 + (x_2 - 1)B_2 + \cdots + (x_k - 1)B_k\}^2 \\ &\quad \times \{(x_1 - 1)C_1 + (x_2 - 1)C_2 + \cdots + (x_k - 1)C_k\}^3 \end{aligned} \quad (3.10)$$

$$\begin{aligned} \text{i.e., } M &= \{(x_1 - 1)A_1 + (x_2 - 1)A_2 + \cdots + (x_k - 1)A_k\} \\ &\quad \times \left\{ \sum_{i=1}^k (x_i - 1)^2 B_i^2 + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^k (x_i - 1)(x_j - 1)B_i B_j \right\} \\ &\quad \times \left\{ \sum_{i=1}^k (x_i - 1)^3 C_i^3 + 3 \sum_{\substack{i,j=1 \\ i \neq j}}^k (x_i - 1)^2 (x_j - 1)C_i^2 C_j \right. \\ &\quad \left. + 6 \sum_{\substack{i,j,l=1 \\ i \neq j \neq l}}^k (x_i - 1)(x_j - 1)(x_l - 1)C_i C_j C_l \right\} \end{aligned} \quad (3.11)$$

where all the A_i 's, B_i 's, C_i 's, for $i = 1, 2, \dots, k$, belong to KG with at least one element in every set of A_i 's, B_i 's, C_i 's, for $i = 1, 2, \dots, k$, is a unit in KG . Now, by Theorem 2.3, nilpotency index of $\Delta(P)$ as well as J in this case is $(kp - k + 1)$. Let, if possible, $V_1 \equiv 1 \pmod{J^7}$, that is, let $M \in J^7$.

Let $I = \{1, 2, \dots, k\}$. Let $I_A = \{t \in I \mid A_t \text{ is a unit}\}$, $I_B = \{t \in I \mid B_t \text{ is a unit}\}$ and $I_C = \{t \in I \mid C_t \text{ is a unit}\}$. Whenever there is an element i in the intersection of any of these sets, we apply a trick by adjusting the powers of the element $(x_i - 1)$ to get an element contradicting the nilpotency index of $\Delta(P)$. Now the following mutually exclusive cases may arise:

Case (I) $I_C \cap I_B \neq \emptyset$. We consider the following mutually exclusive subcases:

(a) $I_C \cap I_B \cap I_A \neq \emptyset$. Let $r \in I_C \cap I_B \cap I_A$. We examine the term

$$\left\{ (x_r - 1)^{p-7} \prod_{\substack{i=1 \\ i \neq r}}^k (x_i - 1)^{p-1} \right\} M,$$

and find that $M \in J^7$ would imply

$$A_r B_r^2 C_r^3 (x_1 - 1)^{p-1} (x_2 - 1)^{p-1} \dots (x_k - 1)^{p-1} \in J^{kp-k-6} J^7 = (0)$$

which is a contradiction to the nilpotency index of $\Delta(P)$, as $A_r B_r^2 C_r^3$ is a unit in KG .

- (b) $I_C \cap I_B \cap I_A = \emptyset$. Pick $m \in I_C \cap I_B$ and $m_A \in I_A$, so $m \notin I_A$. We examine the term

$$\left\{ (x_m - 1)^{p-6} (x_{m_A} - 1)^{p-2} \prod_{\substack{i=1 \\ i \neq m, m_A}}^k (x_i - 1)^{p-1} \right\} M$$

and find that $M \in J^7$ would imply

$$A_{m_A} B_m^2 C_m^3 (x_1 - 1)^{p-1} (x_2 - 1)^{p-1} \dots (x_k - 1)^{p-1} \in J^{kp-k-6} J^7 = (0)$$

which is a contradiction to the nilpotency index of $\Delta(P)$, as $A_{m_A} B_m^2 C_m^3$ is a unit in KG .

Case (II) $I_C \cap I_B = \emptyset$. Again, this has the following subcases:

- (a) $I_C \cap I_A \neq \emptyset$. Pick $l \in I_C \cap I_A$ and $l_B \in I_B$, so $l \notin I_B$ and $l_B \notin I_C$. We examine the term

$$\left\{ (x_l - 1)^{p-5} (x_{l_B} - 1)^{p-3} \prod_{\substack{i=1 \\ i \neq l, l_B}}^k (x_i - 1)^{p-1} \right\} M$$

and find that $M \in J^7$ would imply

$$A_l B_{l_B}^2 C_l^3 (x_1 - 1)^{p-1} (x_2 - 1)^{p-1} \dots (x_k - 1)^{p-1} \in J^{kp-k-6} J^7 = (0)$$

which is a contradiction to the nilpotency index of $\Delta(P)$, as $A_l B_{l_B}^2 C_l^3$ is a unit in KG .

- (b) $I_C \cap I_A = \emptyset$. This has the following subcases.

- (i) $I_B \cap I_A \neq \emptyset$. Let $n \in I_B \cap I_A$, and $n_C \in I_C$, so $n \notin I_C$ and $n_C \notin I_A \cup I_B$. We examine the term

$$\left\{ (x_n - 1)^{p-4} (x_{n_C} - 1)^{p-4} \prod_{\substack{i=1 \\ i \neq n, n_C}}^k (x_i - 1)^{p-1} \right\} M$$

and find that $M \in J^7$ would imply

$$A_n B_{n_C}^2 C_{n_C}^3 (x_1 - 1)^{p-1} (x_2 - 1)^{p-1} \dots (x_k - 1)^{p-1} \in J^{kp-k-6} J^7 = (0)$$

which is a contradiction to the nilpotency index of $\Delta(P)$, as $A_n B_{n_C}^2 C_{n_C}^3$ is a unit in KG .

- (ii) $I_B \cap I_A = \emptyset$, so that I_A , I_B and I_C are pairwise disjoint. Let $d \in I_A$, $e \in I_B$ and $f \in I_C$. Examining the term

$$\left\{ (x_d - 1)^{p-2} (x_e - 1)^{p-3} (x_f - 1)^{p-4} \prod_{\substack{i=1 \\ i \neq d, e, f}}^k (x_i - 1)^{p-1} \right\} M$$

and find that $M \in J^7$ would imply

$$A_d B_e^2 C_f^3 (x_1 - 1)^{p-1} (x_2 - 1)^{p-1} \dots (x_k - 1)^{p-1} \in J^{kp-k-6} J^7 = (0)$$

which is a contradiction to the nilpotency index of $\Delta(P)$, as $A_d B_e^2 C_f^3$ is a unit in KG .

Therefore, $V_1 \not\equiv 1 \pmod{J^7}$, which implies that V_1 is a nontrivial element in $(U^{(3)}, U') = 1$, a contradiction to our given condition. So, when P is elementary abelian, we get that $G = P \times H$.

Now, let P be any p -group. Assume $(P, h) \neq 1$ for some $h \in H$. As the Frattini subgroup $\Phi(P)$ is a characteristic subgroup of P , we have $h\Phi(P) = \Phi(P)$ and hence h induces an automorphism on $P/\Phi(P)$. Now $P/\Phi(P)$ is elementary abelian (by [15], Theorem 5.1.3). Now, we have already proved that h induces the identity automorphism on the elementary abelian group $P/\Phi(P)$. Hence by Theorem 2.4, h induces the identity automorphism on P as well. Hence we get, $G = P \times H$.

Proposition 3.2. *Let $\text{Char } K = p \geq 17$ and let G be a finite p -group such that U satisfies $(U^{(3)}, U') = 1$, then G is abelian.*

PROOF. A finite p -group is nilpotent and torsion. If G is non-abelian then by Result 2.6, the derived length of U for $p \geq 17$ is $\lceil \log_2(p+1) \rceil \geq \lceil \log_2(17+1) \rceil \approx \lceil 4.16 \rceil = 5$. Thus U can only satisfy $(U^{(3)}, U') = 1$, if G is abelian. $\square$

• **Conclusion:**

Combining lemma 3.1 and Proposition 3.2, we find that when $\text{Char } K \geq 17$ and G is a group of odd order such that U satisfies $(U^{(3)}, U') = 1$, then G is abelian. Hence, Theorem 1.1 is proved.

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