

## Characterization of additive maps $\xi$ -Lie derivable at zero on von Neumann Algebras

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**Abstract.** Let  $\mathcal{M}$  be any von Neumann algebra with the center  $\mathcal{Z}(\mathcal{M})$ . For any scalar  $\xi$ , denote by  $[A, B]_\xi = AB - \xi BA$  the  $\xi$ -Lie product of  $A, B \in \mathcal{M}$ . Assume that  $L : \mathcal{M} \rightarrow \mathcal{M}$  is an additive map. It is shown that, if  $\mathcal{M}$  has no central summands of type  $I_1$  or type  $I_2$ , then  $L$  satisfies  $L([A, B]) = [L(A), B] + [A, L(B)]$  whenever  $[A, B] = 0$  if and only if there exists an element  $Z_0 \in \mathcal{Z}(\mathcal{M})$ , an additive map  $h : \mathcal{M} \rightarrow \mathcal{Z}(\mathcal{M})$  and an additive derivation  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  such that  $L(A) = \varphi(A) + h(A) + Z_0 A$  for all  $A \in \mathcal{M}$ ; if  $\mathcal{M}$  has no central summands of type  $I_1$ , then  $L$  satisfies  $L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$  whenever  $[A, B]_\xi = 0$  with  $\xi \neq 1$  if and only if  $L(I) \in \mathcal{Z}(\mathcal{M})$  and there exists an additive derivation  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  such that  $\varphi(\xi A) = \xi \varphi(A)$  and  $L(A) = \varphi(A) + L(I)A$  for all  $A \in \mathcal{M}$ . A result in [22] is improved for prime algebra case.

### 1. Introduction

Let  $\mathcal{R}$  be an associative ring. Recall that an additive map  $\delta$  on  $\mathcal{R}$  is called an additive derivation if  $\delta(AB) = \delta(A)B + A\delta(B)$  for all  $A, B \in \mathcal{R}$ ; is called an additive Jordan derivation if  $\delta(AB + BA) = \delta(A)B + A\delta(B) + \delta(B)A + B\delta(A)$  for all  $A, B \in \mathcal{R}$  (equivalently,  $\delta(A^2) = \delta(A)A + A\delta(A)$  for all  $A \in \mathcal{R}$  if the characteristic of  $\mathcal{R}$  is not 2); is called a Lie derivation if  $\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$  for all  $A, B \in \mathcal{R}$ , where  $[A, B] = AB - BA$  is the Lie product of  $A$  and  $B$ . The structure of derivations, Jordan derivations and Lie derivations had been studied

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intensively for many years (for example, see [1], [3], [4], [7], [10], [16] and the references therein).

Let  $\mathcal{A}$  be an algebra over a field  $\mathbb{F}$ . For a scalar  $\xi \in \mathbb{F}$  and for  $A, B \in \mathcal{A}$ , we say that  $A$  commutes with  $B$  up to a factor  $\xi$  if  $AB = \xi BA$ . The notion of commutativity up to a factor for pairs of operators is an important concept and has been studied in the context of operator algebras and quantum groups. Motivated by this, a binary operation  $[A, B]_\xi = AB - \xi BA$ , called  $\xi$ -Lie product of  $A$  and  $B$ , was introduced in [19]. An additive map  $L : \mathcal{A} \rightarrow \mathcal{A}$  is called an additive  $\xi$ -Lie derivation if  $L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$  for all  $A, B \in \mathcal{A}$ . This conception unifies the above three notions. It is clear that a  $\xi$ -Lie derivation is a derivation if  $\xi = 0$ ; is a Lie derivation if  $\xi = 1$ ; is a Jordan derivation if  $\xi = -1$ . The structure of  $\xi$ -Lie derivations on various operator algebras was also discussed by several authors (see [9], [14], [19], [23], [25]).

Recently, the question of under what conditions an additive map becomes a derivation attracted much attention of many researchers (see [5], [8], [12], [13], [18] and the references therein). For  $\xi$ -Lie derivations, an additive (a linear) map  $L$  on  $\mathcal{A}$  is said to be  $\xi$ -Lie derivable at a point  $Z \in \mathcal{A}$  if  $L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$  for any  $A, B \in \mathcal{A}$  with  $[A, B]_\xi = Z$ . Clearly, this definition is only valid for  $\xi$ -Lie commutators, that is, the elements of the form  $Z = [A, B]_\xi$ . For instance, if  $Z = I$  and  $\xi = 1$ , as the unit  $I$  may not be a commutator in general, there is no sense to define that  $L$  is Lie derivable at  $I$ . Also,  $L$  is a  $\xi$ -Lie derivation if and only if it is  $\xi$ -Lie derivable at every  $\xi$ -Lie commutator  $Z$ . If  $Z \in \mathcal{A}$  satisfies that, for any additive map  $L : \mathcal{A} \rightarrow \mathcal{A}$ ,  $L$  is  $\xi$ -Lie derivable at the point  $Z$  will imply that  $L$  is a  $\xi$ -Lie derivation, we say that  $Z$  is a full  $\xi$ -Lie derivable point of  $\mathcal{A}$ . The following problem is natural.

**Problem.** *How to characterize the additive (linear) maps that are  $\xi$ -Lie derivable at some  $\xi$ -Lie commutator? Are there any  $\xi$ -Lie commutators that are full  $\xi$ -Lie derivable points?*

Since zero is a  $\xi$ -Lie commutator for any  $\xi$  and any algebra, as a start, the above problem has been attacked by several researchers for maps  $\xi$ -Lie derivable at zero. QI and HOU [20] characterized the linear maps between  $\mathcal{J}$ -subspace lattice algebras that Lie derivable at zero. In [22] they consider further the additive maps  $\xi$ -Lie derivable at zero in pure algebra frame. Let  $\mathcal{A}$  be a unital prime algebra over a field  $\mathbb{F}$  containing a non-trivial idempotent  $P$ . Denote by  $\mathcal{Z}(\mathcal{A})$  and  $\mathcal{C}$  the center of  $\mathcal{A}$  and the extended centroid of  $\mathcal{A}$ , respectively. Assume that  $\xi \in \mathbb{F}$  and  $L : \mathcal{A} \rightarrow \mathcal{A}$  is an additive map. QI, CUI and HOU [22] showed that, if  $L$  is  $\xi$ -Lie derivable at zero, then there exists an additive derivation  $\tau : \mathcal{A} \rightarrow \mathcal{C}$  such that (1)

if  $\xi = 0$ , then  $L(I) \in \mathcal{Z}(\mathcal{A})$  and  $L(A) = \tau(A) + L(I)A$  for all  $A \in \mathcal{A}$ ; (2) if  $\xi = 1$  and  $\mathcal{A}$  is of characteristic not 2 with  $\deg(\mathcal{A}) > 2$ , then  $L(A) = \tau(A) + \alpha A + \nu(A)$  for all  $A \in \mathcal{A}$ , where  $\alpha \in \mathcal{C}$  and  $\nu$  is an additive map from  $\mathcal{A}$  into  $\mathcal{C}$ ; (3) if  $\xi = -1$ ,  $L(I) \in \mathcal{Z}(\mathcal{A})$  and  $\mathcal{A}$  is of characteristic not 2, then  $L(A) = \tau(A) + L(I)A$  for all  $A \in \mathcal{A}$ ; (4) if  $\xi \neq 0, \pm 1$ ,  $L(I) \in \mathcal{Z}(\mathcal{A})$  and  $L(\xi A) = \xi L(A)$  for each  $A \in \mathcal{A}$ , then  $L(A) = \tau(A) + L(I)A$  for all  $A \in \mathcal{A}$ . Since factor von Neumann algebras are prime, as a consequence of the result for prime algebras, all additive maps  $\xi$ -Lie derivable at zero on factor von Neumann algebras are characterized. However the proof in [22] for factor von Neumann algebras is not valid anymore for non-factor von Neumann algebras. So, it is natural to ask what happens if the concerned von Neumann algebra is not a factor.

Let  $X$  be a Banach space with  $\dim X \geq 3$  and  $\mathcal{B}(X)$  the algebra of all bounded linear operators acting on  $X$ . We mention here that, LU and JING in [15] introduced another kind of Lie derivable at zero product (idempotent product) for a linear map and showed that, if  $\delta : \mathcal{B}(X) \rightarrow \mathcal{B}(X)$  is a linear map satisfying  $\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$  for any  $A, B \in \mathcal{B}(X)$  with  $AB = 0$  (resp.  $AB = P$ , where  $P$  is a fixed nontrivial idempotent), then  $\delta = \tau + \nu$ , where  $\tau$  is a derivation of  $\mathcal{B}(X)$  and  $\nu : \mathcal{B}(X) \rightarrow \mathbb{C}I$  is a linear map vanishing at commutators  $[A, B]$  with  $AB = 0$  (resp.  $AB = P$ ), particularly,  $\delta$  is a Lie derivation if  $\nu$  vanishes on all commutators. Later, this result was generalized to the additive maps on triangular algebras, prime rings and general von Neumann algebras in [11], [21] and [24] respectively. Let  $\mathcal{M}$  be a von Neumann algebra without central summands of type  $I_1$  and  $L : \mathcal{M} \rightarrow \mathcal{M}$  be an additive map. In [24], QI and HOU showed that,  $L$  satisfies  $L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$  for any  $A, B$  with  $AB = 0$  if and only if there exists an additive derivation  $\varphi$  and an additive map  $f : \mathcal{M} \rightarrow \mathcal{Z}(\mathcal{M})$  that vanishes each commutator  $[A, B]$  whenever  $AB = 0$ , such that (1)  $\xi = 1$ ,  $L = \varphi + f$ ; (2)  $\xi = -1$ ,  $L = \varphi$ ; (3)  $\xi = 0$ ,  $L(A) = \varphi(A) + L(I)A$  for all  $A$ ; (4)  $\xi \notin \{0, \pm 1\}$ ,  $\varphi(\xi A) = \xi L(A)$  and  $L(A) = \varphi(A) + L(I)A$  for all  $A$ .

The purpose of the present paper is to give a complete characterization of additive maps  $\xi$ -Lie derivable at zero on von Neumann algebras without central summands of type  $I_1$  for any scalar  $\xi$ . We remark here that the question of characterizing additive maps  $\delta$  that are  $\xi$ -Lie derivable at zero is relatively more difficult than the question of characterizing additive maps satisfying  $\delta([A, B]_\xi) = [\delta(A), B]_\xi + [A, \delta(B)]_\xi$  for any  $A, B \in \mathcal{M}$  with  $AB = 0$  since it is more difficult to find  $A, B$  satisfying  $[A, B]_\xi = 0$ , of course, the conclusions are also different.

The paper is organized as follows. Let  $\mathcal{M}$  be a von Neumann algebra with the center  $\mathcal{Z}(\mathcal{M})$  and  $L : \mathcal{M} \rightarrow \mathcal{M}$  an additive map. In Section 2, we show that, if  $\mathcal{M}$

has no central summands of type  $I_1$  or type  $I_2$ , then  $L$  is Lie derivable at zero if and only if there exists an element  $Z_0 \in \mathcal{Z}(\mathcal{M})$ , an additive derivation  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  and an additive map  $h : \mathcal{M} \rightarrow \mathcal{Z}(\mathcal{M})$  such that  $L(A) = \varphi(A) + h(A) + Z_0A$  for all  $A \in \mathcal{M}$  (Theorem 2.1). There are counterexamples to illustrate that the condition “ $\mathcal{M}$  has no central summands of type  $I_1$  or type  $I_2$ ” can not be replaced simply by “ $\mathcal{M}$  has no central summands of type  $I_1$ ”. Section 3 is devoted to discussing additive maps  $\xi$ -Lie derivable at zero with  $\xi \neq 1$ . Assume that  $\mathcal{M}$  is a von Neumann algebra without central summands of type  $I_1$ . It is shown that  $L$  satisfies  $L(AB - \xi BA) = L(A)B - \xi BL(A) + AL(B) - \xi L(B)A$  whenever  $AB - \xi BA = 0$  if and only if  $L(I) \in \mathcal{Z}(\mathcal{M})$  and there exists an additive derivation  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  with  $\varphi(\xi A) = \xi\varphi(A)$  for each  $A$  such that  $L(A) = \varphi(A) + L(I)A$  for all  $A \in \mathcal{M}$  (Theorem 3.1). Thus, zero is not a full  $\xi$ -Lie derivable point, though they have a structure very close to  $\xi$ -Lie derivation. From these results, one can easily get a characterization of additive  $\xi$ -Lie derivations. As a consequence, our approach also enables us to improve the result (4) in [22] mentioned in the third paragraph by omitting the assumptions “ $L(I) \in \mathcal{Z}(\mathcal{A})$  and  $L(\xi A) = \xi L(A)$  for each  $A \in \mathcal{A}$ ”.

## 2. Additive maps Lie derivable at zero

In this section, we discuss additive maps Lie derivable at zero on von Neumann algebras. The following is our main result in this section.

**Theorem 2.1.** *Let  $\mathcal{M}$  be a von Neumann algebra without central summands of type  $I_1$  or type  $I_2$ . Suppose that  $L : \mathcal{M} \rightarrow \mathcal{M}$  is an additive map. Then  $L$  is Lie derivable at zero, that is,  $L([A, B]) = [L(A), B] + [A, L(B)]$  for any  $A, B \in \mathcal{M}$  with  $[A, B] = 0$ , if and only if there exists an element  $Z_0 \in \mathcal{Z}(\mathcal{M})$ , the center of  $\mathcal{M}$ , an additive derivation  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  and an additive map  $h : \mathcal{M} \rightarrow \mathcal{Z}(\mathcal{M})$  such that  $L(A) = \varphi(A) + h(A) + Z_0A$  for all  $A \in \mathcal{M}$ .*

We remark that the conditions “without central summands of type  $I_1$  or type  $I_2$ ” in Theorem 2.1 can not be deleted simply because (i) every additive map on a commutative ring is a Lie derivation; and (ii) it is shown in [20, Proposition 2.5] that a linear map  $L$  on 2 by 2 matrix algebra is Lie derivable at zero if and only if  $L(I) = \lambda I$  for some scalar  $\lambda$ .

To prove Theorem 2.1, we need several lemmas.

**Lemma 2.2** ([2, Lemma 2]). *Let  $\mathcal{M}$  be a von Neumann algebra with no*

central summands of type  $I_1$  or type  $I_2$ . Then the ideal  $\mathcal{I}$  of  $\mathcal{M}$  generated algebraically by  $\{[A^2, C]B[A, C] - [A, C]B[A^2, C] : A, B, C \in \mathcal{M}\}$  is equal to  $\mathcal{M}$ .

A ring  $\mathcal{R}$  is said to be semiprime if, for any  $A \in \mathcal{R}$ ,  $A\mathcal{R}A = \{0\}$  will imply that  $A = 0$ ; to be torsion-free if, for any  $A \in \mathcal{R}$  and any positive integer  $n$ ,  $nA = 0$  will imply that  $A = 0$ . Every von Neumann algebra is semiprime and torsion-free.

**Lemma 2.3** ([2, Lemma 6]). *Let  $\mathcal{R}$  be a semiprime torsion-free ring and  $G$  an additive group. Suppose that maps  $\epsilon : G \times G \rightarrow \mathcal{R}$  and  $\tau : G \times G \times G \rightarrow \mathcal{R}$  are additive in each argument. If  $\epsilon(A, A)\mathcal{R}\tau(A, A, A) = \{0\}$  for every  $A \in G$ , then  $\epsilon(B, B)\mathcal{R}\tau(A, A, A) = \{0\}$  for all  $A, B \in G$ .*

Recall that a map  $q$  from a ring  $\mathcal{R}$  into itself is commuting if  $[q(A), A] = 0$  for all  $A \in \mathcal{R}$ ; is a trace of a biadditive map if there exists a biadditive map  $g : \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$  such that  $q(A) = g(A, A)$  for all  $A \in \mathcal{R}$ .

The following lemma is crucial for proving our main result.

**Lemma 2.4** ([2, Theorem 2]). *Let  $\mathcal{M}$  be a von Neumann algebra without central summands of type  $I_1$  or type  $I_2$ . Let  $q$  be a trace of a biadditive map. If  $q$  is commuting, then  $q(A) = \lambda A^2 + \mu(A)A + \nu(A)$  for all  $A \in \mathcal{M}$ , where  $\lambda \in \mathcal{Z}(\mathcal{M})$  and  $\mu, \nu$  are maps of  $\mathcal{M}$  into  $\mathcal{Z}(\mathcal{M})$  with  $\mu$  additive.*

Now we are at a position to give our proof of Theorem 2.1.

**PROOF OF THEOREM 2.1.** The “if” part is obvious. For the “only if” part, assume that  $L : \mathcal{M} \rightarrow \mathcal{M}$  is an additive map Lie derivable at zero, that is,

$$[L(A), B] + [A, L(B)] = 0 \quad \text{for all } A, B \in \mathcal{M} \text{ with } [A, B] = 0. \quad (2.1)$$

Take  $B=A^2$  in equation (2.1) and we get  $L(A)A^2 - A^2L(A) + AL(A^2) - L(A^2)A = 0$ . This yields

$$[L(A^2) - L(A)A - AL(A), A] = 0 \quad \text{for all } A \in \mathcal{M}. \quad (2.2)$$

For any  $A, B \in \mathcal{M}$ , write  $\delta(A, B) = L(AB) - L(A)B - AL(B)$ . It is obvious that  $\delta : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$  is a biadditive map and  $\delta(A, A)$  is a trace of the biadditive map  $\delta$  by equation (2.2). It follows from Lemma 2.4 that there exist an element  $Z \in \mathcal{Z}(\mathcal{M})$ , an additive map  $\mu : \mathcal{M} \rightarrow \mathcal{Z}(\mathcal{M})$  and a map  $\nu : \mathcal{M} \rightarrow \mathcal{Z}(\mathcal{M})$  such that  $\delta(A, A) = ZA^2 + \mu(A)A + \nu(A)$ , that is,

$$L(A^2) - L(A)A - AL(A) = ZA^2 + \mu(A)A + \nu(A) \quad \text{holds for all } A \in \mathcal{M}. \quad (2.3)$$

Now define two maps  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  and  $\epsilon : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$  as follows:

$$\varphi(A) = L(A) + \frac{1}{2}\mu(A) + ZA \quad \text{for all } A \in \mathcal{M}$$

and

$$\epsilon(A, B) = \varphi(AB + BA) - \varphi(A)B - A\varphi(B) - \varphi(B)A - B\varphi(A) \quad \text{for all } A, B \in \mathcal{M}.$$

Clearly,  $\varphi$  is additive and  $\epsilon$  is biadditive. By the definition of  $\varphi$  and equation (2.3), we have

$$\begin{aligned} \varphi(A^2) &= L(A^2) + \frac{1}{2}\mu(A^2) + ZA^2 \\ &= L(A)A + AL(A) + \mu(A)A + \nu(A) + \frac{1}{2}\mu(A^2) + 2ZA^2 \end{aligned}$$

and

$$\varphi(A)A + A\varphi(A) = L(A)A + AL(A) + \mu(A)A + 2ZA^2.$$

The above two equations imply  $\varphi(A^2) - \varphi(A)A - A\varphi(A) = \nu(A) + \frac{1}{2}\mu(A^2) \in \mathcal{Z}(\mathcal{M})$ . Replacing  $A$  by  $A + B$  in the relation, one obtains

$$\varphi(AB + BA) - \varphi(A)B - A\varphi(B) - \varphi(B)A - B\varphi(A) \in \mathcal{Z}(\mathcal{M}),$$

which implies that  $\epsilon$  maps  $\mathcal{M} \times \mathcal{M}$  into  $\mathcal{Z}(\mathcal{M})$ .

**Claim.**  $\epsilon(A, A) = 0$  for all  $A \in \mathcal{M}$ .

For any  $A \in \mathcal{M}$ , by the definition of  $\epsilon$ , we have

$$\begin{aligned} 2\varphi(A^4) &= 2\varphi(A^2A^2) = 2\varphi(A^2)A^2 + 2A^2\varphi(A^2) + \epsilon(A^2, A^2) \\ &= 2\varphi(A)A^3 + 2A\varphi(A)A^2 + 2A^2\varphi(A)A + 2A^3\varphi(A) + 2\epsilon(A, A)A^2 + \epsilon(A^2, A^2) \end{aligned}$$

and

$$\begin{aligned} 4\varphi(A^4) &= 2\varphi(A^3A + AA^3) \\ &= 2\varphi(A^3)A + 2A^3\varphi(A) + 2\varphi(A)A^3 + 2A\varphi(A^3) + 2\epsilon(A^3, A) \\ &= \varphi(A^2A + AA^2)A + 2A^3\varphi(A) + 2\varphi(A)A^3 + A\varphi(A^2A + AA^2) + 2\epsilon(A^3, A) \\ &= 4\varphi(A)A^3 + 4A\varphi(A)A^2 + 4A^2\varphi(A)A + 4A^3\varphi(A) \\ &\quad + 2\epsilon(A^3, A) + 2\epsilon(A, A)A^2 + 2\epsilon(A^2, A)A. \end{aligned}$$

Comparing the above two equations gives

$$\epsilon(A, A)A^2 - \epsilon(A^2, A)A = \epsilon(A^3, A) - \epsilon(A^2, A^2) \in \mathcal{Z}(\mathcal{M}) \quad (2.4)$$

for all  $A \in \mathcal{M}$ . Take any  $A, C \in \mathcal{M}$ . By equation(2.4), it is easily checked that  $\epsilon(A, A)[A^2, C] = \epsilon(A^2, A)[A, C]$ . It follows that

$$\epsilon(A, A)([A^2, C]X[A, C] - [A, C]X[A^2, C]) = 0 \quad \text{for all } A, C, X \in \mathcal{M}. \quad (2.5)$$

Now fix  $X$  and  $C$ . Define a map  $\phi : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$  as follows:

$$\phi(A_1, A_2, A_3) = [A_1 A_2, C]X[A_3, C] - [A_1, C]X[A_2 A_3, C] \quad \text{for all } A_1, A_2, A_3 \in \mathcal{M}.$$

It is obvious that  $\phi$  is additive in each argument and

$$\phi(A, A, A) = [A^2, C]X[A, C] - [A, C]X[A^2, C].$$

Thus, by equation (2.5), one gets  $\epsilon(A, A)\phi(A, A, A) = 0$ , and so  $\epsilon(A, A)\mathcal{M}\phi(A, A, A) = \{0\}$  for each  $A \in \mathcal{M}$  as  $\epsilon$  maps into the center. It follows from Lemma 2.3 that  $\epsilon(B, B)\mathcal{M}\phi(A, A, A) = \{0\}$  for all  $A, B \in \mathcal{M}$ . Now by using Lemma 2.2, we obtain that  $\epsilon(B, B) = 0$  for all  $B \in \mathcal{M}$ , as desired.

Note that  $\epsilon(A, A) = 2(\varphi(A^2) - \varphi(A)A - A\varphi(A))$  for every  $A \in \mathcal{M}$ . So by claim,  $\varphi(A^2) = \varphi(A)A + A\varphi(A)$  holds for all  $A \in \mathcal{M}$ , that is,  $\varphi$  is an additive Jordan derivation. Since every additive Jordan derivation on a 2-torsion free semiprime ring is an additive derivation ([1, Theorem 1]) and every von Neumann algebra is semiprime,  $\varphi$  is in fact an additive derivation. Let  $h = -\frac{1}{2}\mu$  and  $Z_0 = -Z$ . It follows from the definition of  $\varphi$  that

$$L(A) = \varphi(A) + Z_0 A + h(A) \quad (2.6)$$

for all  $A$ , where  $Z_0 \in \mathcal{Z}(\mathcal{M})$ ,  $\varphi$  is an additive derivation and  $h : \mathcal{M} \rightarrow \mathcal{Z}(\mathcal{M})$  is an additive map. The proof of the theorem is complete.  $\square$

By Theorem 2.1, we get a characterization of Lie derivations, which is also given in [24, Corollary 2.5] as a corollary of [24, Theorem 2.1].

**Corollary 2.5.** *Let  $\mathcal{M}$  be a von Neumann algebra without central summands of type  $I_1$  or type  $I_2$ . Suppose that  $L : \mathcal{M} \rightarrow \mathcal{M}$  is an additive map. Then  $L$  is a Lie derivation if and only if there exists an additive derivation  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  and an additive map  $h : \mathcal{M} \rightarrow \mathcal{Z}(\mathcal{M})$  vanishing on each commutator such that  $L(A) = \varphi(A) + h(A)$  for all  $A \in \mathcal{M}$ .*

PROOF. The “if” part is obvious. For the “only if” part, assume that  $L$  is an additive Lie derivation. Then,  $L$  is Lie derivable at zero and hence, by Theorem 2.1, has the form equation (2.6). Thus we have

$$\begin{aligned} L(AB - BA) &= \varphi(AB - BA) + h(AB - BA) + Z_0(AB - BA) \\ &= \varphi(A)B + A\varphi(B) - \varphi(B)A - B\varphi(A) + h(AB - BA) + Z_0(AB - BA) \end{aligned} \quad (2.7)$$

and on the other hand,

$$\begin{aligned} L(AB - BA) &= L(A)B - BL(A) + AL(B) - L(B)A \\ &= \varphi(A)B + A\varphi(B) - \varphi(B)A - B\varphi(A) + 2Z_0(AB - BA) \end{aligned} \quad (2.8)$$

for all  $A, B \in \mathcal{M}$ . Comparing equations (2.7)–(2.8), we see that  $Z_0(AB - BA) = h(AB - BA) \in \mathcal{Z}(\mathcal{M})$  for any  $A, B$ . Note that, for any  $A \in \mathcal{M}$  and any projection  $P \in \mathcal{M}$ ,  $PA(I - P) = [P, P + PA(I - P)]$  is a commutator. Thus  $h(PA(I - P)) = Z_0PA(I - P) = PZ_0A(I - P) \in \mathcal{Z}(\mathcal{M})$  holds for any projection  $P$  and any element  $A$  in  $\mathcal{M}$ . It follows that  $PZ_0A(I - P) = PPZ_0A(I - P) = PZ_0A(I - P)P = 0$  for all projection  $P$  and all  $A$  in  $\mathcal{M}$ . This forces that  $Z_0A$  commutes with each projection and hence is an element in the center of  $\mathcal{M}$ . So we have  $Z_0\mathcal{M} \subseteq \mathcal{Z}(\mathcal{M})$ . Since  $\mathcal{M}$  has no central summand of type  $I_1$ , we must have  $Z_0 = 0$ , and consequently,  $L = \varphi + h$  with  $h$  vanishing on all commutators.  $\square$

By Corollary 2.5, the additive maps Lie derivable at zero are very close to Lie derivations, but, not Lie derivations in general. Thus zero is not a full Lie derivable point of the von Neumann algebra  $\mathcal{M}$ .

### 3. Additive maps $\xi$ -Lie derivable at zero with $\xi \neq 1$

In this section, we will give a characterization of additive maps  $\xi$ -Lie derivable at zero on von Neumann algebras without central summands of type  $I_1$ . Here  $\xi \neq 1$ .

The following is the main result of this section.

**Theorem 3.1.** *Let  $\mathcal{M}$  be a von Neumann algebra without central summands of type  $I_1$ . Suppose that  $L: \mathcal{M} \rightarrow \mathcal{M}$  is an additive map and  $\xi$  is a scalar with  $\xi \neq 1$ . Then  $L$  is  $\xi$ -Lie derivable at zero (that is,  $L$  satisfies  $L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$  for any  $A, B \in \mathcal{M}$  with  $[A, B]_\xi = 0$ ) if and only if  $L(I) \in \mathcal{Z}(\mathcal{M})$  and there exists an additive derivation  $\varphi: \mathcal{M} \rightarrow \mathcal{M}$  such that  $\varphi(\xi A) = \xi\varphi(A)$  and  $L(A) = \varphi(A) + L(I)A$  for all  $A \in \mathcal{M}$ .*

Before proving Theorem 3.1, we need some notations. Let  $\mathcal{M}$  be any von Neumann algebra and  $A \in \mathcal{M}$ . Recall that the central carrier of  $A$ , denoted by  $\overline{A}$ , is the intersection of all central projections  $P$  such that  $PA = A$ . If  $A$  is self-adjoint, then the core of  $A$ , denoted by  $\underline{A}$ , is  $\sup\{S \in \mathcal{Z}(\mathcal{M}) : S = S^*, S \leq A\}$ . Particularly, if  $A = P$  is a projection, it is clear that  $\underline{P}$  is the largest central projection  $\leq P$ . A projection  $P$  is called core-free if  $\underline{P} = 0$ . It is easy to see that  $\underline{P} = 0$  if and only if  $\overline{I - P} = I$ .

We first give two useful lemmas which are needed to prove Theorem 3.1.

**Lemma 3.2** ([17, Lemma 4]). *Let  $\mathcal{M}$  be a von Neumann algebra without central summands of type  $I_1$ . Then each nonzero central projection  $C \in \mathcal{M}$  is the carrier of a core-free projection in  $\mathcal{M}$ . Particularly, there exists a nonzero core-free projection  $P \in \mathcal{M}$  with  $\overline{P} = I$ .*

In fact,  $\mathcal{M}$  is a von Neumann algebra without central summands of type  $I_1$  if and only if it has a projection  $P$  with  $\underline{P} = 0$  and  $\overline{P} = I$ .

**Lemma 3.3** ([17]). *Let  $\mathcal{M}$  be a von Neumann algebra. For projections  $P, Q \in \mathcal{M}$ , if  $\overline{P} = \overline{Q} \neq 0$  and  $P + Q = I$ , then  $T \in \mathcal{M}$  commutes with  $PXQ$  and  $QXP$  for all  $X \in \mathcal{M}$  implies  $T \in \mathcal{Z}(\mathcal{M})$ .*

For more properties of core-free projections, see [17].

**PROOF OF THEOREM 3.1.** The “if” part is easily checked. We only need to give the proof of the “only if” part.

Assume that  $\xi \neq 1$  and  $L: \mathcal{M} \rightarrow \mathcal{M}$  is an additive map satisfying  $L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$  for any  $A, B \in \mathcal{M}$  with  $[A, B]_\xi = 0$ . We will prove the “only if” part by several claims.

By Lemma 3.2, we can find a core-free projection  $P \in \mathcal{M}$  with  $\overline{P} = I$ . In the sequel fix such a projection  $P$ . By the definitions of core and central carrier, we have  $\underline{P} = I - P = 0$  and  $\overline{I - P} = I$ . For the convenience, write  $P_1 = P$ ,  $P_2 = I - P$  and  $\mathcal{M}_{ij} = P_i \mathcal{M} P_j$ ,  $i, j \in \{1, 2\}$ . Then  $\mathcal{M} = \mathcal{M}_{11} + \mathcal{M}_{12} + \mathcal{M}_{21} + \mathcal{M}_{22}$ .

**Claim 1.**  $P_1 L(I) P_2 = P_2 L(I) P_1 = 0$  and  $P_2 L(P_1) P_2 = P_1 L(P_2) P_1 = 0$ .

Since  $[P_1, P_2]_\xi = [P_2, P_1]_\xi = 0$ , we have  $[L(P_1), P_2]_\xi + [P_1, L(P_2)]_\xi = 0$  and  $[L(P_2), P_1]_\xi + [P_2, L(P_1)]_\xi = 0$ , that is,

$$L(P_1)P_2 - \xi P_2 L(P_1) + P_1 L(P_2) - \xi L(P_2)P_1 = 0 \quad (3.1)$$

and

$$L(P_2)P_1 - \xi P_1 L(P_2) + P_2 L(P_1) - \xi L(P_1)P_2 = 0. \quad (3.2)$$

Multiplying by  $P_1$  and  $P_2$  from the left and the right respectively in equation (3.1), and multiplying by  $P_2$  and  $P_1$  from the left and the right respectively in equation (3.2), one gets  $P_1 L(P_1) P_2 + P_1 L(P_2) P_2 = 0$  and  $P_2 L(P_2) P_1 + P_2 L(P_1) P_1 = 0$ , which imply

$$P_1 L(I) P_2 = 0 \quad \text{and} \quad P_2 L(I) P_1 = 0.$$

Multiplying by  $P_2$  and  $P_1$  from both sides in equation (3.1) and equation (3.2), respectively, one gets  $P_2 L(P_1) P_2 - \xi P_2 L(P_1) P_2 = 0$  and  $P_1 L(P_2) P_1 - \xi P_1 L(P_2) P_1 = 0$ . It follows from the assumption  $\xi \neq 1$  that

$$P_2 L(P_1) P_2 = 0 \quad \text{and} \quad P_1 L(P_2) P_1 = 0.$$

The claim holds.

Now define a map  $\delta : \mathcal{M} \rightarrow \mathcal{M}$  by  $\delta(A) = L(A) + SA - AS$  for each  $A \in \mathcal{M}$ , where  $S = P_1L(P_1)P_2 - P_2L(P_1)P_1$ . It is easily verified that  $\delta$  is also an additive map  $\xi$ -Lie derivable at zero, that is,  $\delta$  satisfies

$$\delta([A, B]_\xi) = [\delta(A), B]_\xi + [A, \delta(B)]_\xi \quad \text{for } A, B \in \mathcal{M} \text{ with } [A, B]_\xi = 0.$$

Moreover,  $P_1\delta(I)P_2 = P_2\delta(I)P_1 = P_2\delta(P_1)P_2 = P_1\delta(P_2)P_1 = 0$  by Claim 1. Thus we get

$$\begin{aligned} \delta(P_1) &= L(P_1) + SP_1 - P_1S = P_1L(P_1)P_1 \\ &= P_1\delta(P_1)P_1 - P_1(SP_1 - P_1S)P_1 = P_1\delta(P_1)P_1 \in \mathcal{M}_{11} \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} \delta(P_2) &= L(P_2) + SP_2 - P_2S = P_2L(P_2)P_2 \\ &= P_2\delta(P_2)P_2 - P_2(SP_2 - P_2S)P_2 = P_2\delta(P_2)P_2 \in \mathcal{M}_{22}. \end{aligned} \quad (3.4)$$

**Claim 2.**  $\delta(\mathcal{M}_{ii}) \subseteq \mathcal{M}_{ii}$ ,  $i = 1, 2$ .

We only give the proof for  $\mathcal{M}_{11}$ . The proof for  $\mathcal{M}_{22}$  is similar.

Take any  $A_{11} \in \mathcal{M}_{11}$ . Since  $[A_{11}, P_2]_\xi = 0$ , we have  $[\delta(A_{11}), P_2]_\xi + [A_{11}, \delta(P_2)]_\xi = 0$ . This and equation (3.4) yield

$$\delta(A_{11})P_2 - \xi P_2\delta(A_{11}) = 0. \quad (3.5)$$

Multiplying by  $P_1$  from the left side in equation (3.5), one gets

$$P_1\delta(A_{11})P_2 = 0; \quad (3.6)$$

multiplying by  $P_2$  from both sides in equation (3.5), one gets  $(1-\xi)P_2\delta(A_{11})P_2 = 0$ , which implies

$$P_2\delta(A_{11})P_2 = 0. \quad (3.7)$$

Note that  $[P_2, A_{11}]_\xi = 0$ . Then  $[\delta(P_2), A_{11}]_\xi + [P_2, \delta(A_{11})]_\xi = 0$ , which and equation (3.4) give  $P_2\delta(A_{11}) - \xi\delta(A_{11})P_2 = 0$ . Then, multiplying  $P_1$  from the right side in this equation, one gets

$$P_2\delta(A_{11})P_1 = 0. \quad (3.8)$$

Combining equations (3.6)–(3.8), we achieve that  $\delta(A_{11}) \in \mathcal{M}_{11}$ . So the claim is true.

**Claim 3.** For any  $A_{ij} \in \mathcal{M}_{ij}$ ,  $1 \leq i \neq j \leq 2$ , the following statements hold.

(1) If  $\xi \neq -1$ , then  $\delta(A_{ij}) \in \mathcal{M}_{ij}$ .

(2) If  $\xi = -1$ , then  $P_i \delta(A_{ij}) P_i = P_j \delta(A_{ij}) P_j = 0$  and  $\delta(A_{ij}) A_{ij} + A_{ij} \delta(A_{ij}) = 0$ .

For any  $A_{ij} \in \mathcal{M}_{ij}$  ( $i \neq j$ ), since  $[A_{ij}, A_{ij}]_\xi = 0$ , we have  $[\delta(A_{ij}), A_{ij}]_\xi + [A_{ij}, \delta(A_{ij})]_\xi = 0$ , that is,

$$\begin{aligned} \delta(A_{ij}) A_{ij} - \xi A_{ij} \delta(A_{ij}) + A_{ij} \delta(A_{ij}) - \xi \delta(A_{ij}) A_{ij} \\ = (1 - \xi)(\delta(A_{ij}) A_{ij} + A_{ij} \delta(A_{ij})) = 0. \end{aligned}$$

It follows from the assumption  $\xi \neq 1$  that

$$\delta(A_{ij}) A_{ij} + A_{ij} \delta(A_{ij}) = 0 \quad \text{for all } A_{ij} \in \mathcal{M}_{ij}. \quad (3.9)$$

Since  $[P_i + A_{ij}, A_{ij} - P_j]_\xi = 0$ , by equations (3.3)–(3.4) and equation (3.9), we have

$$\begin{aligned} 0 &= [\delta(P_i) + \delta(A_{ij}), A_{ij} - P_j]_\xi + [P_i + A_{ij}, \delta(A_{ij}) - \delta(P_j)]_\xi \\ &= \delta(P_i) A_{ij} - \delta(A_{ij}) P_j + \xi P_j \delta(A_{ij}) + P_i \delta(A_{ij}) - A_{ij} \delta(P_j) - \xi \delta(A_{ij}) P_i. \end{aligned}$$

Multiplying by  $P_i$  and  $P_j$  from both sides in the above equation, respectively, and noting that equations (3.3)–(3.4), one gets  $(1 - \xi) P_i \delta(A_{ij}) P_i = 0$  and  $(1 - \xi) P_j \delta(A_{ij}) P_j = 0$ , and so

$$P_i \delta(A_{ij}) P_i = 0 \quad \text{and} \quad P_j \delta(A_{ij}) P_j = 0. \quad (3.10)$$

Now, combining equations (3.9) and (3.10), we see that (2) is true.

To prove (1), one needs to check further that  $P_j \delta(A_{ij}) P_i = 0$  whenever  $\xi \neq -1$ . In fact, since  $[A_{ij}, P_i + \xi P_j]_\xi = 0$ , by Claim 2, we have

$$\begin{aligned} 0 &= [\delta(A_{ij}), P_i + \xi P_j]_\xi + [A_{ij}, \delta(P_i) + \delta(\xi P_j)]_\xi \\ &= \delta(A_{ij}) P_i + \xi \delta(A_{ij}) P_j - \xi P_i \delta(A_{ij}) - \xi^2 P_j \delta(A_{ij}) + A_{ij} \delta(\xi P_j) - \xi \delta(P_i) A_{ij}. \end{aligned}$$

Multiplying by  $P_j$  and  $P_i$  from the left and the right side respectively in the above equation, one obtains  $(1 - \xi^2) P_j \delta(A_{ij}) P_i = 0$ , which implies  $P_j \delta(A_{ij}) P_i = 0$  as  $\xi \neq -1$ . This and equation (3.10) yield  $\delta(A_{ij}) = P_i \delta(A_{ij}) P_j \in \mathcal{M}_{ij}$  whenever  $\xi \neq -1$ , and so (1) holds.

**Claim 4.** For any  $A_{ij} \in \mathcal{M}_{ij}$  ( $1 \leq i \neq j \leq 2$ ), we have  $\delta(P_i) A_{ij} = A_{ij} \delta(P_j)$ .

Take any  $A_{ij} \in \mathcal{M}_{ij}$  ( $1 \leq i \neq j \leq 2$ ). Since  $[P_i + A_{ij}, A_{ij} - P_j]_\xi = 0$ , by equations (3.3)–(3.4) and Claim 3, we have

$$0 = [\delta(P_i) + \delta(A_{ij}), A_{ij} - P_j]_\xi + [P_i + A_{ij}, \delta(A_{ij}) - \delta(P_j)]_\xi$$

$$\begin{aligned}
&= \delta(P_i)A_{ij} + \delta(A_{ij})A_{ij} - \delta(A_{ij})P_j - \xi A_{ij}\delta(A_{ij}) + \xi P_j\delta(A_{ij}) \\
&\quad + P_i\delta(A_{ij}) + A_{ij}\delta(A_{ij}) - A_{ij}\delta(P_j) - \xi\delta(A_{ij})P_i - \xi\delta(A_{ij})A_{ij} \\
&= P_i\delta(P_i)A_{ij} + \xi P_j\delta(A_{ij})P_i - A_{ij}\delta(P_j)P_j - \xi P_j\delta(A_{ij})P_i.
\end{aligned}$$

This implies that  $P_i\delta(P_i)A_{ij} - A_{ij}\delta(P_j)P_j = 0$ , that is,  $\delta(P_i)A_{ij} = A_{ij}\delta(P_j)$  holds for all  $A_{ij} \in \mathcal{M}_{ij}$ .

**Claim 5.**  $\delta(\xi I) = \xi\delta(I)$ .

For any  $A_{ij} \in \mathcal{M}_{ij}$  ( $1 \leq i \neq j \leq 2$ ), since  $[\xi P_i + P_j, A_{ij}]_\xi = 0$ , by Claim 2, we have

$$\begin{aligned}
0 &= [\delta(\xi P_i) + \delta(P_j), A_{ij}]_\xi + [\xi P_i + P_j, \delta(A_{ij})]_\xi \\
&= \delta(\xi P_i)A_{ij} - \xi A_{ij}\delta(P_j) + \xi P_i\delta(A_{ij}) + P_j\delta(A_{ij}) - \xi^2\delta(A_{ij})P_i - \xi\delta(A_{ij})P_j.
\end{aligned}$$

Multiplying by  $P_i$  and  $P_j$  from the left and the right side respectively in the above equation, by Claim 2 and Claim 4, one can get

$$\delta(\xi P_i)A_{ij} = P_i\delta(\xi P_i)A_{ij} = \xi A_{ij}\delta(P_j)P_j = \xi A_{ij}\delta(P_j) = \xi\delta(P_i)A_{ij}.$$

That is,  $(\delta(\xi P_i) - \xi\delta(P_i))P_iAP_j = 0$  for all  $A \in \mathcal{M}$ . Note that  $\overline{P_j} = I$ . It follows from the definition of the central carrier that  $\text{span}\{TP_j(x) : T \in \mathcal{M}, x \in H\}$  is dense in  $H$ . So  $\delta(\xi P_i) = \xi\delta(P_i)$  for  $i = 1, 2$ . Thus we obtain

$$\delta(\xi I) = \delta(\xi P_1 + \xi P_2) = \xi\delta(P_1) + \xi\delta(P_2) = \xi\delta(I).$$

**Claim 6.**  $\delta(I) \in \mathcal{Z}(\mathcal{M})$ .

By Claim 4, we have proved that, for any  $A_{12} \in \mathcal{M}_{12}$  and  $A_{21} \in \mathcal{M}_{21}$ ,

$$\delta(P_1)A_{12} = A_{12}\delta(P_2) \quad \text{and} \quad \delta(P_2)A_{21} = A_{21}\delta(P_1).$$

So, by using equations (3.3)–(3.4), we obtain

$$\delta(I)A_{12} = \delta(P_1)A_{12} = A_{12}\delta(P_2) = A_{12}\delta(I)$$

and

$$\delta(I)A_{21} = \delta(P_2)A_{21} = A_{21}\delta(P_1) = A_{21}\delta(I).$$

It follows from Lemma 3.3 that  $\delta(I) \in \mathcal{Z}(\mathcal{M})$ . The claim is true.

Now, note that  $\delta(A) = L(A) + SA - AS$  for each  $A \in \mathcal{M}$ . By Claim 6, one has proved that

$$L(I) = \delta(I) \in \mathcal{Z}(\mathcal{M}). \tag{3.11}$$

**Claim 7.** If  $\xi \neq -1$ , then there exists an additive derivation  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  satisfying  $\varphi(\xi A) = \xi\varphi(A)$  for each  $A \in \mathcal{M}$  such that  $L(A) = \varphi(A) + L(I)A$  for all  $A \in \mathcal{M}$ .

We will complete the proof of Claim 7 by several steps.

**Step 1.** For any  $A_{ii} \in \mathcal{M}_{ii}$  and  $B_{ij} \in \mathcal{M}_{ij}$ , we have  $\delta(A_{ii}B_{ij}) = \delta(A_{ii})B_{ij} + A_{ii}\delta(B_{ij}) - A_{ii}B_{ij}\delta(I)$ ,  $1 \leq i \neq j \leq 2$ .

Let  $1 \leq i \neq j \leq 2$ . For any  $A_{ii} \in \mathcal{M}_{ii}$  and  $B_{ij} \in \mathcal{M}_{ij}$ , since  $[A_{ii} + A_{ii}B_{ij}, B_{ij} - P_j]_\xi = 0$ , by Claim 2 and Claim 3(1), we have

$$\begin{aligned} 0 &= [\delta(A_{ii}) + \delta(A_{ii}B_{ij}), B_{ij} - P_j]_\xi + [A_{ii} + A_{ii}B_{ij}, \delta(B_{ij}) - \delta(P_j)]_\xi \\ &= \delta(A_{ii})B_{ij} - \delta(A_{ii}B_{ij}) + A_{ii}\delta(B_{ij}) - A_{ii}B_{ij}\delta(P_j). \end{aligned}$$

Note that  $A_{ii}B_{ij}\delta(P_j) = A_{ii}B_{ij}\delta(I)$  by equations (3.3)–(3.4). So Step 1 holds.

**Step 2.** For any  $A_{ii} \in \mathcal{M}_{ii}$  and  $B_{ij} \in \mathcal{M}_{ij}$ , we have  $\delta(A_{ij}B_{jj}) = \delta(A_{ij})B_{jj} + A_{ij}\delta(B_{jj}) - A_{ij}B_{jj}\delta(I)$ ,  $1 \leq i \neq j \leq 2$ .

As  $[A_{ij}B_{jj} + B_{jj}, A_{ij} - P_i]_\xi = 0$ , the assertion of Step 2 follows from Claim 2 and Claim 3(1) immediately.

**Step 3.** For any  $A_{ii}, B_{ii} \in \mathcal{M}_{ii}$ , we have  $\delta(A_{ii}B_{ii}) = \delta(A_{ii})B_{ii} + A_{ii}\delta(B_{ii}) - A_{ii}B_{ii}\delta(I)$ ,  $i = 1, 2$ .

Let  $i \neq j$ . Take any  $A_{ii}, B_{ii} \in \mathcal{M}_{ii}$  and any  $S_{ij} \in \mathcal{M}_{ij}$ . By Step 1, we get

$$\delta(A_{ii}B_{ii}S_{ij}) = \delta(A_{ii}B_{ii})S_{ij} + A_{ii}B_{ii}\delta(S_{ij}) - A_{ii}B_{ii}S_{ij}\delta(I)$$

and

$$\begin{aligned} \delta(A_{ii}B_{ii}S_{ij}) &= \delta(A_{ii})B_{ii}S_{ij} + A_{ii}\delta(B_{ii}S_{ij}) - A_{ii}B_{ii}S_{ij}\delta(I) \\ &= \delta(A_{ii})B_{ii}S_{ij} + A_{ii}\delta(B_{ii})S_{ij} + A_{ii}B_{ii}\delta(S_{ij}) - 2A_{ii}B_{ii}S_{ij}\delta(I). \end{aligned}$$

Comparing the above two equations, and by Claims 2 and 6, one obtains that

$$(\delta(A_{ii}B_{ii}) - \delta(A_{ii})B_{ii} - A_{ii}\delta(B_{ii}) + A_{ii}B_{ii}\delta(I))SP_j = 0$$

holds for all  $S \in \mathcal{M}$ . It follows from the fact  $\overline{P}_j = I$  that

$$\delta(A_{ii}B_{ii}) = \delta(A_{ii})B_{ii} + A_{ii}\delta(B_{ii}) - A_{ii}B_{ii}\delta(I).$$

**Step 4.** For any  $A_{ij} \in \mathcal{M}_{ij}$  and  $B_{ji} \in \mathcal{M}_{ji}$ , we have  $\delta(A_{ij}B_{ji}) = \delta(A_{ij})B_{ji} + A_{ij}\delta(B_{ji}) - A_{ij}B_{ji}\delta(I)$ ,  $1 \leq i \neq j \leq 2$ .

Take any  $A_{ij} \in \mathcal{M}_{ij}$  and  $B_{ji} \in \mathcal{M}_{ji}$ . Note that

$$[-A_{12}B_{21} + A_{12} + B_{21} - P_2, P_1 + A_{12} + B_{21} + B_{21}A_{12}]_\xi = 0.$$

By the definition of  $\delta$ , Claim 2, Claim 3(1), Claim 4, Claim 6 and Steps 1-2, one gets

$$\begin{aligned}
0 &= [-\delta(A_{12}B_{21}) + \delta(A_{12}) + \delta(B_{21}) - \delta(P_2), P_1 + A_{12} + B_{21} + B_{21}A_{12}]_\xi \\
&\quad + [-A_{12}B_{21} + A_{12} + B_{21} - P_2, \delta(P_1) + \delta(A_{12}) + \delta(B_{21}) + \delta(B_{21}A_{12})]_\xi \\
&= -\delta(A_{12}B_{21}) - \delta(A_{12}B_{21})A_{12} + \delta(A_{12})B_{21} + \delta(A_{12})B_{21}A_{12} + \delta(B_{21}) \\
&\quad + \delta(B_{21})A_{12} - \delta(P_2)B_{21} - \delta(P_2)B_{21}A_{12} + \xi\delta(A_{12}B_{21}) - \xi\delta(A_{12}) - \xi A_{12}\delta(B_{21}) \\
&\quad + \xi A_{12}\delta(P_2) + \xi B_{21}\delta(A_{12}B_{21}) - \xi B_{21}\delta(A_{12}) - \xi B_{21}A_{21}\delta(B_{21}) + \xi B_{21}A_{12}\delta(P_2) \\
&\quad - A_{12}B_{21}\delta(P_1) - A_{12}B_{21}\delta(A_{12}) + A_{12}\delta(B_{21}) + A_{12}\delta(B_{21}A_{12}) + B_{21}\delta(P_1) \\
&\quad + B_{21}\delta(A_{12}) - \delta(B_{21}) - \delta(B_{21}A_{12}) + \xi\delta(P_1)A_{12}B_{21} - \xi\delta(P_1)A_{12} - \xi\delta(A_{12})B_{21} \\
&\quad + \xi\delta(A_{12}) + \xi\delta(B_{21})A_{12}B_{21} - \xi\delta(B_{21})A_{12} - \xi\delta(B_{21}A_{12})B_{21} + \xi\delta(B_{21}A_{12}) \\
&= (\xi - 1)\delta(A_{12}B_{21}) + (1 - \xi)\delta(A_{12})B_{21} + (1 - \xi)A_{12}\delta(B_{21}) + (\xi - 1)A_{12}B_{21}\delta(I) \\
&\quad + (\xi - 1)\delta(B_{21}A_{12}) + (1 - \xi)\delta(B_{21})A_{12} + (1 - \xi)B_{21}\delta(A_{12}) + (\xi - 1)B_{21}A_{12}\delta(I).
\end{aligned}$$

As  $\xi \neq 1$ , the above equation implies that

$$\delta(A_{12}B_{21}) = \delta(A_{12})B_{21} + A_{12}\delta(B_{21}) - A_{12}B_{21}\delta(I)$$

and

$$\delta(B_{21}A_{12}) = \delta(B_{21})A_{12} - B_{21}\delta(A_{12}) + B_{21}A_{12}\delta(I).$$

**Step 5.** For any  $A, B \in \mathcal{M}$ , we have  $\delta(AB) = \delta(A)B + A\delta(B) - \delta(I)AB$ .

For any  $A = A_{11} + A_{12} + A_{21} + A_{22}$ ,  $B = B_{11} + B_{12} + B_{21} + B_{22} \in \mathcal{M}$ , by the additivity of  $\delta$ , Claim 6 and Steps 1-4, one can easily check that  $\delta(AB) = \delta(A)B + A\delta(B) - \delta(I)AB$ .

**Step 6.** There exists an additive derivation  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  satisfying  $\varphi(\xi A) = \xi\varphi(A)$  for each  $A$  such that  $L(A) = \varphi(A) + L(I)A$  for all  $A \in \mathcal{M}$ .

Define a map  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  by  $\varphi(A) = L(A) - L(I)A = \delta(A) - SA + AS - L(I)A$  for all  $A \in \mathcal{M}$ . Obviously,  $\varphi$  is additive. Moreover, for any  $A, B \in \mathcal{M}$ , by Step 5 and equation (3.11), one achieves

$$\begin{aligned}
\varphi(AB) &= \delta(AB) - SAB + ABS - L(I)AB \\
&= \delta(A)B + A\delta(B) - 2L(I)AB - SAB + ABS \\
&= (\delta(A) - L(I)A - SA + AS)B + A(\delta(B) - L(I)B + BS - SB) \\
&= \varphi(A)B + A\varphi(B),
\end{aligned}$$

that is,  $\varphi$  is an additive derivation.

Now we show that  $\varphi(\xi A) = \xi\varphi(A)$  for each  $A \in \mathcal{M}$ . In fact, since  $\delta(\xi I) = \xi\delta(I)$  (Claim 5), we get  $\varphi(\xi I) = \xi\varphi(I) = 0$ . Hence, for any  $A \in \mathcal{M}$ , we have

$$\varphi(\xi A) = \varphi(\xi IA) = \varphi(\xi I)A + \xi\varphi(A) = \xi\varphi(A).$$

Step 6, combining equation (3.11), ensures that Claim 7 holds.

**Claim 8.** If  $\xi = -1$ , then there exists an additive derivation  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  such that  $L(A) = \varphi(A) + L(I)A$  for all  $A \in \mathcal{M}$ .

In this case, we will first show that  $\delta(A^2) = \delta(A)A + A\delta(A) - \delta(I)A^2$  for all  $A \in \mathcal{M}$ . This will be done by several steps.

**Step 1.** For any  $A_{ij} \in \mathcal{M}_{ij}$  and  $A_{jj} \in \mathcal{M}_{jj}$  ( $1 \leq i \neq j \leq 2$ ), we have

$$\delta(A_{ij}A_{jj}) = \delta(A_{ij})A_{jj} + A_{ij}\delta(A_{jj}) + A_{jj}\delta(A_{ij}) - \delta(I)A_{ij}A_{jj}.$$

Let  $1 \leq i \neq j \leq 2$ . Taking any  $A_{ij} \in \mathcal{M}_{ij}$ ,  $A_{jj} \in \mathcal{M}_{jj}$ , and noting that

$$[A_{ij}A_{jj} + A_{jj}, A_{ij} - P_i]_{-1} = (A_{ij}A_{jj} + A_{jj})(A_{ij} - P_i) + (A_{ij} - P_i)(A_{ij}A_{jj} + A_{jj}) = 0,$$

by Claim 2, we have

$$\begin{aligned} 0 &= [\delta(A_{ij}A_{jj}) + \delta(A_{jj}), A_{ij} - P_i]_{-1} + [A_{ij}A_{jj} + A_{jj}, \delta(A_{ij}) - \delta(P_i)]_{-1} \\ &= \delta(A_{ij}A_{jj})A_{ij} - \delta(A_{ij}A_{jj})P_i + A_{ij}\delta(A_{ij}A_{jj}) + A_{ij}\delta(A_{jj}) - P_i\delta(A_{ij}A_{jj}) \\ &\quad + A_{ij}A_{jj}\delta(A_{ij}) + A_{jj}\delta(A_{ij}) + \delta(A_{ij})A_{ij}A_{jj} + \delta(A_{ij})A_{jj} - \delta(P_i)A_{ij}A_{jj}. \end{aligned} \quad (3.12)$$

Multiplying by  $P_i$  and  $P_j$  from the left and the right side in equation (3.12) respectively, by Claim 2 and Claim 3(2), we get

$$\begin{aligned} P_i\delta(A_{ij}A_{jj})P_j &= P_i\delta(A_{ij})A_{jj} + A_{ij}\delta(A_{jj}) - \delta(P_i)A_{ij}A_{jj} \\ &= \delta(A_{ij})A_{jj} + A_{ij}\delta(A_{jj}) - \delta(I)A_{ij}A_{jj}. \end{aligned}$$

Similarly, multiplying by  $P_j$  and  $P_i$  from the left and the right side in equation (3.12) respectively, one can get

$$P_j\delta(A_{ij}A_{jj})P_i = A_{jj}\delta(A_{ij})P_i = A_{jj}\delta(A_{ij}).$$

So

$$\begin{aligned} \delta(A_{ij}A_{jj}) &= P_i\delta(A_{ij}A_{jj})P_j + P_j\delta(A_{ij}A_{jj})P_i \\ &= \delta(A_{ij})A_{jj} + A_{ij}\delta(A_{jj}) + A_{jj}\delta(A_{ij}) - \delta(I)A_{ij}A_{jj}. \end{aligned}$$

**Step 2.** For any  $A_{ii} \in \mathcal{M}_{ii}$  and  $A_{ij} \in \mathcal{M}_{ij}$  ( $1 \leq i \neq j \leq 2$ ), we have

$$\delta(A_{ii}A_{ij}) = \delta(A_{ii})A_{ij} + A_{ii}\delta(A_{ij}) + \delta(A_{ij})A_{ii} - \delta(I)A_{ii}A_{ij}.$$

As  $[A_{ii}A_{ij} + A_{ii}, A_{ij} - P_j]_{-1} = 0$ , by the same argument as that of Step 1, one can check that this step is true.

**Step 3.** For any  $A_{ii} \in \mathcal{M}_{ii}$ , we have  $\delta(A_{ii}^2) = \delta(A_{ii})A_{ii} + A_{ii}\delta(A_{ii}) - A_{ii}^2\delta(I)$ ,  $i = 1, 2$ .

Let  $j \neq i$ . For any  $A_{ii} \in \mathcal{M}_{ii}$  and any  $S_{ij} \in \mathcal{M}_{ij}$ , by using Step 2 and calculating  $\delta(A_{ii}A_{ii}S_{ij})$  by two different ways, one can easily check that the step is true.

**Step 4.** For any  $A_{ij} \in \mathcal{M}_{ij}$  and  $A_{ji} \in \mathcal{M}_{ji}$  ( $1 \leq i \neq j \leq 2$ ), we have

$$\delta(A_{ij}A_{ji}) = \delta(A_{ij})A_{ji} + A_{ij}\delta(A_{ji}) - A_{ij}A_{ji}\delta(I).$$

In fact, for any  $A_{ij} \in \mathcal{M}_{ij}$  and  $A_{ji} \in \mathcal{M}_{ji}$  ( $1 \leq i \neq j \leq 2$ ), since

$$[A_{ij}A_{ji} + A_{ij} + A_{ji} + P_j, P_i + A_{ij} - A_{ji} + A_{ji}A_{ij}]_{-1} = 0,$$

the assertion of this step follows from Claim 2, Claim 3(2) and Steps 1–3 in Claim 8.

**Step 5.** For any  $A \in \mathcal{M}$ , we have  $\delta(A^2) = \delta(A)A + A\delta(A) - \delta(I)A^2$ .

For any  $A = A_{11} + A_{12} + A_{21} + A_{22} \in \mathcal{M}$ , by the additivity of  $\delta$ , Claim 5 and Steps 1–4 in Case 2, it is easily checked that  $\delta(A^2) = \delta(A)A + A\delta(A) - \delta(I)A^2$ .

**Step 6.** There exists an additive derivation  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  such that  $L(A) = \varphi(A) + L(I)A$  holds for all  $A \in \mathcal{M}$ .

Define a map  $\varphi : \mathcal{M} \rightarrow \mathcal{M}$  by  $\varphi(A) = L(A) - L(I)A = \delta(A) - SA + AS - L(I)A$  for all  $A \in \mathcal{M}$ . It is clear that  $\varphi$  is additive and  $L(A) = \varphi(A) + L(I)A$  for each  $A$ . Moreover, by using Step 5 of Claim 8, one can check that  $\varphi$  is a Jordan derivation, that is,  $\varphi(A^2) = \varphi(A)A + A\varphi(A)$  for all  $A \in \mathcal{M}$ . By [1],  $\varphi$  is an additive derivation. This and equation (3.11) imply that Claim 8 holds.

Since  $\varphi(-1A) = -\varphi(A)$  for each  $A$  as  $\varphi$  is additive, Claims 7–8 and equation (3.11) together ensure that Theorem 3.1 is true, finishing the proof.  $\square$

By Theorem 3.1 we get a characterization of additive  $\xi$ -Lie derivations immediately.

**Corollary 3.4.** *Let  $\mathcal{M}$  be a von Neumann algebra without central summands of type  $I_1$ . Suppose that  $L : \mathcal{M} \rightarrow \mathcal{M}$  is an additive map and  $\xi$  is a scalar with  $\xi \neq 1$ . Then the following statements are equivalent.*

- (1)  $L$  is a  $\xi$ -Lie derivation.
- (2)  $L$  is a derivation with  $L(\xi A) = \xi L(A)$  for all  $A \in \mathcal{M}$ .
- (3)  $L$  is  $\xi$ -Lie derivable at zero and  $L(I) = 0$ .

Finally, we remark that, by a similar approach to the proof of Theorem 3.1, we can improve further the main result [22, Theorem 2.1] omitting the assumptions  $L(I) \in \mathcal{Z}(\mathcal{A})$  and  $L(\xi A) = \xi L(A)$  for each  $A$  in the prime algebra case.

**Theorem 3.5.** *Let  $\mathcal{A}$  be a unital prime algebra over a field  $\mathbb{F}$  containing a non-trivial idempotent  $P$ . Denote by  $\mathcal{Z}(\mathcal{A})$  and  $\mathcal{C}$  the center of  $\mathcal{A}$  and the extended centroid of  $\mathcal{A}$ , respectively. Assume that  $\xi \in \mathbb{F}$  and  $L : \mathcal{A} \rightarrow \mathcal{A}$  is an additive map. If  $L$  is  $\xi$ -Lie derivable at zero, then there exists an additive derivation  $\tau : \mathcal{A} \rightarrow \mathcal{C}$  such that*

- (1) *if  $\xi = 0$ , then  $L(I) \in \mathcal{Z}(\mathcal{A})$  and  $L(A) = \tau(A) + L(I)A$  for all  $A \in \mathcal{A}$ ;*
- (2) *if  $\xi = 1$  and  $\mathcal{A}$  is of characteristic not 2 with  $\deg(\mathcal{A}) > 2$ , then  $L(A) = \tau(A) + \alpha A + \nu(A)$  for all  $A \in \mathcal{A}$ , where  $\alpha \in \mathcal{C}$  and  $\nu$  is an additive map from  $\mathcal{A}$  into  $\mathcal{C}$ ;*
- (3) *if  $\xi = -1$  and  $\mathcal{A}$  is of characteristic not 2, then  $L(I) \in \mathcal{Z}(\mathcal{A})$  and  $L(A) = \tau(A) + L(I)A$  for all  $A \in \mathcal{A}$ ;*
- (4) *if  $\xi \neq 0, \pm 1$ , then  $L(I) \in \mathcal{Z}(\mathcal{A})$ ,  $L(\xi A) = \xi L(A)$  and  $L(A) = \tau(A) + L(I)A$  for all  $A \in \mathcal{A}$ .*

PROOF. Assume that  $L : \mathcal{A} \rightarrow \mathcal{A}$  is  $\xi$ -Lie derivable at zero. We need only to show that, for  $\xi \neq 1$ ,  $L(I) \in \mathcal{Z}(\mathcal{A})$ ,  $L(\xi A) = \xi L(A)$ .

Using the same notations as that in the proof of Theorem 3.1, but replacing  $\mathcal{M}$  by the prime algebra  $\mathcal{A}$ , Claims 1–4 are still true. Thus, similar to the argument in Claim 5, we have

$$\delta(\xi P_i)A_{ij} = P_i\delta(\xi P_i)A_{ij} = \xi A_{ij}\delta(P_j)P_j = \xi A_{ij}\delta(P_j) = \xi\delta(P_i)A_{ij}.$$

So  $(\delta(\xi P_i) - \xi\delta(P_i))P_iAP_j = 0$  for all  $A \in \mathcal{A}$ . Since  $\mathcal{A}$  is prime, we must have  $\delta(\xi P_i) - \xi\delta(P_i) = (\delta(\xi P_i) - \xi\delta(P_i))P_i = 0$ , that is,  $\delta(\xi P_i) = \xi\delta(P_i)$  for  $i = 1, 2$ . Thus we obtain

$$\delta(\xi I) = \delta(\xi P_1 + \xi P_2) = \xi\delta(P_1) + \xi\delta(P_2) = \xi\delta(I).$$

Then, a similar argument as in Claim 6 gives

$$\delta(I)A_{12} = \delta(P_1)A_{12} = A_{12}\delta(P_2) = A_{12}\delta(I).$$

This implies that

$$\delta(P_1)AP_2 = P_1A\delta(P_2) \quad \text{for all } A \in \mathcal{A}.$$

Again, as  $\mathcal{A}$  is prime, it follows from [6, Theorem A.7] that there exists some central element  $\lambda \in \mathcal{C}$  such that  $\delta(P_i) = \lambda P_i$ . Hence  $\delta(I) = \delta(P_1) + \delta(P_2) =$

$\lambda(P_1 + P_2) = \lambda \in \mathcal{C} \cap \mathcal{A} = \mathcal{Z}(\mathcal{A})$ . Thus  $L(I) = \delta(I) \in \mathcal{Z}(\mathcal{A})$  and  $L(\xi I) = \xi L(I)$ .

Now, similar to the proof of Claim 8, and using the primeness of  $\mathcal{A}$  where needs, one can show that, for any  $A \in \mathcal{A}$ , we have  $\varphi(\xi A) = \varphi(\xi I A) = \varphi(\xi I) A + \xi \varphi(A) = \xi \varphi(A)$ . This, together with the facts proved before, entails that  $L(\xi A) = \xi L(A)$  for all  $A \in \mathcal{A}$ .  $\square$

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