

Parallelism of normal Jacobi operator for real hypersurfaces in complex two-plane Grassmannians

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Abstract. In this paper, we give a partial classification of $\mathfrak{D}^\perp$ -invariant real hypersurfaces in complex two-plane Grassmannians with Reeb parallel normal Jacobi operator.

1. Introduction

The Jacobi fields along geodesics of a given Riemannian manifold $(\tilde{M}, \tilde{g})$ satisfy a well-known differential equation. This classical differential equation naturally inspires the so-called Jacobi operator. That is, if $\tilde{R}$ is the Riemannian curvature tensor of $\tilde{M}$, and X is any tangent vector field to $\tilde{M}$, the Jacobi operator with respect to X at $p \in \tilde{M}$ is defined by

$$(\tilde{R}_X Y)(p) = (\tilde{R}(Y, X)X)(p)$$

for any $Y \in T_p \tilde{M}$, becomes a self adjoint endomorphism of the tangent bundle $T\tilde{M}$ of $\tilde{M}$. Clearly, each tangent vector field X to $\tilde{M}$ provides a Jacobi operator with respect to X .

In the geometry of real hypersurfaces in complex space forms or in quaternionic space forms there have been many characterizations of homogeneous hypersurfaces of type (A_1) , (A_2) , (B) , (C) , (D) and (E) in complex projective

Mathematics Subject Classification: Primary: 53C40; Secondary: 53C15.

Key words and phrases: real hypersurfaces, complex two-plane Grassmannians, Hopf hypersurface, normal Jacobi operator, Reeb parallel normal Jacobi operator.

This work was supported by grant Proj. No. NRF-2015-R1A2A1A-01002459 from National Research Foundation of Korea.

space $\mathbb{C}P^n$, of type (A_0) , (A_1) , (A_2) and (B) in complex hyperbolic space $\mathbb{C}H^n$ or of type (A_1) , (A_2) and (B) in quaternionic projective space $\mathbb{H}P^n$, which are completely classified by TAKAGI ([14]), CECIL and RYAN ([6]), KIMURA ([9]), MONTIAL and ROMERO ([11]) and MARTINEZ and PÉREZ ([10]), respectively.

In quaternionic space forms BERNDT ([2]) has introduced the notion of normal Jacobi operator

$$\bar{R}_N = \bar{R}(X, N)N \in \text{End } T_x M, \quad x \in M$$

for real hypersurfaces M in a quaternionic projective space $\mathbb{H}P^n$ or in a quaternionic hyperbolic space $\mathbb{H}H^n$, where $\bar{R}$ denotes the Riemannian curvature tensor of $\mathbb{H}P^n$ and $\mathbb{H}H^n$ respectively. The almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ are defined by $\xi_i = -J_i N$, $i = 1, 2, 3$, where $\{J_1, J_2, J_3\}$ denote a canonical local basis of quaternionic Kähler structure on $\mathbb{H}P^n$ and N a unit normal vector field of M in $\mathbb{H}P^n$. He has also shown that the curvature adaptedness, that is, the normal Jacobi operator $\bar{R}_N$ commutes with the shape operator A , is equivalent to the fact that the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp = \text{span}\{\xi_1, \xi_2, \xi_3\}$ are invariant by the shape operator A of M , that is, $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$, where $T_x M = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$. And he gave a complete classification of curvature adapted real hypersurfaces in non-flat quaternionic space forms with the assumption of constant principal curvatures in the hyperbolic case (See [2]).

Now let us consider a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ which consists of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The ambient space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometric structure. It was known that the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$. By using such kinds of two natural geometric structures, many geometers have investigated some characterizations for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. Among them, BERNDT and SUH ([4], [5]) and SUH ([13]) have shown some examples of two kinds of tubes, which said to be of type (A) and of type (B), and have given a characterization of type (A) (resp. of type (B)) by the isometric Reeb flow in [5] (resp. contact hypersurfaces in [13]).

As one of examples BERNDT and SUH [4] considered two natural geometric conditions for hypersurfaces in $G_2(\mathbb{C}^{m+2})$ that $[\xi] = \text{span}\{\xi\}$ and $\mathfrak{D}^\perp = \text{span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator. By using such conditions and the result in ALEKSEEVSKII [1], they have proved the following

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

- (A) M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or
- (B) m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.

The structure vector field ξ , $\xi = -JN$, of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be a *Reeb* vector field. If the *Reeb* vector field ξ of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is invariant by the shape operator, M is said to be a *Hopf hypersurface*. In such a case the integral curves of the *Reeb* vector field ξ are geodesics (See [5]).

The Riemannian curvature tensor $\bar{R}(X, Y)Z$ for any tangent vector fields X , Y and Z on $G_2(\mathbb{C}^{m+2})$ is explicitly defined in [3]. In a paper [12] due to PÉREZ, JEONG and SUH, we have introduced a notion of normal Jacobi operator $\bar{R}_N$ for hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ in such a way that

$$\bar{R}_N X = \bar{R}(X, N)N \in \text{End } T_x M, \quad x \in M,$$

for any tangent vector field X on M , where $\bar{R}$ and N respectively denote the Riemannian curvature tensor and a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. Related to such a normal Jacobi operator $\bar{R}_N$, JEONG, KIM and SUH [7] obtained a non-existence theorem for *Hopf hypersurfaces* in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator, that is, $(\nabla_X \bar{R}_N)Y = 0$, $\forall X \in TM$, where ∇ denotes the induced Riemannian connection on M .

Motivated by this fact, in such a paper we consider more general notion of parallelism weaker than the notion of parallel normal Jacobi operator. So we consider a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with *Reeb parallel* normal Jacobi operator, that is, $\nabla_\xi \bar{R}_N = 0$. The normal Jacobi operator $\bar{R}_N$ is said to be *Reeb parallel* on M if the covariant derivative of the normal Jacobi operator $\bar{R}_N$ along the direction of the *Reeb* vector ξ identically vanishes, that is, $\nabla_\xi \bar{R}_N = 0$. Here the meaning of *Reeb parallel* normal Jacobi operator $\bar{R}_N$ gives that every eigenspaces of the normal Jacobi operator $\bar{R}_N$ are *parallel* along the integral curve γ of the *Reeb* vector field ξ in M . Here the eigenspaces of the normal Jacobi operator $\bar{R}_N$ are said to be *parallel* along the curve γ if they are *invariant* under the *parallel displacement* along the curve γ in M .

Related to such a *Reeb parallel* normal Jacobi operator $\bar{R}_N$ in section 3 we prove an important theorem for $\mathfrak{D}^\perp$ -invariant real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ as follows:

Main Theorem. *Let M be a $\mathfrak{D}^\perp$ -invariant real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with *Reeb parallel* normal Jacobi operator. If the distribution $\mathfrak{D}$ and $\mathfrak{D}^\perp$*

components of the Reeb vector field are eigenvectors of the shape operator at every point, then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius $r \in (0, \pi/\sqrt{8})$.

As a corollary of this theorem, together with the result in [7], we may assert the following

Corollary. *There do not exist any connected $\mathfrak{D}^\perp$ -invariant real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator if the distribution $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components of the Reeb vector field are eigenvectors of the shape operator at every point.*

In the sequel we will use some notations as in [4] and [5].

2. Reeb parallel normal Jacobi operator

In this section we want to derive some formulas related to the *Reeb parallel* normal Jacobi operator from the curvature tensor $\bar{R}(X, Y)Z$ of $G_2(\mathbb{C}^{m+2})$. Moreover, we will show whether the hypersurfaces of type (A) or (B) in Theorem A have Reeb parallel normal Jacobi operator, that is, result mentioned in our main theorem satisfy the assumption of *Reeb parallel*, that is, $\nabla_\xi \bar{R}_N = 0$. From now, unless otherwise stated, let us follow the notations such as η , η_ν , ϕ , and ϕ_ν in [4], [5], [7], [12] and [13].

Then first the normal Jacobi operator $\bar{R}_N$ can be defined in such a way that

$$\begin{aligned} \bar{R}_N(X) &= \bar{R}(X, N)N = X + 3\eta(X)\xi + 3\sum_{\nu=1}^3 \eta_\nu(X)\xi_\nu \\ &\quad - \sum_{\nu=1}^3 \{ \eta_\nu(\xi)(\phi_\nu \phi X - \eta(X)\xi_\nu) - \eta_\nu(\phi X)\phi_\nu \xi \}. \end{aligned} \quad (2.1)$$

We used these standard notations such as η , η_ν , ϕ , ϕ_ν in (2.1). These were used in [8]. Of course, we know that the normal Jacobi operator $\bar{R}_N$ is a symmetric endomorphism of $T_x M$, $x \in M$ ([8], [12]).

A real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator, that is, $\nabla_X \bar{R}_N = 0$ for any tangent vector field X on M , satisfies the following

$$\begin{aligned} 0 &= (\nabla_X \bar{R}_N)Y = 3g(\phi AX, Y)\xi + 3\eta(Y)\phi AX \\ &\quad + 3\sum_{\nu=1}^3 \{ g(\phi_\nu AX, Y)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \} \\ &\quad - \sum_{\nu=1}^3 [2\eta_\nu(\phi AX)(\phi_\nu \phi Y - \eta(Y)\xi_\nu) - g(\phi_\nu AX, \phi Y)\phi_\nu \xi \\ &\quad - \eta(Y)\eta_\nu(AX)\phi_\nu \xi - \eta_\nu(\phi Y)(\phi_\nu \phi AX - g(AX, \xi)\xi_\nu)] \end{aligned}$$

(See [8]).

From this, by putting $X = \xi$ and replacing Y by X , we have

$$\begin{aligned} 0 &= (\nabla_\xi \bar{R}_N)X = 3g(\phi A\xi, X)\xi + 3\eta(X)\phi A\xi \\ &\quad + 3\sum_{\nu=1}^3 \{g(\phi_\nu A\xi, X)\xi_\nu + \eta_\nu(X)\phi_\nu A\xi\} \\ &\quad - \sum_{\nu=1}^3 [2\eta_\nu(\phi A\xi)(\phi_\nu \phi X - \eta(X)\xi_\nu) - g(\phi_\nu A\xi, \phi X)\phi_\nu \xi \\ &\quad - \eta(X)\eta_\nu(A\xi)\phi_\nu \xi - \eta_\nu(\phi X)(\phi_\nu \phi A\xi - g(A\xi, \xi)\xi_\nu)] \end{aligned} \quad (2.2)$$

for any tangent vector field X on M in $G_2(\mathbb{C}^{m+2})$. And by putting $X = \xi$ into (2.2), we have

$$\begin{aligned} 0 &= (\nabla_\xi \bar{R}_N)\xi = 3\phi A\xi + 5\sum_{\nu=1}^3 \eta_\nu(\phi A\xi)\xi_\nu \\ &\quad + 3\sum_{\nu=1}^3 \eta_\nu(\xi)\phi_\nu A\xi + \sum_{\nu=1}^3 \eta_\nu(A\xi)\phi_\nu \xi. \end{aligned} \quad (2.3)$$

Now we check whether the normal Jacobi operator $\bar{R}_N$ for hypersurfaces of type (A) or of type (B) is Reeb parallel or not. By using (2.2), (2.3) and from Proposition 3 in [4], we check for a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ whether it has Reeb parallel normal Jacobi operator or not as follows:

Case I: $\xi = \xi_1 \in T_\alpha$.

Then by (2.3) we have

$$(\nabla_\xi \bar{R}_N)\xi = 4\alpha \sum_{\nu=1}^3 \eta_\nu(\xi)\phi_\nu \xi = 4\alpha \phi_1 \xi_1 = 0.$$

Case II: $\xi_2, \xi_3 \in T_\beta$.

Putting $X = \xi_2$ into (2.2) and using $\phi\xi_2 = -\xi_3, \phi\xi_3 = \xi_2$, we have

$$\begin{aligned} (\nabla_\xi \bar{R}_N)\xi_2 &= 3\sum_{\nu=1}^3 \{g(\phi_\nu A\xi, \xi_2)\xi_\nu + \eta_\nu(\xi_2)\phi_\nu A\xi\} \\ &\quad - \sum_{\nu=1}^3 \{-g(\phi_\nu A\xi, \phi\xi_2)\phi_\nu \xi + \eta_\nu(\phi\xi_2)g(A\xi, \xi)\xi_\nu\} \\ &= 3\alpha(g(\phi_2\xi, \xi_2)\xi_2 + g(\phi_3\xi, \xi_2)\xi_3) + 3\alpha\phi_2\xi + \alpha\phi_2\xi + \alpha\xi_3 = 0. \end{aligned}$$

Similarly, by putting $X = \xi_3$ into (2.2) we know that $(\nabla_\xi \bar{R}_N)\xi_3 = 0$.

Case III: $X_i \in T_\lambda, i = 1, \dots, 2(m-1)$.

Then by Proposition 3 in [4] we know that the eigenspace T_λ has the property that $\phi X = \phi_1 X$ for any $X \in T_\lambda$. Moreover, it is invariant by the structure tensor ϕ , that is $\phi T_\lambda \subset T_\lambda$. Because for any $X_i \in \mathfrak{D}$ such that $\phi X_i = \phi_1 X_i$ we have $\phi\phi X_i =$

$-X_i$ and $\phi_1\phi X_i = \phi_1^2 X_i = -X_i$. Then $\phi\phi X_i = \phi_1\phi X_i$. So it follows that $\phi X_i \in T_\lambda$. From this, together with (2.2), we have $(\nabla_\xi \bar{R}_N)X_i = 0$, $i = 1, \dots, 2(m-1)$.

Case IV: $Y_i \in T_\mu$, $i = 1, \dots, 2(m-1)$.

The eigenspace T_μ has the property that $\phi Y = -\phi_1 Y$ for any $Y \in T_\mu$. Moreover, such an eigenspace T_μ is ϕ -invariant, that is, $\phi T_\mu \subset T_\mu$. In fact, suppose $Y_i \in \mathfrak{D}$ such that $\phi Y_i = -\phi_1 Y_i$. Then $\phi Y_i \in \mathfrak{D}$ and satisfies $\phi\phi Y_i = -Y_i$ and $\phi_1\phi Y_i = -\phi_1^2 Y_i = Y_i$. So it follows that $\phi Y_i \in T_\mu$. Then also by using (2.2) we have $(\nabla_\xi \bar{R}_N)Y_i = 0$, $i = 1, \dots, 2(m-1)$.

Then by these Cases I, II, III and IV we know that a real hypersurface of type (A) in Theorem A has Reeb parallel normal Jacobi operator $\bar{R}_N$ for $\xi \in \mathfrak{D}^\perp$.

Next, we check whether the normal Jacobi operator $\bar{R}_N$ for hypersurfaces of type (B) is Reeb parallel or not. Now let us consider a unit eigenvector $X \in T_\beta$ from Proposition 2 in [4]. In other words, we can substitute $X = \xi_\mu \in T_\beta$ into (2.2). Then it follows that

$$\begin{aligned} 0 &= (\nabla_\xi \bar{R}_N)\xi_\mu = 3 \sum_{\nu=1}^3 \{g(\phi_\nu A\xi, \xi_\mu)\xi_\nu + \eta_\nu(\xi_\mu)\phi_\nu A\xi\} \\ &\quad + \sum_{\nu=1}^3 g(\phi_\nu A\xi, \phi\xi_\mu)\phi_\nu \xi = 4\alpha\phi_\mu \xi. \end{aligned}$$

Since $\alpha = -2\tan(2r)$ is non zero for some $r \in (0, \pi/4)$, we have $\phi_\mu \xi = 0$. But $g(\phi_\mu \xi, \phi_\mu \xi) = 1$, which makes a contradiction. So we know that the normal Jacobi operator $\bar{R}_N$ for hypersurfaces of type (B) in $G_2(\mathbb{C}^{m+2})$ can not be Reeb parallel when the Reeb vector ξ belongs to the distribution $\mathfrak{D}$.

3. Proof of Main Theorem

Now let us consider a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with Reeb parallel normal Jacobi operator, that is, $(\nabla_\xi \bar{R}_N)X = 0$ for any tangent vector field $X \in TM$.

We assert the following

Lemma 3.1. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel normal Jacobi operator. If the distribution $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components of the Reeb vector field are eigenvectors of the shape operator at every point, then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

PROOF. Let us assume that $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit $X_0 \in \mathfrak{D}$, non-zero functions $\eta(X_0)$ and $\eta(\xi_1)$.

From this, together with formula (2.3), we have

$$\begin{aligned} 0 &= 3(\eta(X_0)\phi AX_0 + \eta(\xi_1)\phi A\xi_1) + 5\sum_{\nu=1}^3 g(\xi_\nu, \eta(X_0)\phi AX_0 \\ &\quad + \eta(\xi_1)\phi A\xi_1)\xi_\nu + 3\eta(\xi_1)(\eta(X_0)\phi_1 AX_0 + \eta(\xi_1)\phi_1 A\xi_1) \\ &\quad + \sum_{\nu=1}^3 g(\xi_\nu, \eta(X_0)AX_0 + \eta(\xi_1)A\xi_1)\phi_\nu \xi. \end{aligned} \quad (3.1)$$

And we consider the distribution $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components of the Reeb vector field are eigenvectors of the shape operator at every point. Then this gives the following

$$AX_0 = g(AX_0, X_0)X_0, \quad A\xi_1 = g(A\xi_1, \xi_1)\xi_1.$$

By using this in (3.1), we have

$$\begin{aligned} 0 &= 3\{\eta(X_0)g(AX_0, X_0)\phi X_0 + \eta(\xi_1)g(A\xi_1, \xi_1)\phi \xi_1\} \\ &\quad + 5\sum_{\nu=1}^3 g(\xi_\nu, \eta(X_0)g(AX_0, X_0)\phi X_0 + \eta(\xi_1)g(A\xi_1, \xi_1)\phi \xi_1)\xi_\nu \\ &\quad + 3\eta(\xi_1)\eta(X_0)g(AX_0, X_0)\phi_1 X_0 + \eta(\xi_1)g(A\xi_1, \xi_1)\phi_1 \xi. \end{aligned}$$

From this, by the assumption of $\eta(X_0)\eta(\xi_1) \neq 0$, we have

$$0 = g(A\xi_1, \xi_1)\phi_1 X_0,$$

where we have used the following

$$\phi \xi_1 = \eta(X_0)\phi_1 X_0, \quad \phi X_0 = -\eta(\xi_1)\phi_1 X_0, \quad \eta_\nu(\phi_1 X_0) = 0 \quad (\nu = 1, 2, 3).$$

So we assert that $g(A\xi_1, \xi_1) = 0$.

From this, we obtain the following

$$A\xi = \eta(X_0)g(AX_0, X_0)X_0.$$

So we put $A\xi = \sigma X_0$, where $\sigma = \eta(X_0)g(AX_0, X_0)$.

On the other hand, we may put $X = X_0 \in \mathfrak{D}$ in (2.2). Then we have the following

$$0 = \sigma\eta(X_0)\eta(\xi_1)\phi_1 X_0.$$

From this, it follows that $\sigma = 0$, where we have used the assumption of $\eta(X_0)\eta(\xi_1) \neq 0$. Then we say that the geodesic Reeb flow is vanishing, that is, $A\xi = 0$.

Since M is Hopf, we can use the result due to BERNT and SUH (see [5], p. 92).

Then we have the following

$$0 = \sum_{\nu=1}^3 \eta_\nu(\xi)\phi \xi_\nu = \eta_1(\xi)\phi \xi_1 = \eta(\xi_1)\eta(X_0)\phi_1 X_0.$$

We have $\phi_1 X_0 = 0$, where we have used the assumption of $\eta(X_0)\eta(\xi_1) \neq 0$. This makes a contradiction, so the result follows. $\square$

According to Lemma 3.1 we divide two cases such that $\xi \in \mathfrak{D}$ or $\xi \in \mathfrak{D}^\perp$. First, we consider the case that ξ belongs to the distribution $\mathfrak{D}^\perp$.

Lemma 3.2. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel normal Jacobi operator. If the Reeb vector ξ belongs to the distribution $\mathfrak{D}^\perp$, then M becomes a Hopf hypersurface.*

PROOF. Now let us consider the case $\xi \in \mathfrak{D}^\perp$. So we may put $\xi = \xi_1$. Then we know that

$$\phi_2\xi = -\xi_3, \quad \phi_3\xi = \xi_2, \quad \eta_2(\phi A\xi) = \eta_3(A\xi), \quad \eta_3(\phi A\xi) = -\eta_2(A\xi).$$

By using these formulas and together with (2.3) we get the following

$$0 = 3\phi A\xi + 6\eta_3(A\xi)\xi_2 - 6\eta_2(A\xi)\xi_3 + 3\phi_1 A\xi. \quad (3.2)$$

From this, by taking an inner product with ξ_3 and ξ_2 , respectively, we obtain the following

$$\eta_2(A\xi) = 0, \quad \eta_3(A\xi) = 0. \quad (3.3)$$

Then substituting (3.3) into (3.2) implies

$$0 = \phi A\xi + \phi_1 A\xi.$$

From this, if we apply the structure tensor ϕ , we have

$$0 = -A\xi + \eta(A\xi)\xi + \phi\phi_1 A\xi. \quad (3.4)$$

And we have

$$\phi\phi_1 A\xi = \phi_1\phi A\xi = \phi_1\nabla_\xi\xi = -(\nabla_\xi\phi_1)\xi = q_2(\xi)\xi_2 + q_3(\xi)\xi_3 - A\xi + \eta(A\xi)\xi. \quad (3.5)$$

Now substituting (3.5) into (3.4), we have

$$0 = -2A\xi + 2\eta(A\xi)\xi + q_2(\xi)\xi_2 + q_3(\xi)\xi_3. \quad (3.6)$$

On the other hand, by using assumption we have

$$\nabla_\xi\xi = \nabla_\xi\xi_1.$$

So we have

$$\phi A\xi = q_3(\xi)\xi_2 - q_2(\xi)\xi_3 + \phi_1 A\xi.$$

From this, by taking an inner product with ξ_2 and ξ_3 , respectively, we obtain the following

$$q_3(\xi) = 2\eta_3(A\xi), \quad q_2(\xi) = 2\eta_2(A\xi).$$

And by using this, together with formula (3.3) we have

$$q_3(\xi) = 0, \quad q_2(\xi) = 0.$$

Then substituting these formulas into (3.6) gives

$$A\xi = \eta(A\xi)\xi.$$

This means that a real hypersurface M satisfying Reeb parallel normal Jacobi operator and $\xi \in \mathfrak{D}^\perp$ becomes a Hopf hypersurface in this case. $\square$

Next in the latter case we consider that ξ belongs to the distribution $\mathfrak{D}$.

Lemma 3.3. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel normal Jacobi operator. If the Reeb vector ξ belongs to the distribution $\mathfrak{D}$, then M becomes a Hopf hypersurface.*

PROOF. Now let us consider the case $\xi \in \mathfrak{D}$. Then by using (2.3) we have

$$0 = 3\phi A\xi + 5 \sum_{\nu=1}^3 \eta_\nu(\phi A\xi)\xi_\nu + \sum_{\nu=1}^3 \eta_\nu(A\xi)\phi_\nu\xi. \quad (3.7)$$

From this, by taking an inner product with ξ_μ , $\mu = 1, 2, 3$, we have

$$\eta_\mu(\phi A\xi) = 0, \quad \mu = 1, 2, 3. \quad (3.8)$$

By taking an inner product with $\phi_\mu\xi$, $\mu = 1, 2, 3$, into (3.8), we have

$$\eta_\mu(A\xi) = 0, \quad \mu = 1, 2, 3. \quad (3.9)$$

Applying (3.8) and (3.9) into (3.7) gives

$$0 = \phi A\xi.$$

From this, if we apply the structure tensor ϕ , we have

$$A\xi = \eta(A\xi)\xi.$$

This means that a real hypersurface M with Reeb parallel normal Jacobi operator and $\xi \in \mathfrak{D}$ becomes also a Hopf hypersurface. $\square$

Remark. If M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ and the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$, the normal Jacobi operator $\bar{R}_N$ becomes Reeb parallel. Even in such a case the condition of Hopf does not give us any meaning when we consider the Reeb parallel normal Jacobi operator. From such a point of a view, instead of Hopf we have considered the notion of $\mathfrak{D}^\perp$ -invariant real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ in our Main Theorem.

Then summing up Lemmas 3.1, 3.2, 3.3 and together with Theorem A ([4]), we know that a $\mathfrak{D}^\perp$ -invariant real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel normal Jacobi operator is locally congruent to of type (A) or of type (B) if the distribution $\mathfrak{D}$ and $\mathfrak{D}^\perp$ components of the Reeb vector field are eigenvectors of the shape operator at every point.

The converse part of our main result is checked in section 2, in which a real hypersurface of type (A) in Theorem A satisfies Reeb parallel normal Jacobi operator for $\xi \in \mathfrak{D}^\perp$. But we can easily verify that the normal Jacobi operator $\bar{R}_N$ for hypersurfaces of type (B) in $G_2(\mathbb{C}^{m+2})$ can not be Reeb parallel for $\xi \in \mathfrak{D}$. From this, we have completed the proof of our Main Theorem in the introduction.

ACKNOWLEDGEMENTS. The authors would like to express their thanks to the referee for his/her valuable suggestions and comments to develop the first version of our manuscript.

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(Received September 23, 2014; revised February 10, 2015)