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On p-hypercyclically embedded subgroups of finite groups

By YUEMEI MAO (Hefei), XIAOYU CHEN (Nanjing) and WENBIN GUO (Hefei)

Abstract. Let G be a finite group and p a prime. A normal subgroup E of G is said to be p-hypercyclically embedded in G if every p-chief factor of G below E is cyclic. We say that a subgroup H of G is generalized $S\Phi$-supplemented in G if G has a subnormal subgroup T such that $G = HT$ and $(H \cap T)H_{sG}/H_{sG} \leq \Phi(H/H_{sG})$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G. In this paper, some new characterizations of p-hypercyclically embeddability of normal subgroups of a finite group are obtained based on the assumption that some primary subgroups are generalized $S\Phi$-supplemented in G.

1. Introduction

Throughout this paper, all groups considered are finite. G always denotes a group, p denotes a prime, and $|G|_p$ denotes the order of a Sylow p-subgroup of G.

A normal subgroup E of G is said to be hypercyclically embedded (resp. p-hypercyclically embedded) in G if every chief factor (resp. p-chief factor) of G below E is cyclic. The hypercyclically embedded subgroups have a great influence on the structure of a group, and some important classes of groups can be characterized in terms of hypercyclically embedded subgroups. For example, if all subgroups of G of prime order or order 4 are hypercyclically embedded in G,

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then G is supersoluble (HUPPERT [12], DOERK [5], see also [23]). A group G is quasisupersoluble (i.e. for every non-cyclic chief factor H/K of G, every automorphism of H/K induced by an element of G is inner) if and only if it has a normal hypercyclically embedded subgroup E such that G/E is semisimple (see [10, Theorem C]). Some recent results in this topic can be found in, for example, [2], [9], [11], [22], [23], [24], [25].

Recall that a subgroup H of G is said to be s-permutable in G if H permutes with every Sylow subgroup of G. A subgroup H of G is said to be weakly s-permutable in G [21] if G has a subnormal subgroup T such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G. A subgroup H of a group G is called $S\Phi$-supplemented [17] (or Φ-s-supplemented [16]) in G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq \Phi(H)$, where $\Phi(H)$ is the Frattini subgroup of H. Note that H_{sG} is normal in H. We now introduce the following concept which is closely related to the above two concepts.

Definition 1.1. A subgroup H of G is said to be generalized $S\Phi$-supplemented in G if there exists a subnormal subgroup T of G such that $G = HT$ and $(H \cap T)H_{sG}/H_{sG} \leq \Phi(H/H_{sG})$.

It is easy to see that weakly s-permutable subgroups and $S\Phi$-supplemented subgroups of G are all generalized $S\Phi$-supplemented in G. But the following examples show that the converse does not hold in general.

Example 1.2. Let $G = Q_8 = \langle a, b \,|\, a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1} \rangle$ and $H = \langle b^2 \rangle$. Then, clearly, H is s-permutable in G and H has the unique supplement G in G. Hence H is generalized $S\Phi$-supplemented in G. But H is not $S\Phi$-supplemented in G because $\Phi(H) = 1$.

Example 1.3. Let $G = S_5$ be the symmetric group of degree 5 and $H = \langle (1234) \rangle$. Then $H_{sG} = H_G = 1$. Since $G = HA_5$ and $H \cap A_5 = \Phi(H) = \langle (13)(24) \rangle$, H is generalized $S\Phi$-supplemented in G, but H is not weakly s-permutable in G.

A class of groups $\mathfrak{F}$ is called a formation if it is closed under taking homomorphic images and subdirect products. The $\mathfrak{F}$-residual of G, denoted by $G^{\mathfrak{F}}$, is the smallest normal subgroup of G with quotient in $\mathfrak{F}$. Let $Z_{\mathfrak{F}}(G)$ (resp. $Z_{p\mathfrak{F}}(G)$) denote the $\mathfrak{F}$-hypercentre (resp. $p\mathfrak{F}$-hypercentre) of G, that is, the product of all normal subgroups H of G such that all chief factors (resp. p-chief factors) L/K of G below H is $\mathfrak{F}$-hypercentral (i.e. $L/K \rtimes G/C_G(L/K) \in \mathfrak{F}$ (see [8, Chap. 1])). Let $\mathfrak{U}$ denote the classes of all supersoluble groups. Then $Z_{\mathfrak{U}}(G)$ (resp. $Z_{p\mathfrak{U}}(G)$) is the product of all normal hypercyclically embedded (resp. p-hypercyclically

embedded) subgroups of G. Moreover, the generalized Fitting subgroup $F^*(G)$ (resp. the generalized p-Fitting subgroup $F_p^*(G)$) of G is the maximal quasinilpotent subgroup (resp. the maximal p-quasinilpotent subgroup) of G (for details, see [14, Chap. X] and [15]).

In the present paper, we will give a new characterization of p-hypercyclically embedded subgroups of G by using the generalized $S\Phi$-supplemented subgroups. Our main result is the following.

Theorem 1.4. *Let E and X be normal subgroups of G such that $F_p^*(E) \leq X \leq E$ and P a Sylow p-subgroup of X. If P has a subgroup D such that $1 < |D| < |P|$, and all subgroups H of P with $|H| = |D|$ and all cyclic subgroups of P of order 4 (when P is a non-abelian 2-group and $|D| = 2$) are generalized $S\Phi$-supplemented in G, then $E \leq Z_{p\mathfrak{U}}(G)$.*

The following example illustrates that the converse of Theorem 1.4 does not hold.

Example 1.5. Let $G = \langle a, b \,|\, a^5 = 1, b^4 = 1, b^{-1}ab = a^2 \rangle$ and $H = \langle b^2 \rangle$. Then clearly, G is 2-supersoluble, $H_{sG} = 1$ and $\Phi(H) = 1$. If H is generalized $S\Phi$-supplemented in G, then there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T = 1$. This implies that $\langle b \rangle = H(\langle b \rangle \cap T)$, and so $H \leq \langle b \rangle \leq T$. This contradiction shows that H is not generalized $S\Phi$-supplemented in G.

The proof of Theorem 1.4 consists of a large number of steps. The following propositions are the main stages of it.

Proposition 1.6. *Let P be a normal p-subgroup of G. If P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with $|H| = |D|$ and all cyclic subgroups of P of order 4 (when P is a non-abelian 2-group and $|D| = 2$) are generalized $S\Phi$-supplemented in G, then $P \leq Z_{\mathfrak{U}}(G)$.*

Proposition 1.7. *Let E be a normal subgroup of G and P a Sylow p-subgroup of E. If every cyclic subgroup of P of order p or 4 (when P is a non-abelian 2-group) is generalized $S\Phi$-supplemented in G, then $E \leq Z_{p\mathfrak{U}}(G)$.*

Proposition 1.8. *Let E be a normal subgroup of G and P a Sylow p-subgroup of E. If every maximal subgroup of P is generalized $S\Phi$-supplemented in G, then either $E \leq Z_{p\mathfrak{U}}(G)$ or $|E|_p = p$.*

Note that Propositions 1.6–1.8 are independently interesting since they cover main results of many papers among which one can find recent publications (for example, [17], [16], [19]). We prove Theorem 1.4 and Propositions 1.6–1.8 in Section 3. Some applications of these results will be discussed in Section 4.

All unexplained notation and terminology are standard, as in [6], [7], [8].

2. Preliminaries

Lemma 2.1 (see [8, Chap. 1, Lemma 5.34]). *Let $H \leq G$, $K \leq G$ and $N \trianglelefteq G$.*

(1) *If H is s-permutable in G, then H is subnormal in G.*

(2) *If H is s-permutable in G, then HN/N is s-permutable in G/N.*

(3) *If H is a p-group, then H is s-permutable in G if and only if $O^p(G) \leq N_G(H)$.*

(4) *If H is s-permutable in G, then $H \cap K$ is s-permutable in K.*

Lemma 2.2 (see [21, Lemma 2.8] or [8, Chap. 3, Lemma 3.6]). *Let $H \leq K \leq G$. Then:*

(1) *H_{sG} is an s-permutable subgroup of G;*

(2) *$H_{sG} \leq H_{sK}$;*

(3) *If $H \trianglelefteq G$, then $(K/H)_{s(G/H)} = K_{sG}/H$.*

Lemma 2.3. *Let $H \leq K \leq G$ and $N \trianglelefteq G$. Suppose that H is generalized $S\Phi$-supplemented in G. Then:*

(1) *H is generalized $S\Phi$-supplemented in K.*

(2) *If either $N \leq H$ or $(|H|, |N|) = 1$, then HN/N is generalized $S\Phi$-supplemented in G/N.*

Proof. By the hypothesis, G has a subnormal subgroup T such that $G = HT$ and$(H \cap T)H_{sG}/H_{sG} \leq \Phi(H/H_{sG})$. Let $V/H_{sG} = \Phi(H/H_{sG})$.

(1) By Dedekind's identity, $K = H(T \cap K)$. Then by Lemma 2.2(2), $H_{sG} \leq H_{sK}$, and so $(H \cap T)H_{sK}/H_{sK} \leq VH_{sK}/H_{sK} \leq \Phi(H/H_{sK})$. Hence H is generalized $S\Phi$-supplemented in K.

(2) Clearly, $G/N = (HN/N)(TN/N)$ and $H_{sG}N/N \leq (HN/N)_{sG} = (HN)_{sG}/N$ by Lemma 2.2(3). Also, by Lemma 2.1(4), $(HN)_{sG} = ((HN)_{sG} \cap H)N \leq H_{sG}N$. This implies that $(HN)_{sG} = H_{sG}N$. Since either $N \leq H$ or $(|H|, |N|) = 1$, $HN \cap TN = (H \cap T)N$, and so $(HN \cap TN)(HN)_{sG}/(HN)_{sG} \leq VN/H_{sG}N \leq \Phi(HN/H_{sG}N)$. This shows that HN/N is generalized $S\Phi$-supplemented in G/N. □

Let P be a p-group. If P is not a non-abelian 2-group, then we use $\Omega(P)$ to denote the subgroup $\Omega_1(P)$. Otherwise, $\Omega(P) = \Omega_2(P)$.

Lemma 2.4 (see [4, Lemma 2.12]). *Let P be a normal p-subgroup of G and C a Thompson critical subgroup of P (see [7, p. 186]). If $P/\Phi(P) \leq Z_{\mathfrak{U}}(G/\Phi(P))$ or $C \leq Z_{\mathfrak{U}}(G)$ or $\Omega(P) \leq Z_{\mathfrak{U}}(G)$, then $P \leq Z_{\mathfrak{U}}(G)$.*

Lemma 2.5 (see [3, Lemma 2.10]). *Let C be a Thompson critical subgroup of a nontrivial p-group P.*

(1) *If p is odd, then the exponent of $\Omega_1(C)$ is p.*

(2) *If P is an abelian 2-group, then the exponent of $\Omega_1(C)$ is 2.*

(3) *If $p = 2$, then the exponent of $\Omega_2(C)$ is at most 4.*

Lemma 2.6 (see [1, Theorem 2.1.6]). *Let G be a p-supersoluble group. Then the derived subgroup G' of G is p-nilpotent. In particular, if $O_{p'}(G) = 1$, then G has a unique Sylow p-subgroup.*

Lemma 2.7 (see [25, Lemma 2.13]). *Let $\mathfrak{F}$ be a formation and E a normal subgroup of G. Then $E \leq Z_{p\mathfrak{F}}(G)$ if and only if $F_p^*(E) \leq Z_{p\mathfrak{F}}(G)$.*

Lemma 2.8 (see [20, Lemma 2.6]). *Let V be an s-permutable subgroup of G of order 4.*

(1) *If $V = A \times B$, where $|A| = |B| = 2$ and A is s-permutable in G, then B is s-permutable in G.*

(2) *If $V = \langle x \rangle$ is cyclic, then $\langle x^2 \rangle$ is s-permutable in G.*

Lemma 2.9 (see [22, Theorem B]). *Let $\mathfrak{F}$ be any formation and E a normal subgroup of G. If $F^*(E) \leq Z_{\mathfrak{F}}(G)$, then $E \leq Z_{\mathfrak{F}}(G)$.*

3. Proof of main results

For a p-subgroup H of G, we know that $\Phi(H/H_{sG}) = \Phi(H)H_{sG}/H_{sG}$ (see [13, Chap. 3, Theorem 3.14(c)]). Therefore, if H is a generalized $S\Phi$-supplemented p-subgroup of G, then there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq \Phi(H)H_{sG}$.

Proof of Proposition 1.6. Suppose that the assertion is false and let (G, P) be a counterexample for which $|G| + |P|$ is minimal. Then:

(1) $|D| > p$.

If $|D| = p$, then by the hypothesis, every cyclic subgroup of P of order p or 4 (when P is a non-abelian 2-group) is generalized $S\Phi$-supplemented in G. Let P/R be a chief factor of G. Clearly, (G, R) satisfies the hypothesis of the proposition. The choice of (G, P) implies that $R \leq Z_{\mathfrak{U}}(G)$. If $|P/R| = p$, then $P \leq Z_{\mathfrak{U}}(G)$, a contradiction. Hence $|P/R| > p$. Suppose that $L \trianglelefteq G$ and $L < P$. Then, similarly as above, we have that $L \leq Z_{\mathfrak{U}}(G)$. If $L \not\leq R$, then $P = RL \leq Z_{\mathfrak{U}}(G)$, a contradiction. Hence $L \leq R$. This shows that G has a unique normal subgroup R

such that P/R is a chief factor of G. Let C be a Thompson critical subgroup of P. Note that C is characteristic in P (see [7, Chap. 5, Theorem 3.11]). If $\Omega(C) < P$, then $\Omega(C) \leq R \leq Z_{\mathfrak{U}}(G)$. It follows from Lemma 2.4 that $P \leq Z_{\mathfrak{U}}(G)$, which is impossible. Hence $P = C = \Omega(C)$. Then by Lemma 2.5, the exponent of P is p or 4 (when P is a non-abelian 2-group).

Obviously, $P/R \cap Z(G_p/R) > 1$, where G_p is a Sylow p-subgroup of G. Suppose that $V/R \leq P/R \cap Z(G_p/R)$ and $|V/R| = p$. Let $x \in V \backslash R$ and $H = \langle x \rangle$. Then $V = HR$ and $|H| = p$ or 4. If $H = H_{sG}$, then by Lemma 2.2(1), H is s-permutable in G, and so $V/R = HR/R \trianglelefteq G/R$ by Lemma 2.1(2)(3). But since P/R is a chief factor of G, we have that $P = V$. It follows that $P/R = V/R$ is cyclic, and so $P \leq Z_{\mathfrak{U}}(G)$, a contradiction. Hence $H \neq H_{sG}$ and so $H_{sG} \leq \Phi(H)$. By the hypothesis, there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq \Phi(H)$. In this case, $P \cap T < P$, and so $(P \cap T)^G = (P \cap T)^P < P$. This means that $(P \cap T)^G \leq R$, and so $P = H(P \cap T) = HR = V$, also a contradiction. Hence $|D| > p$.

(2) $|D| < |P|/p$.

Suppose that $p|D| = |P|$. By the hypothesis, every maximal subgroup of P is generalized $S\Phi$-supplemented in G. Let N be a minimal normal subgroup of G contained in P. Then by Lemma 2.3(2), $(G/N, P/N)$ satisfies the hypothesis of the proposition. The choice of (G, P) yields that $P/N \leq Z_{\mathfrak{U}}(G/N)$. If $|N| = p$, then $P \leq Z_{\mathfrak{U}}(G)$, which is impossible. Hence $|N| > p$. Suppose that G has another minimal normal subgroup L contained in P such that $N \neq L$. With a similar discussion as above, we have that $P/L \leq Z_{\mathfrak{U}}(G/L)$. It follows that $NL/L \leq Z_{\mathfrak{U}}(G/L)$, and so $|N| = p$, a contradiction. Thus G has a unique minimal normal subgroup N contained in P.

If $\Phi(P) = 1$, then P is elementary abelian. Let N_1 be a maximal subgroup of N such that N_1 is normal in some Sylow p-subgroup G_p of G, and let S be a complement of N in P. Then $P_1 = N_1 S$ is a maximal subgroup of P. By [13, Chap. 3, Lemma 3.3], $\Phi(P_1) \leq \Phi(P) = 1$. Therefore, there exists a subnormal subgroup T of G such that $G = P_1 T$ and $P_1 \cap T \leq (P_1)_{sG}$. Then $G = PT$ and $P = P_1(P \cap T)$. It is easy to see that $1 \neq P \cap T \trianglelefteq G$. Hence $N \leq P \cap T$, and so $P_1 \cap N \leq P_1 \cap T \leq (P_1)_{sG}$. It follows that $N_1 = P_1 \cap N = (P_1)_{sG} \cap N$ is s-permutable in G. By Lemma 2.1(3), $N_1 \trianglelefteq G$, and so $|N| = p$, a contradiction. Thus $\Phi(P) \neq 1$. Then $N \leq \Phi(P)$. Since $P/N \leq Z_{\mathfrak{U}}(G/N)$, $P/\Phi(P) \leq Z_{\mathfrak{U}}(G/\Phi(P))$. Applying Lemma 2.4, we obtain that $P \leq Z_{\mathfrak{U}}(G)$. The contradiction completes the proof of (2).

(3) *Final contradiction.*

We shall show that all subgroups H of P with $|H| = |D|$ are s-permutable in G. By the hypothesis, G has a subnormal subgroup T such that $G = HT$ and $H \cap T \leq \Phi(H)H_{sG}$. If $T < G$, then there exists a normal subgroup M of G such that $T \leq M$ and $|G : M| = p$. Since $|P : P \cap M| = |PM : M| = p$, $P \cap M$ is a maximal subgroup of P and so $|D| < |P \cap M|$ by (2). Clearly, $P \cap M \trianglelefteq G$. Then $(G, P \cap M)$ satisfies the hypothesis of the proposition. The choice of (G, P) yields that $P \cap M \leq Z_{\mathfrak{U}}(G)$. Consequently, $P \leq Z_{\mathfrak{U}}(G)$, which is impossible. Hence $T = G$. This implies that $H = H_{sG}$ is s-permutable in G by Lemma 2.2(1). Then by [24, Theorem], $P \leq Z_{\mathfrak{U}}(G)$. The final contradiction ends the proof. □

Proof of Proposition 1.7. Suppose that the assertion is false and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. We now proceed via the following steps.

(1) $O_{p'}(E) = 1$.

If $O_{p'}(E) \neq 1$, then by Lemma 2.3(2), $(G/O_{p'}(E), E/O_{p'}(E))$ satisfies the hypothesis of the proposition. The choice of (G, E) implies that $E/O_{p'}(E) \leq Z_{p\mathfrak{U}}(G/O_{p'}(E)) = Z_{p\mathfrak{U}}(G)/O_{p'}(E)$, and so $E \leq Z_{p\mathfrak{U}}(G)$, a contradiction. Hence $O_{p'}(E) = 1$.

(2) $E = G$.

Suppose that $E < G$. Then by Lemma 2.3(1), (E, E) satisfies the hypothesis of the proposition. The choice of (G, E) implies that E is p-supersoluble. By (1) and Lemma 2.6, we see that $P \trianglelefteq E$. Thus $P \trianglelefteq G$. Then by Proposition 1.6, we have $P \leq Z_{\mathfrak{U}}(G)$. Consequently, $E \leq Z_{p\mathfrak{U}}(G)$, which is absurd. Therefore, $E = G$.

(3) *$Z_{p\mathfrak{U}}(G)$ is the unique normal subgroup of G such that $G/Z_{p\mathfrak{U}}(G)$ is a chief factor of G, $G^{\mathfrak{U}} = G$ and $O_p(G) = Z(G) = Z_{\mathfrak{U}}(G)$ is the Sylow p-subgroup of $Z_{p\mathfrak{U}}(G)$.*

Let G/K be a chief factor of G. Obviously, (G, K) satisfies the hypothesis of the proposition. By the choice of the (G, E), $K \leq Z_{p\mathfrak{U}}(G)$, and so $K = Z_{p\mathfrak{U}}(G)$. This shows that $Z_{p\mathfrak{U}}(G)$ is the unique normal subgroup of G such that $G/Z_{p\mathfrak{U}}(G)$ is a chief factor of G. By Proposition 1.6, $O_p(G) \leq Z_{\mathfrak{U}}(G) \leq Z_{p\mathfrak{U}}(G)$. Then by (1), (2) and Lemma 2.6, $O_p(G)$ is the Sylow p-subgroup of $Z_{p\mathfrak{U}}(G)$. If $G^{\mathfrak{U}} < G$, then $G^{\mathfrak{U}} \leq Z_{p\mathfrak{U}}(G)$. So $G^{\mathfrak{U}} \cap O_p(G)$ is the Sylow p-subgroup of $G^{\mathfrak{U}}$. Let $P_1 = G^{\mathfrak{U}} \cap O_p(G)$. Note that $(G/P_1)/(G^{\mathfrak{U}}/P_1)$ is supersoluble and $G^{\mathfrak{U}}/P_1$ is a p'-group. Hence G/P_1 is p-supersoluble, and so G is p-supersoluble because $P_1 \leq Z_{\mathfrak{U}}(G)$, a contradiction. Thus $G^{\mathfrak{U}} = G$. It follows from [6, Chap. IV, Theorem 6.10] that

$Z_{\mathfrak{U}}(G) \leq Z(G)$. Since $O_{p'}(Z(G)) \leq O_{p'}(G) = 1$ by (1) and (2), $Z(G) \leq O_p(G)$. Therefore, $O_p(G) = Z(G) = Z_{\mathfrak{U}}(G)$.

(4) *Final contradiction.*

By (3), we have that $G' = G$. If P is abelian, then by (3) and [13, Chap. VI, Theorem 14.3], $Z(G)=1$. Hence by (3), $Z_{p\mathfrak{U}}(G)$ is a p'-group. Then by (1) and (2), $Z_{p\mathfrak{U}}(G) = 1$, and so G is simple by (3) again. Let x be an element of G of order p. Then by the hypothesis, G has a subnormal subgroup T such that $G = \langle x\rangle T$ and $\langle x\rangle \cap T \leq \langle x\rangle_{sG}$. In this case, clearly, $T = G$ and so $\langle x\rangle$ is s-permutable in G by Lemma 2.2(1). Then $\langle x\rangle$ is subnormal in G by Lemma 2.1(1). So $G = \langle x\rangle$, which is impossible. Thus P is non-abelian.

By [13, Chap. IV, Satz 5.5], we see that there exists a cyclic subgroup H of P of order p or 4 which is not contained in $Z(G)$. Then by the hypothesis, H is generalized $S\Phi$-supplemented in G. Thus G has a subnormal subgroup T such that $G = HT$ and $H\cap T \leq \Phi(H)H_{sG}$. If $T < G$, then G has a normal subgroup M such that $T \leq M$ and $|G : M| = p$. It is easy to see that (G, M) satisfies the hypothesis of the proposition. The choice of (G, E) implies that $M \leq Z_{p\mathfrak{U}}(G)$, and so $G \leq Z_{p\mathfrak{U}}(G)$, which is impossible. Hence $T = G$. Then $H = H_{sG}$ is s-permutable in G by Lemma 2.2(1). Since $H \not\leq Z(G)$ and $Z(G)$ is the Sylow p-subgroup of $Z_{p\mathfrak{U}}(G)$ by (3), $H \not\leq Z_{p\mathfrak{U}}(G)$. Hence by (3) and Lemma 2.1(3), we have that $G = (HZ_{p\mathfrak{U}}(G))^G = (HZ_{p\mathfrak{U}}(G))^P \leq PZ_{p\mathfrak{U}}(G)$. But since $G/Z_{p\mathfrak{U}}(G)$ is a chief factor of G, $|G/Z_{p\mathfrak{U}}(G)| = p$. This shows that G is p-supersoluble, a contradiction. This completes the proof. □

Proof of Proposition 1.8. Suppose that the assertion is false and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. Then:

(1) $O_{p'}(E) = 1$ *and* $E = G$.

See steps (1) and (2) in the proof of Proposition 1.7.

(2) *Let N be a minimal normal subgroup of G. Then either G/N is p-supersoluble or $|G/N|_p = p$.*

Suppose that M/N is a maximal subgroup of PN/N. Then there exists a maximal subgroup P_1 of P such that $M = P_1N$ and $P \cap N = P_1 \cap N$. By the hypothesis, G has a subnormal subgroup T such that $G = P_1T$ and $P_1 \cap T \leq \Phi(P_1)(P_1)_{sG}$. Clearly, $(|N : P_1\cap N|, |N : T\cap N|) = 1$. Hence $N = (P_1\cap N)(T\cap N)$, and so $P_1N \cap TN = (P_1 \cap T)N$. By discussing similarly as in the proof of Lemma 2.3(2), $M/N = P_1N/N$ is generalized $S\Phi$-supplemented in G/N. This shows that $(G/N, G/N)$ satisfies the hypothesis of the proposition. The choice of (G, E) implies that either G/N is p-supersoluble or $|G/N|_p = p$. Hence (2) holds.

(3) *If* $PN < G$, *then* $N \leq O_p(G)$.

By Lemma 2.3(1), (PN, PN) satisfies the hypothesis of the proposition, and so the choice of (G, E) implies that either PN is p-supersoluble or $|PN|_p = p$. Then by (1), $N \leq O_p(G)$.

(4) N *is the unique minimal normal subgroup of* G.

Let N and L be two distinct minimal normal subgroups of G. By (2), we may discuss the following three possible cases.

(i) If G/N and G/L are all p-supersoluble, then G is p-supersoluble, a contradiction.

(ii) Without loss of generality, we may assume that G/N is p-supersoluble and $|G/L|_p = p$. Since LN/N is a minimal normal subgroup of G/N and $p||L|$ by (1), $|L| = |LN/N| = p$, and so $|P| = p^2$. Then by (1), $|N|_p = |P \cap N| = p$ and N is a non-abelian simple group. Let $N_1 = P \cap N$. Then $(N_1)_{sG} = 1$ by Lemma 2.1(1). By the hypothesis, N_1 is generalized $S\Phi$-supplemented in G. Thus G has a subnormal subgroup T such that $G = N_1T$ and $N_1 \cap T = 1$. Thus $T \trianglelefteq G$. It follows that either $N \cap T = 1$ or $N \leq T$. For the former case, we have $N = N \cap N_1T = N_1$, a contradiction. For the latter case, it follows that $N_1 = 1$, which is impossible.

(iii) Suppose that $|G/N|_p = p$ and $|G/L|_p = p$. Without loss of generality, we may assume that N and L are non-abelian simple groups. Then $P = (P \cap N)(P \cap L)$, and so $|P| = p^2$. Then with a similar discussion as above, we can derive a contradiction. Hence (4) holds.

(5) $N \not\leq \Phi(P)$.

Suppose that $N \leq \Phi(P)$. Then $N \leq \Phi(G)$. By (2), either G/N is p-supersoluble or $|G/N|_p = p$. But the former case is clearly impossible. Hence we may assume that $|G/N|_p = p$. Then $|P/N| = p$. This implies that P is cyclic, and so $|N| = p$. Then $|P| = p^2$. We show that G/N is a non-abelian simple group. Let $A/N = O_{p'}(G/N)$. Then $A \cap P \leq N \leq \Phi(P)$, and so A is p-nilpotent by [13, Chap. IV, Satz 4.7]. It follows from (1) that $A = N$. Thus $O_{p'}(G/N) = 1$. Suppose that K/N is a chief factor of G. Then $|K/N|_p = p$, and so $P \leq K$. Obviously, (G, K) satisfies the hypothesis of the proposition. If $K < G$, the choice of (G, E) yields that $K \leq Z_{p\mathfrak{U}}(G)$. Thus G is p-supersoluble. This contradiction shows that $G = K$. Then G/N is a non-abelian simple group. Since $|N| = p$, $G/C_G(N)$ is abelian, and so $C_G(N) = G$. It follows that $N \leq Z(G)$, which contradicts [13, Chap. VI, Satz 14.3].

(6) $O_p(G) = 1$.

Suppose that $O_p(G) \neq 1$. By (4), $N \leq O_p(G)$. If G/N is p-supersoluble, then $N \not\leq \Phi(G)$. Therefore there exists a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Since $P = N(P \cap M)$, P has a maximal subgroup P_1 containing $P \cap M$ and $P = NP_1$. If $(P_1)_{sG} \neq 1$, then by (4), Lemma 2.1(3) and Lemma 2.2(1), $N \leq ((P_1)_{sG})^G = ((P_1)_{sG})^P \leq P_1$, a contradiction. Thus $(P_1)_{sG} = 1$. Then by the hypothesis, there exists a subnormal subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq \Phi(P_1)$. Note that $N \leq O^p(G) \leq T$ by (4). Thus $P_1 \cap N \leq \Phi(P_1)$. This induces that $P_1 = (P_1 \cap N)(P \cap M) = P \cap M$. Hence $P_1 \cap N = 1$, and so $|N| = p$, a contradiction. Now assume that $|G/N|_p = p$. Then $|P/N| = p$. By (5), P has a maximal subgroup P_2 such that $P = P_2N$. With a similar argument as above, we have that $(P_2)_{sG} = 1$. Therefore, by the hypothesis, there exists a subnormal subgroup T of G such that $G = P_2T$ and $P_2 \cap T \leq \Phi(P_2)$. Then clearly, $N \leq T$, and so $|G : T| = p$. This implies that $T \trianglelefteq G$ and T/N is a p'-group. Thus G/N is p-supersoluble. This case has been dealt with in the above. Hence we have (6).

(7) *Final contradiction.*

By (3) and (6), we have that $G = PN$. If $P \leq N$, then G is a non-abelian simple group. Let P_1 be a maximal subgroup of P. Then P_1 is generalized $S\Phi$-supplemented in G. It follows that $P_1 = (P_1)_{sG}$ is s-permutable in G by Lemma 2.2(1), and so $P_1 = 1$ by Lemma 2.1(1). Thus $|G|_p = |P| = p$. This contradiction shows that P has a maximal subgroup P_2 such that $P \cap N \leq P_2$. Then $(P_2)_{sG} = 1$ by (6), Lemma 2.1(1) and Lemma 2.2(1). Hence, by the hypothesis, G has a subnormal subgroup T such that $G = P_2T$ and $P_2 \cap T \leq \Phi(P_2) \leq \Phi(P)$. By [21, Lemma 2.5(7)], we have $O^p(G) \leq T$. Hence by (4), $N \leq O^p(G) \leq T$, and thereby $P \cap N = P_2 \cap N \leq \Phi(P)$. Then by [13, Chap. IV, Satz 4.7], N is p-nilpotent, and so N is a p-group by (1), which contradicts (6). The proof is thus completed. □

Proof of Theorem 1.4. Suppose that the result is false and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. We now proceed via the following steps.

(1) $O_{p'}(E) = 1$ *and* $X = E = G$.

Suppose that $X < E$. Then clearly, $F_p^*(X) = F_p^*(E)$. Hence (G, X) satisfies the hypothesis of the theorem. The choice of (G, E) implies that $F_p^*(E) \leq X \leq Z_{p\mathfrak{U}}(G)$, and so $E \leq Z_{p\mathfrak{U}}(G)$ by Lemma 2.7. This contradiction shows that $X = E$. With a similar argument as in steps (1) and (2) in the proof of Proposition 1.7, we have that $O_{p'}(E) = 1$ and $E = G$.

(2) $p < |D| < |P|/p$.

It follows immediately from Propositions 1.7 and 1.8.

(3) *If $H \leq P$ and $|H| = |D|$, then H is s-permutable in G.*

By the hypothesis, G has a subnormal subgroup T such that $G = HT$ and $H \cap T \leq \Phi(H)H_{sG}$. If $T < G$, then there exists a normal subgroup M of G such that $T \leq M$ and $|G : M| = p$. Hence by (2), (G, M) satisfies the hypothesis of the theorem. The choice of (G, E) implies that $M \leq Z_{p\mathfrak{U}}(G)$, and so G is p-supersoluble, a contradiction. Thus $T = G$. It follows that $H = H_{sG}$ is s-permutable in G by Lemma 2.2(1).

(4) *Final contradiction.*

Let N be a minimal normal subgroup of G. Then by (1), $p \mid |N|$. If $N \not\leq O_p(G)$, then we may take a subgroup H of P such that $|H| = |D|$ and $H \cap N \neq 1$. By (3) and Lemma 2.1(1), $H \cap N \leq O_p(N) = 1$, a contradiction. Hence $N \leq O_p(G)$. If $|N| > |D|$, then N has a subgroup H such that $H \trianglelefteq P$ and $|H| = |D|$. By (3) and Lemma 2.1(3), $H \trianglelefteq G$, a contradiction. Now assume that $|N| = |D|$. Then by (2), there exists a subgroup V of P such that $N < V < P$, $V \trianglelefteq P$ and $|V : N| = p$. If $\Phi(V) = N$, then V is cyclic, and so $|N| = p$, which contradicts (2). Thus $\Phi(V) < N$. It follows that N has a subgroup N_1 such that $\Phi(V) \leq N_1 < N$, $N_1 \trianglelefteq P$ and $|N : N_1| = p$. Then V has a subgroup H such that $|H| = |D|$ and $H \cap N = N_1$. By (3), N_1 is s-permutable in G, and so $N_1 \trianglelefteq G$ by Lemma 2.1(3). Thus $N_1 = 1$, which implies that $|N| = |D| = p$, which contradicts (2). Therefore, we have that $|N| < |D|$.

If $p > 2$ or $p = 2$ and P/N is abelian or $p = 2$ and $|D| > 2|N|$, then by Lemma 2.3(2), we see that $(G/N, G/N)$ satisfies the hypothesis of the theorem. Now assume that $p = 2$, P/N is non-abelian and $|D| = 2|N|$. Then P is non-abelian. By (3) and Lemma 2.1(2), all subgroups of P/N of order 2 are s-permutable in G/N. Let L/N be a cyclic subgroup of order 4 of P/N. If $N \leq \Phi(L)$, then L is cyclic, and so $|D| = 2|N| = 4$. By (3), all subgroups of P of order 4 are s-permutable in G. For any subgroup K of P of order 2 with $K \neq N$, NK is s-permutable in G. Thus by Lemma 2.8, K is s-permutable in G. Now by Proposition 1.7, we have that G is p-supersoluble, a contradiction. Hence we may assume that $N \not\leq \Phi(L)$. Then there exists a maximal subgroup L_1 of L such that $L = L_1N$. Since $|L_1| = |D|$, $L/N = L_1N/N$ is s-permutable in G/N by (3) and Lemma 2.1(2). This shows that $(G/N, G/N)$ also satisfies the hypothesis of the theorem. Hence, by the choice of (G, E), G/N is p-supersoluble. Then clearly, N is the unique normal subgroup of G and $N \not\leq \Phi(G)$. It follows that G has a maximal subgroup M such that $G = N \rtimes M$. Since $O_p(G) \cap M = 1$,

$N = O_p(G)$, and so $|N| \geq |D|$ by (3) and Lemma 2.1(1). The final contradiction completes the proof. □

4. Further applications

By Theorem 1.4, we can prove the following corollaries.

Corollary 4.1. *Let E be a normal subgroup of G and P a Sylow p-subgroup of E, where $(|E|, p-1) = 1$. If P has a subgroup D such that $1 < |D| < |P|$, and all subgroups H of P with $|H| = |D|$ and all cyclic subgroups of P of order 4 (when P is a non-abelian 2-group and $|D| = 2$) are generalized $S\Phi$-supplemented in G, then E is p-nilpotent.*

Proof. By Theorem 1.4, $E \leq Z_{p\mathfrak{U}}(G)$, and so E is p-supersoluble. Since $(|E|, p-1) = 1$, we see that E is p-nilpotent. □

Corollary 4.2. *Let E and X be normal subgroups of G such that $F^*(E) \leq X \leq E$. If for any non-cyclic Sylow subgroup P of X, P has a subgroup D such that $1 < |D| < |P|$, and all subgroups H of P with $|H| = |D|$ and all cyclic subgroups of P of order 4 (when P is a non-abelian 2-group and $|D| = 2$) are generalized $S\Phi$-supplemented in G, then $E \leq Z_{\mathfrak{U}}(G)$.*

Proof. By Lemma 2.3(2) and Corollary 4.1, we have that X has a Sylow tower of supersoluble type. If P is cyclic, then clearly, $X \leq Z_{p\mathfrak{U}}(G)$. Now assume that P is non-cyclic. Then by Theorem 1.4, $X \leq Z_{p\mathfrak{U}}(G)$ also holds. Therefore, $F^*(E) \leq X \leq Z_{\mathfrak{U}}(G)$, and so $E \leq Z_{\mathfrak{U}}(G)$ by Lemma 2.9. □

Corollary 4.3. *Let E be a normal subgroup of G such that G/E is p-nilpotent and P a Sylow p-subgroup of E such that $N_G(P)$ is p-nilpotent. If P has a subgroup D such that $1 < |D| < |P|$, and all subgroups H of P with $|H| = |D|$ and all cyclic subgroups of P of order 4 (when P is a non-abelian 2-group and $|D| = 2$) are generalized $S\Phi$-supplemented in G, then G is p-nilpotent.*

Proof. Suppose that the result is false and let (G, E) be a counterexample for which $|G| + |E|$ is minimal. Assume that $O_{p'}(E) \neq 1$. Since

$$N_{G/O_{p'}(E)}(PO_{p'}(E)/O_{p'}(E)) = N_G(P)O_{p'}(E)/O_{p'}(E), \quad (G/O_{p'}(E), E/O_{p'}(E))$$

satisfies the hypothesis of the corollary by Lemma 2.3(2). The choice of (G, E) implies that $G/O_{p'}(E)$ is p-nilpotent, and so G is p-nilpotent, a contradiction.

Hence $O_{p'}(E) = 1$. Note that by Theorem 1.4, E is p-supersoluble. Then by Lemma 2.6, $P \trianglelefteq G$. Hence $G = N_G(P)$ is p-nilpotent, a contradiction. □

Note that Corollaries 4.1–4.3 generalize many known results, for example, [16, Theorems 3.1, 3.6, 3.11, 4.1, 4.3 and 4.4], [17, Theorems 3.1–3.5], [18, Theorems 1.2 and 1.3], [19, Theorems 3.1 and 3.2], [20, Theorem 1.4], [21, Theorems 1.3 and 1.4], [24, Theorem]. Moreover, we point out that [16, Theorem 3.9] and [19, Theorem 3.3] follow directly from Proposition 1.8.

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YUEMEI MAO
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF SCIENCE AND
TECHNOLOGY OF CHINA
HEFEI 230026
P. R. CHINA
AND
SCHOOL OF MATHEMATICS AND
COMPUTER SCIENCE
UNIVERSITY OF DATONG OF SHANXI
DATONG 037009
P. R. CHINA

E-mail: maoym@mail.ustc.edu.cn

XIAOYU CHEN
SCHOOL OF MATHEMATICAL SCIENCES
AND INSTITUTE OF MATHEMATICS
NANJING NORMAL UNIVERSITY
NANJING 210023
P. R. CHINA

E-mail: jelly@njnu.edu.cn

WENBIN GUO
SCHOOL OF MATHEMATICAL SCIENCES
UNIVERSITY OF SCIENCE AND
TECHNOLOGY OF CHINA
HEFEI 230026
P. R. CHINA

E-mail: wbguo@ustc.edu.cn

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