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Pexiderization of some logarithmic functional equations

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Dedicated to the 60th birthday of Zsolt Páles

Abstract. We study some new logarithmic functional equations and their Pexiderizations on different structures.

1. Introduction

The functional equation

$$f(xy) = f(x) + f(y) \tag{CL}$$

with function $f : \mathbb{R}_+ \to \mathbb{R}$ (or with function $f : \mathbb{R}_0 \to \mathbb{R}$) is usually called the Cauchy logarithmic functional equation. Here $\mathbb{R}_+$ is the set of positive elements in real numbers $\mathbb{R}$ and $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$.

Several works appeared on functional equations satisfied by logarithmic function, referred to as logarithmic functional equation.

In [8] and [4], for function $f : \mathbb{R}_+ \to \mathbb{R}$, it is proved that functional equation

$$f(x+y) - f(x) - f(y) = f\left(\frac{1}{x} + \frac{1}{y}\right) \tag{1}$$

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and (CL) are equivalent in the sense that each solution of one equation is also solution of the other.

In [9], the authors add the functional equation

$$f(x+y) - f(xy) = f\left(\frac{1}{x} + \frac{1}{y}\right) \tag{2}$$

to the above list of equivalent equations by proving that (2) and (CL) are equivalent. In addition, the Pexider generalizations of (1) and (2) are considered in [9] in form

$$f(x+y) - g(x) - h(y) = k\left(\frac{1}{x} + \frac{1}{y}\right) \tag{3}$$

and

$$f(x+y) - g(xy) = h\left(\frac{1}{x} + \frac{1}{y}\right), \tag{4}$$

respectively for functions f, g, h, $k : \mathbb{R}_+ \to \mathbb{R}$.

In [3], the author gave a simple way to find the general solution of (4).

Then in [5], the equivalence of equations (1) and (2) was proved for function $f : K_0 \to A$, where $K_0 = K \setminus \{0\}$ (K is a field excluding $\mathbb{Z}_2$) and A is an Abelian group which has no 2-torsion.

In [10], the authors complemented the works [4], [8] and [9] mentioned above by solving a few other logarithmic functional equations in Pexider form.

Here we study two new logarithmic functional equations and their Pexiderizations for functions mapping $\mathbb{R}_+$ or $\mathbb{T}_+$ (where $\mathbb{T}_+$ is the set of positive elements in an ordered field $\mathbb{T}$) into $\mathbb{R}$ or into a uniquely 2-divisible Abelian group A.

2. The first new logarithmic equation

It is easy to see that any solution of (CL) is a solution of the functional equation

$$f(x+y) + f\left(\frac{x+y}{xy}\right) = f\left(\frac{(x+y)^2}{xy}\right) \tag{5}$$

for function $f : \mathbb{R}_+ \to \mathbb{R}$. We will prove the equivalence of equations (5) and (CL).

First, we present the general solution of the Pexiderized version

$$f(x+y) + g\left(\frac{x+y}{xy}\right) = h\left(\frac{(x+y)^2}{xy}\right) \tag{6}$$

of (5) for $x, y \in \mathbb{R}_+$.

Theorem 2.1. *The functions $f, g : \mathbb{R}_+ \to \mathbb{R}$ and $h : D = \{t \in \mathbb{R}_+ | t \geqslant 4\} \to \mathbb{R}$ satisfy functional equation* (6) *for all $x, y \in \mathbb{R}_+$ if and only if they have the form*

$$\begin{aligned} f(x) &= l(x) + a \qquad (x \in \mathbb{R}_+), \\ g(x) &= l(x) + b \qquad (x \in \mathbb{R}_+), \\ h(x) &= l(x) + a + b \qquad (x \in D), \end{aligned} \tag{7}$$

where $l : \mathbb{R}_+ \to \mathbb{R}$ is a logarithmic function (i.e. satisfies (CL) *for all $x, y \in \mathbb{R}_+$) and $a, b \in \mathbb{R}$ are arbitrary constants.*

Proof. Assume that the functions f, g, h satisfy equation (6) for all $x, y \in \mathbb{R}_+$. Set $x + y = t$, $(x + y)/(xy) = s$ in (6) to get

$$f(t) + g(s) = h(ts) \qquad (t, s \in \mathbb{R}_+, t \cdot s \geqslant 4). \tag{8}$$

Let $t, s \in \mathbb{R}_+$ be arbitrary, then there exists $u \in \mathbb{R}_+$ such that $uts \geqslant 4$. Then we have, by (8),

$$h(uts) = f(ut) + g(s) \quad \text{and} \quad h(uts) = f(u) + g(ts),$$

so we get

$$g(ts) - g(s) = f(ut) - f(u) := \alpha(t) \qquad (t, s \in \mathbb{R}_+).$$

Hence $g(ts) = \alpha(t) + g(s)$ for all $t, s \in \mathbb{R}_+$. Thus (see [1]) there exists a logarithmic function $l : \mathbb{R}_+ \to \mathbb{R}$ such that

$$g(t) = l(t) + b \quad \text{and} \quad \alpha(t) = l(t) \qquad (t \in \mathbb{R}_+), \tag{9}$$

where $b \in \mathbb{R}$ is a constant.

Putting $x = y = t/2$ in (6), we obtain

$$f(t) = -g\left(\frac{4}{t}\right) + h(4) = -l\left(\frac{4}{t}\right) - b + h(4) = l(t) + a \qquad (t \in \mathbb{R}_+), \tag{10}$$

where $a \in \mathbb{R}$ is a constant.

Finally, we get from (6), (10) and (9) that

$$\begin{aligned} h\left(\frac{(x+y)^2}{xy}\right) &= f(x+y) + g\left(\frac{x+y}{xy}\right) \\ &= l(x+y) + a + l\left(\frac{x+y}{xy}\right) + b = l\left(\frac{(x+y)^2}{xy}\right) + a + b \qquad (x, y \in \mathbb{R}_+), \end{aligned}$$

so we have $h(t) = l(t) + a + b$ for all $t \geqslant 4$, since $t = (x+y)^2/(xy) = x/y + y/x + 2 \geqslant 4$.

The converse can be easily obtained by a simple calculation. $\square$

Corollary 2.1. *Functional equation* (5) *and* (CL) *are equivalent.*

Proof. In case $f = g = h$, Theorem 2.1 implies that $a = b = 0$, thus the only solution of (5) is the logarithmic function $f(x) = l(x)$ for all $x \in \mathbb{R}_+$. The converse is easy to check. □

To generalize Theorem 2.1, we need the following

Lemma 2.1 (see [6])**.** *If S is a non-empty set and $f, g : \mathbb{T}_+ \to S$ are functions such that*

$$f(x+y) = g(xy) \qquad (x, y \in \mathbb{T}_+), \tag{11}$$

then f and g are constant.

Proof. Let $\mu \in \mathbb{T}_+$ $(\mu > 1)$ be arbitrary. Replacing x by μx and y by $(1/\mu)y$ in (11), we find that

$$f\left(\mu x + \frac{1}{\mu} y\right) = g(xy) = f(x+y) \qquad (x, y \in \mathbb{T}_+). \tag{12}$$

Let $u \in T_+$ be arbitrary and choose $x, y \in \mathbb{T}_+$ and $\mu > 1$ such that $\mu x + (1/\mu)y = 1$ and $x + y = u$. This system of equations is satisfied if and only if

$$x = \frac{\mu - u}{\mu^2 - 1} \quad \text{and} \quad y = \frac{\mu^2 u - \mu}{\mu^2 - 1}.$$

Let $\mu = u + 1/u$, then $x = u/(u^4 + u^2 + 1)$, $y = (u^5 + u^3)/(u^4 + u^2 + 1)$. Putting these in (12), we have

$$f(u) = f(1) \qquad (u \in \mathbb{T}_+).$$

Thus f is constant on $\mathbb{T}_+$, and hence so is g. □

Remark 2.1. In the case $\mathbb{T}_+ = \mathbb{R}_+$ this lemma was proved in [2].

Lemma 2.2. *Let A be an Abelian group. The functions f, g, $K : \mathbb{T}_+ \to A$ satisfy functional equation*

$$f(x+y) + g\left(\frac{x+y}{xy}\right) = K\left(\frac{x}{y}\right) \qquad (x, y \in \mathbb{T}_+) \tag{13}$$

if and only if

$$\begin{aligned} f(x) &= l(x) + a \qquad (x \in \mathbb{T}_+), \\ g(x) &= l(x) + b \qquad (x \in \mathbb{T}_+), \\ K(x) &= l\left(x + \frac{1}{x} + 2\right) + a + b \qquad (x \in \mathbb{T}_+), \end{aligned} \tag{14}$$

where the function $l : \mathbb{T}_+ \to A$ satisfies the Cauchy logarithmic equation (CL) *for all $x, y \in \mathbb{T}_+$, and $a, b \in A$ are arbitrary constants.*

Proof. Assume that f, g, K satisfy (13) for all $x, y \in \mathbb{T}_+$. Replacing x and y by $x/2$ in (13), we have that

$$f(x) + g\left(\frac{4}{x}\right) = K(1) \qquad (x \in \mathbb{T}_+), \tag{15}$$

which gives that

$$g\left(\frac{x+y}{xy}\right) = -f\left(\frac{4xy}{x+y}\right) + K(1) \qquad (x, y \in \mathbb{T}_+). \tag{16}$$

Applying (16) in (6), we get

$$f(x+y) - f\left(\frac{4xy}{x+y}\right) + K(1) = K\left(\frac{x}{y}\right) \qquad (x, y \in \mathbb{T}_+). \tag{17}$$

Now let $\lambda \in \mathbb{T}_+$ be arbitrary. The equation (17) shows that

$$\Delta_\lambda f(x+y) = \Delta_\lambda f\left(\frac{4xy}{x+y}\right) \qquad (x, y \in \mathbb{T}_+), \tag{18}$$

where the function $\Delta_\lambda f : \mathbb{T}_+ \to A$ is defined by

$$\Delta_\lambda f(x) = f(\lambda x) - f(x) \qquad (x \in \mathbb{T}_+).$$

Replace x by $x(x+y)/4$ and y by $y(x+y)/4$ in (18), we obtain that

$$F_\lambda(x+y) = \Delta_\lambda f(xy) \qquad (x, y \in \mathbb{T}_+) \tag{19}$$

with function $F_\lambda : \mathbb{T}_+ \to A$ defined by

$$F_\lambda(x) = \Delta_\lambda f\left[\left(\frac{x}{2}\right)^2\right] \qquad (x \in \mathbb{T}_+).$$

From (19), by Lemma 2.1, we can infer that the function $\Delta_\lambda f$ is constant, that is for all $\lambda \in \mathbb{T}_+$ there exists a constant $c(\lambda) \in A$ such that $\Delta_\lambda f(x) = c(\lambda)$ for all $x \in \mathbb{T}_+$. This equality and the definition of $\Delta_\lambda f$ imply that

$$f(\lambda x) - f(x) = c(\lambda) \qquad (\lambda, x \in \mathbb{T}_+). \tag{20}$$

From (20), by the substitution $x = 1$, we obtain that $c(\lambda) = f(\lambda) - f(1)$ for all $\lambda \in \mathbb{T}_+$. This equation and (20) give that $f(\lambda x) = f(x) + f(\lambda) - f(1)$ for all

$x, \lambda \in \mathbb{T}_+$, which shows that there exists a logarithmic function $l : \mathbb{T}_+ \to A$, such that

$$f(x) = l(x) + a \qquad (x \in \mathbb{T}_+), \tag{21}$$

where $a = f(1) \in A$ is a constant.

Now (15) implies that

$$g(x) = -f\left(\frac{4}{x}\right) + K(1) = -l\left(\frac{4}{x}\right) - a + K(1) = l(x) + b \qquad (x \in \mathbb{T}_+), \tag{22}$$

with constant $b = K(1) - a - l(4) \in A$, thus f and g is of the form as in (14).

Finally, setting $y = 1$ in (13), from (21) and (22) we get that

$$\begin{aligned} K(x) &= f(x+1) + g\left(\frac{x+1}{x}\right) = l(x+1) + a + l\left(\frac{x+1}{x}\right) + b \\ &= l\left(\frac{(x+1)^2}{x}\right) + a + b = l\left(x + \frac{1}{x} + 2\right) + a + b \qquad (x, y \in \mathbb{T}_+), \end{aligned}$$

which completes the proof. The converse is easy to check. □

Theorem 2.2. *Let $D = \{t \in \mathbb{T}_+ | \exists u \in \mathbb{T}_+ : t = u + \frac{1}{u} + 2\}$ and A be an Abelian group. The functions f, $g : \mathbb{T}_+ \to A$, $h : D \to A$ satisfy functional equation* (6) *for all $x, y \in T_+$ if and only if*

$$\begin{aligned} f(x) &= l(x) + a \qquad (x \in \mathbb{T}_+), \\ g(x) &= l(x) + b \qquad (x \in \mathbb{T}_+), \\ h(x) &= l(x) + a + b \qquad (x \in D), \end{aligned} \tag{23}$$

where the function $l : \mathbb{T}_+ \to A$ satisfies the Cauchy logarithmic equation (CL) *for all $x, y \in \mathbb{T}_+$, and $a, b \in A$ are arbitrary constants.*

Proof. Assume that functions f, g, h satisfy (6) for all $x, y \in \mathbb{T}_+$. Then one can easily see that functions f, g and function $K : D \to A$ defined by

$$K(x) = h\left(x + \frac{1}{x} + 2\right) \qquad (x \in \mathbb{T}_+)$$

satisfy the functional equation (13). It follows from Lemma 2.2 that f, g, K are of the form (14). Finally, (14) and the definition of K imply (23) for functions f, g, h. Conversely, functions in (23) indeed satisfy (6). □

Corollary 2.2. *Functional equations* (5) *and* (CL) *are equivalent for function $f : \mathbb{T}_+ \to A$, too.*

Proof. See the proof of Corollary 2.1. □

3. The second new logarithmic equation

One can easily see that any solution of (CL) is a solution of the functional equation

$$f(x(y+1)) + f(y(x+1)) = f(x(x+1)) + f(y(y+1)) \qquad (x, y \in \mathbb{T}_+) \tag{24}$$

for function $f : \mathbb{T}_+ \to A$. We will prove the equivalence of equations (24) and (CL).

To do this, consider the functional equation

$$f(y(x+1)) + g(x(y+1)) = h(x) + h(y) \qquad (x, y \in \mathbb{T}_+) \tag{25}$$

for functions f, g, $h : \mathbb{T}_+ \to A$.

Theorem 3.1. *Let A be a uniquely 2-divisible Abelian group. The functions f, g, $h : \mathbb{T}_+ \to A$ satisfy the functional equation* (25) *if and only if*

$$\begin{aligned} f(x) &= l(x) + a \qquad (x \in \mathbb{T}_+), \\ g(x) &= l(x) + b \qquad (x \in \mathbb{T}_+), \\ h(x) &= l(x(x+1)) + \frac{a+b}{2} \qquad (x \in \mathbb{T}_+), \end{aligned} \tag{26}$$

where $l : \mathbb{T}_+ \to A$ is a logarithmic function (that is l satisfies (CL)*), and $a, b \in A$ are arbitrary constants.*

Proof. Suppose that f, g, $h : T_+ \to A$ satisfy (25) for all $x, y \in \mathbb{T}_+$. Replace y by $1/y$ in (25), we get

$$f\left(\frac{x+1}{y}\right) + g\left(x\left(\frac{1}{y}+1\right)\right) = h(x) + h\left(\frac{1}{y}\right) \qquad (x, y \in \mathbb{T}_+). \tag{27}$$

Interchange x and y in (27) and deduce that

$$f\left(\frac{y+1}{x}\right) + g\left(y\left(\frac{1}{x}+1\right)\right) = h\left(\frac{1}{x}\right) + h(y) \qquad (x, y \in \mathbb{T}_+). \tag{28}$$

Adding equations (27) and (28), we get that

$$\begin{aligned} f\left(\frac{x+1}{y}\right) + f\left(\frac{y+1}{x}\right) + g\left(y\left(\frac{1}{x}+1\right)\right) + g\left(x\left(\frac{1}{y}+1\right)\right) \\ = h(x) + h\left(\frac{1}{x}\right) + h(y) + h\left(\frac{1}{y}\right) \qquad (x, y \in \mathbb{T}_+). \end{aligned} \tag{29}$$

Replace x by x/y and y by $1/y$ in (29), we obtain that

$$f(x+y)+f\left(\frac{y+1}{x}\right)+g\left(x\left(\frac{1}{y}+1\right)\right)+g\left(\frac{x+y}{xy}\right)$$
$$=h\left(\frac{x}{y}\right)+h\left(\frac{y}{x}\right)+h(y)+h\left(\frac{1}{y}\right) \qquad (x,y\in\mathbb{T}_+). \quad (30)$$

Interchange here x and y and add the resulting equation to equation (30) to get

$$2f(x+y)+f\left(\frac{x+1}{y}\right)+f\left(\frac{y+1}{x}\right)$$
$$+g\left(x\left(\frac{1}{y}+1\right)\right)+g\left(y\left(\frac{1}{x}+1\right)\right)+2g\left(\frac{x+y}{xy}\right)$$
$$=2h\left(\frac{x}{y}\right)+2h\left(\frac{y}{x}\right)+h(x)+h\left(\frac{1}{x}\right)+h(y)+h\left(\frac{1}{y}\right) \qquad (x,y\in\mathbb{T}_+). \quad (31)$$

Comparing equations (29) and (31) and using the uniquely 2-divisibility of A, we see that functions f,g,h satisfy the functional equation

$$f(x+y)+g\left(\frac{x+y}{xy}\right)=h\left(\frac{x}{y}\right)+h\left(\frac{y}{x}\right) \qquad (x,y\in\mathbb{T}_+). \quad (32)$$

It follows that the functions f, g and the function $K:\mathbb{T}_+\to A$ defined by

$$K(x)=h(x)+h\left(\frac{1}{x}\right) \qquad (x\in\mathbb{T}_+)$$

satisfy functional equation (13) in Lemma 2.2. Then Lemma 2.2 shows that f and g are of the form (26).

Finally, from (25), with substitution $x=y$, using the uniquely 2-divisibility of A and the form of f, g, we get that

$$h(x)=\frac{1}{2}\left[f(x(x+1))+g(x(x+1))\right]=\frac{1}{2}\left[l(x(x+1))+a+l(x(x+1))+b\right]$$
$$=l(x(x+1))+\frac{a+b}{2} \qquad (x,y\in\mathbb{T}_+),$$

which gives (26) for h, too. The converse is evident again. □

Corollary 3.1 (see [6]). *Let A be a uniquely 2-divisible Abelian group. The function $f:\mathbb{T}_+\to A$ satisfies* (24) *for all $x,y\in\mathbb{T}_+$ if and only if $f(x)=l(x)+a$ for all $x\in\mathbb{T}_+$, where $l:\mathbb{T}_+\to A$ satisfies* (CL) *for all $x,y\in\mathbb{T}_+$, and $a\in A$ is an arbitrary constant. Furthermore,* (24) *with condition $f(1)=0$ and* (CL) *are equivalent for function $f:\mathbb{T}_+\to A$.*

PROOF. f satisfies (25) with $g = f$ and $h(x) = f(x(x+1))$. Thus Theorem 3.1 gives that $f(x) = l(x) + a$ for all $x \in \mathbb{T}_+$, where $l : \mathbb{T}_+ \to A$ satisfies (CL) for all $x, y \in \mathbb{T}_+$, and $a \in A$ is an arbitrary constant. This proves the first part of our Corollary. If $f(1) = 0$, then $a = 0$, thus we have that $f(x) = l(x)$ for all $x \in \mathbb{T}_+$, that is, f satisfies (CL). The converse is easy to see. □

Remark 3.1. In case $\mathbb{T}_+ = \mathbb{R}_+$, $A = \mathbb{R}$, Theorem 3.1 and Corollary 3.1 imply that (24) with condition $f(1) = 0$ and (CL) are equivalent for function $f : \mathbb{R}_+ \to \mathbb{R}$.

4. A third new logarithmic functional equation

As a counterpart of equation (24), we recall our former result [7, Theorem 2]:

Theorem 4.1. *Let A be a uniquely 2-divisible Abelian group. The function $\gamma : \mathbb{T}_+ \to A$ satisfies the functional equation*

$$\gamma\left(\frac{x+1}{y}\right) + \gamma\left(\frac{y+1}{x}\right) = \gamma\left(\frac{x+1}{x}\right) + \gamma\left(\frac{y+1}{y}\right) \qquad (x, y \in \mathbb{T}_+) \qquad (33)$$

if and only if it is of the form

$$\gamma(x) = l(x) + c \qquad (x \in \mathbb{T}_+),$$

where $l : \mathbb{T}_+ \to A$ satisfies (CL) *for all $x, y \in \mathbb{T}_+$, and $c \in A$ is an arbitrary constant.*

Now, one can easily derive from Theorem 4.1 the following

Corollary 4.1. (33), *with condition $\gamma(1) = 0$, and* (CL) *are equivalent for function $f : \mathbb{T}_+ \to A$ (or for function $f : \mathbb{R}_+ \to \mathbb{R}$).*

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