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Marcinkiewicz-like means of two dimensional Vilenkin–Fourier series

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Dedicated to Professor Zsolt Páles on the occasion of his sixtieth birthday

Abstract. Let a be a lacunary sequence of natural numbers. In this paper, among others, we investigate means of two variable Vilenkin–Fourier series of the following kind: $t_n^{\alpha,a} f = \frac{1}{a_n}\sum_{k=0}^{a_n-1} S_{\alpha_1(n,k),\alpha_2(n,k)} f$, and prove the a.e. convergence $t_n^{\alpha,a} f \to f$ for each integrable function f. This immediately implies for the triangle means of the two variable integrable function f the a.e. relation $t_n^{\triangle,a} f = \frac{1}{a_n}\sum_{k=0}^{a_n-1} S_{k,a_n-k} f \to f$ $(n \to \infty)$.

1. Introduction

In 1939, for the two-dimensional quadratical trigonometric Fourier partial sums $S_{j,j}f$ Marcinkiewicz [9] proved that for all $f \in L\log L([0,2\pi]^2)$ the a.e. relation

$$\frac{1}{n}\sum_{j=1}^{n} S_{j,j} f \to f \tag{1}$$

holds as $n \to \infty$. Zhizhiashvili [13] improved this result for $f \in L([0,2\pi]^2)$. Dyachenko [3] proved this result for dimensions greater than 2. In 2001, Weisz [12] proved this result with respect to the Walsh–Paley system. Goginava [6] proved

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the corresponding result for the d-dimensional Walsh–Paley system. The author of this paper investigated [4] the (bounded) Vilenkin situation, proving also the a.e. convergence of the Marcinkiewicz means of integrable functions.

The aim of this paper is to generalize the notion of Marcinkiewicz means with respect to two-dimensional Vilenkin systems, prove (1) for these general means and also to verify this result for another "slightly different" special one with lacunary indices n. An example for this Marcinkiewicz-like means is the triangular means of two dimensional Fourier series.

First, we give a brief introduction to the theory of Vilenkin systems. These orthonormal systems were introduced by N. Ja. Vilenkin in 1947 (see e.g. [11] and [1]) as follows.

Let $m = (m_k, k \in \mathbb{N})$ ($\mathbb{N} = \{0, 1, \dots\}, \mathbb{P} = \mathbb{N} \setminus \{0\}$) be a sequence of integers, each of them not less than 2. Let Z_{m_k} denote the discrete cyclic group of order m_k. That is, Z_{m_k} can be represented by the set $\{0, 1, ..., m_k - 1\}$, with the group operation $\bmod m_k$ addition. Since the group is discrete, every subset is open. The normalized Haar measure on Z_{m_k}, μ_k is defined by $\mu_k(\{j\}) := 1/m_k$ ($j \in \{0, 1, ..., m_k - 1\}$). Let

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}.$$

Then every $x \in G_m$ can be represented by a sequence $x = (x_i, i \in \mathbb{N})$, where $x_i \in Z_{m_i}$ ($i \in \mathbb{N}$). The group operation on G_m (denoted by $+$) is the coordinate-wise addition (the inverse operation is denoted by $-$), the measure (denoted by μ), which is the normalized Haar measure, and the topology are the product measure and topology. Consequently, G_m is a compact Abelian group. If $\sup_{n \in \mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. In this paper we discuss bounded Vilenkin groups, only. That is, $m^* = \sup_n m_n < \infty$. Let $M_0 := 1, M_{n+1} := m_n M_n$ ($n \in \mathbb{N}$) be the so-called generalized powers.

The Vilenkin group is metrizable in the following way:

$$d(x, y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{M_{i+1}} \quad (x, y \in G_m).$$

The topology induced by this metric, the product topology, and the topology given by intervals defined below, are the same. A base for the neighbourhoods of G_m can be given by the intervals:

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \,\text{for}\, i < n\}$$

for $x \in G_m, n \in \mathbb{P}$. Let $0 = (0, i \in \mathbb{N}) \in G_m$ denote the nullelement of G_m and $e_n = (0, \dots, 0, 1, 0, \dots) \in G_m$, where the n-th coordinate of e_n is 1 $(n \in \mathbb{N})$.

Furthermore, let $L^p(G_m)\,(1 \leq p \leq \infty)$ denote the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) on G_m, $\mathcal{A}_n$ the σ-algebra generated by the sets $I_n(x)\,(x \in G_m)$, and E_n the conditional expectation operator with respect to $\mathcal{A}_n\,(n \in \mathbb{N})$.

Let p either be a real not less than 1 or plus infinity. We say that operator T is of type (L^p, L^p) if there exists an absolute constant $C > 0$ for which $\|Tf\|_p \leq C\|f\|_p$ for all $f \in L^p$. T is said to be of weak type (L^1, L^1) if there exists an absolute constant $C > 0$ for which $\mu(Tf > \lambda) \leq C\|f\|_1/\lambda$ for all $\lambda > 0$ and $f \in L^1(G_m)$. It is known that the operator which maps a function f to the maximal function $f^* := \sup |E_n f|$ is of weak type (L^1, L^1), and of type (L^p, L^p) for all $1 < p \leq \infty$ (see e.g. [2]).

Each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0, 1, ..., m_i - 1\},\ i \in \mathbb{N}),$$

where only a finite number of n_i's differ from zero. Later, we also use the notations $n^j := \sum_{i=j}^{\infty} n_i M_i$ and $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$ for positive integers. That is, $M_{|n|} \leq n < M_{|n|+1} \leq m^* M_{|n|}$. The generalized Rademacher functions are defined as

$$r_n(x) := \exp\left(2\pi\imath \frac{x_n}{m_n}\right) \quad (x \in G_m, n \in \mathbb{N}, \imath := \sqrt{-1}).$$

It is known that

$$\sum_{i=0}^{m_n-1} r_n^i(x) = \begin{cases} 0, & \text{if } x_n \neq 0, \\ m_n, & \text{if } x_n = 0 \end{cases} \quad (x \in G_m,\ n \in \mathbb{N}).$$

The n-th Vilenkin function is

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} \quad (n \in \mathbb{N}).$$

The system $\psi := (\psi_n : n \in \mathbb{N})$ is called a Vilenkin system. Each ψ_n is a character of G_m, and all the characters of G_m are of this form. Define the m-adic addition as

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j (\bmod m_j)) M_j \quad (k, n \in \mathbb{N}).$$

Then, $\psi_{k\oplus n}=\psi_k\psi_n$, $\psi_n(x+y)=\psi_n(x)\psi_n(y)$, $\psi_n(-x)=\bar\psi_n(x)$, $|\psi_n|=1(k,n\in\mathbb{N},x,y\in G_m)$.

Define the Fourier coefficients, the partial sums of the Fourier series and the Dirichlet kernels with respect to the Vilenkin system ψ as follows

$$\hat f(n):=\int_{G_m} f\bar\psi_n d\mu,\quad S_nf:=\sum_{k=0}^{n-1}\hat f(k)\psi_k,$$

$$D_n(y,x)=D_n(y-x):=\sum_{k=0}^{n-1}\psi_k(y)\bar\psi_k(x),\quad (n\in\mathbb{N},y,x\in G_m,f\in L^1(G_m)).$$

It is well-known that

$$S_nf(y)=\int_{G_m} f(x)D_n(y-x)d\mu(x)=f*D_n(y)\quad (n\in\mathbb{N},\ y\in G_m,\ f\in L^1(G_m)).$$

It is also well-known [1] that

$$D_{M_n}(x)=\begin{cases}M_n, & \text{if } x\in I_n:=I_n(0),\\ 0, & \text{if } x\notin I_n,\end{cases}$$

$$D_n(x)=\psi_n(x)\sum_{j=0}^{\infty}D_{M_j}(x)\sum_{p=m_j-n_j}^{m_j-1}r_j^p(x),$$

$$S_{M_n}f(x)=M_n\int_{I_n(x)} f d\mu=E_nf(x)\quad (f\in L^1(G_m),n\in\mathbb{N}). \tag{2}$$

Next, we introduce some notation with respect to the theory of two-dimensional Vilenkin systems. Let $\tilde m$ be a sequence like m. The relation between the sequence $(\tilde m_n)$ and $(\tilde M_n)$ is the same as between sequence (m_n) and (M_n). The group $G_m\times G_{\tilde m}$ is called a two-dimensional Vilenkin group. The normalized Haar measure is denoted by μ, just as in the one-dimensional case. It will not cause any misunderstanding. In this paper we also suppose that $m=\tilde m$, and that the generating sequence m is a bounded one.

The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, the Dirichlet kernels, the Marcinkiewicz means, and the Marcinkiewicz kernels with respect to the two-dimensional Vilenkin system are defined as follows:

$$\hat f(n_1,n_2):=\int_{G_m\times G_m} f(x^1,x^2)\bar\psi_{n_1}(x^1)\bar\psi_{n_2}(x^2)d\mu(x^1,x^2),$$

$$\begin{aligned}
S_{n_1,n_2}f(y^1,y^2) &:= \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\hat{f}(k_1,k_2)\psi_{k_1}(y^1)\psi_{k_2}(y^2),\\
D_{n_1,n_2}(y,x) &= D_{n_1}(y^1-x^1)D_{n_2}(y^2-x^2)\\
&:= \sum_{k_1=0}^{n_1-1}\sum_{k_2=0}^{n_2-1}\psi_{k_1}(y^1)\psi_{k_2}(y^2)\bar{\psi}_{k_1}(x^1)\bar{\psi}_{k_2}(x^2),\\
t_nf &:= \frac{1}{n}\sum_{j=0}^{n-1}S_{j,j}f,\\
K_n(y,x) &= K_n(y-x) := \frac{1}{n}\sum_{j=0}^{n-1}D_{j,j}(y-x),\\
(y &= (y^1,y^2), x=(x^1,x^2)\in G_m\times G_m).
\end{aligned}$$

It is also well-known that

$$t_nf(y)=\int_{G_m\times G_m}f(x)K_n(y-x)d\mu(x)=f*K_n(y).$$

For the two-dimensional trigonometric Fourier partial sums $S_{j,j}f$ Marcinkiewicz [9] proved that for all $f\in L\log L([0,2\pi]^2)$ the a.e. relation $t_nf\to f$ as $n\to\infty$. Zhizhiashvili [13] improved this result for $f\in L([0,2\pi]^2)$. In 2000, Weisz [12] verified the result of Zhizhiashvili for the Walsh–Paley system. In 2003, Goginava [6] proved this result with respect to the d-dimensional Walsh–Paley system. In 2004, Gát [4] proved the a.e. convergence of Marcinkiewicz means of integrable functions on two dimensional bounded Vilenkin groups.

2. The result

After then, we turn our attention to the generalization of Marcinkiewicz means. Let $\alpha=(\alpha_1,\alpha_2):\mathbb{N}^2\to\mathbb{N}^2$ be a function. Define the following Marcinkiewicz-like kernels and means:

$$K_n^\alpha(x):=\frac{1}{n}\sum_{k=0}^{n-1}D_{\alpha_1(|n|,k)}(x^1)D_{\alpha_2(|n|,k)}(x^2),$$
$$t_n^\alpha f:=f*K_n^\alpha,\quad (f\in L^1(G_m^2),n\in\mathbb{P}).$$

The main aim of this paper is to give a class of functions α for which we have the a.e. convergence relation $t_n^\alpha f\to f$ for each integrable two-variable function f. The following properties will play a prominent role in the a.e. convergence properties of generalized Vilenkin–Marcinkiewicz means.

$$\#\{l \in \mathbb{N} : \alpha_j(|n|, l) = \alpha_j(|n|, k), l < n\} \le C\,(k < n, j = 1, 2), \tag{3}$$

$$\max\{\alpha_j(|n|, k) : k < n\} \le Cn \quad (k, n \in \mathbb{P}, j = 1, 2). \tag{4}$$

More precisely, we prove:

Theorem 2.1. *Let α satisfy* (3) *and* (4). *Then we have $t_n^\alpha f \to f$ for each $f \in L^1(G_m^2)$.*

We give a corollary of Theorem 2.1.

Corollary 2.2. *Let (a_n) be a lacunary sequence of natural numbers, i.e. $a_{n+1} \ge a_n q$ for some $q > 1$ $(n \in \mathbb{N})$ and α satisfy conditions* (3) *and $\alpha_j(n, k) \le Ca_n$ $(k < a_n,\ j = 1, 2)$ (modified version of condition* (4)*). Then for every integrable function $f \in L^1(G_m^2)$ we have*

$$\frac{1}{a_n}\sum_{k=0}^{a_n-1} S_{\alpha_1(n,k),\alpha_2(n,k)} f(x) \to f(x)$$

for a.e. $x \in G_m^2$.

For the Walsh–Paley case (that is, $m_n = 2$ for all $n \in \mathbb{N}$) Theorem 2.1 and Corollary 2.2 are proved also by the author of this paper [5].

Proof. The proof of this corollary runs as follows. First suppose that $q \ge m^* = \sup_n m_n$. Let $b_n = |a_n| + 1$. In this situation $b_{n+1} = |a_{n+1}| + 1 \ge |qa_n| + 1 \ge |m^* a_n| + 1 \ge |a_n| + 2 = b_n + 1$. Moreover, let

$$\tilde{\alpha}_j(b_n, k) = \begin{cases} \alpha_j(n, k), & \text{if } 0 \le k < a_n, \\ k, & \text{if } a_n \le k < M_{b_n} \end{cases} \qquad (j = 1, 2).$$

Then, $\tilde{\alpha}$ satisfies conditions (3) (trivially) and (4) since $\tilde{\alpha}_j(b_n, k) = \alpha_j(n, k) \le Ca_n \le CM_{b_n}$ $(k < a_n)$ and by Theorem 2.8 (see in this paper below) it follows that for the maximal operator $t_*^{\tilde{\alpha}} f := \sup |t_n^{\tilde{\alpha}} f|$ we have $\operatorname{mes}\{t_*^{\tilde{\alpha}} f \ge \lambda\} \le C\|f\|_1/\lambda$ for all $f \in L^1(G_m^2)$ and $\lambda > 0$. Since

$$\frac{1}{a_n}\sum_{k=0}^{a_n-1} S_{\alpha_1(n,k),\alpha_2(n,k)} f =$$

$$= \frac{M_{b_n}}{a_n}\frac{1}{M_{b_n}}\sum_{k=0}^{M_{b_n}-1} S_{\tilde\alpha_1(b_n,k),\tilde\alpha_2(b_n,k)} f - \frac{M_{b_n}}{a_n}\frac{1}{M_{b_n}}\sum_{k=a_n}^{M_{b_n}-1} S_{k,k} f,$$

and consequently, $|t^{\alpha}_{a_n} f| \le m^*|t^{\tilde\alpha}_{M_{b_n}} f| + m^*|t_{M_{b_n}} f| + m^*|t_{a_n} f|$, then $t^{\alpha}_* f \le C t^{\tilde\alpha}_* f + C t_* f$. The ordinary maximal Marcinkiewicz operator is of weak type (L^1, L^1) and of type (L^p, L^p) $(1 < p \le \infty)$ (see e.g. [4]), and thus so does t^{α}_*. This, by the standard density argument, completes the proof of this corollary for the case of $q \ge m^* = \sup_n m_n$. If this is not the case, then let γ be the smallest natural number for which $q^\gamma \ge m^*$ and divide the sequence $a = (a_n)$ to γ subsequences: $a^j = (a^j_n) = (a_{n\gamma+j})$, $j = 0, \dots, \gamma-1, n \in \mathbb{N}$. Since for each subsequence a^j we have $a^j_{n+1} \ge q^\gamma a^j_n \ge m^* a^j_n$ and consequently $\frac{1}{a^j_n}\sum_{k=0}^{a^j_n-1} S_{\alpha_1(n,k),\alpha_2(n,k)} f(x) \to f(x)$ a.e. for $j = 0, \dots, \gamma$, then the proof of this corollary is complete for every $q > 1$. □

The triangular partial sums of the 2-dimensional Fourier series are defined as

$$S^{\triangle}_k f(x^1, x^2) := \sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1} \hat f(i,j)\psi_i(x^1)\psi_j(x^2).$$

Denote the triangular kernel

$$D^{\triangle}_k(x^1, x^2) := \sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1} \bar\psi_i(x^1)\psi_j(x^2).$$

The Fejér means of the triangular partial sums of the two-dimensional integrable function f (see e.g. [7]) are

$$t^{\triangle}_n f = \frac{1}{n}\sum_{k=0}^{n-1} S^{\triangle}_k f.$$

For the trigonometric system Herriot [8] proved the a.e. (and norm) convergence $t^{\triangle}_n f \to f$ $(f \in L^1)$. His method cannot be adopted for the Walsh and Vilenkin systems, since for the time being there is no kernel formula available for these systems. The first result in this a.e. convergence issue of triangular means is due to Goginava and Weisz [7]. They proved for the Walsh–Paley system and each integrable function the a.e. convergence relation $t^{\triangle}_{2^n} f \to f$. That is, we have the subsequence $(t^{\triangle}_{2^n})$ of the whole sequence of the triangular mean operators. This result for every lacunary sequence (a_n) (instead of (2^n)) follows from a result of Gát [5]. For bounded Vilenkin systems Corollary 2.2 gives this a.e. relation for lacunary triangular means with $\alpha_1(n,k)=k$, $\alpha_2(n,k)=a_n-k$. To demonstrate this, see also some calculations with respect to the triangle kernels.

$$\begin{aligned}
K_n^{\triangle}(x^1,x^2) &= \frac{1}{n}\sum_{k=0}^{n-1} D_k^{\triangle}(x^1,x^2) = \frac{1}{n}\sum_{k=1}^{n-1}\sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1}\psi_i(x^1)\psi_j(x^2)\\
&= \frac{1}{n}\sum_{k=1}^{n-1}\sum_{i=0}^{k-1}\psi_i(x^1)D_{k-i}(x^2) = \frac{1}{n}\sum_{k=1}^{n-1}\sum_{i=1}^{k}\psi_{k-i}(x^1)D_i(x^2)\\
&= \frac{1}{n}\sum_{i=1}^{n-1}\sum_{k=i}^{n-1}\psi_{k-i}(x^1)D_i(x^2) = \frac{1}{n}\sum_{i=1}^{n-1} D_{n-i}(x^1)D_i(x^2).
\end{aligned}$$

That is, we proved the following corollary.

Corollary 2.3. *Let (a_n) be a lacunary sequence of natural numbers, i.e. $a_{n+1}\ge a_n q$ for some $q>1$. Then for every integrable function $f\in L^1(G_m^2)$ we have*

$$t_{a_n}^{\triangle} f(x) = \frac{1}{a_n}\sum_{k=0}^{a_n-1} S_{k,a_n-k}f(x) \to f(x)$$

for a.e. $x\in G_m^2$.

Now, we turn our attention to the proof of the convergence theorem. Our first main aim is to prove that the operator $t_*^{\alpha}f := \sup_{n\in\mathbb{P}}|t_n^{\alpha}f|$ is of weak type (L^1,L^1). In order to have this, we need a sequence of lemmas. The first, which (one might say) is the very base of the proof of Theorem 2.1, is the most difficult one. However, the techniques of its proof will also be used in the proof of the forthcoming lemmas.

Denote for $k\in\mathbb{N}$ $J_k = I_k\setminus I_{k+1}$ and recall that $n^s := \sum_{k=s}^{\infty} n_k M_k$ $(n,s\in\mathbb{N})$; $n^0=n, n^{|n|+1}=0$.

Lemma 2.4. *Let $a\in\mathbb{N}$ and*

$$J_{a,1} = \bigcup_{t^1=0}^{a-1}\bigcup_{t^2=t^1}^{\infty} J_{t^1}\times J_{t^2} \subset G_m\times G_m.$$

Then

$$\int_{J_{a,1}} \sup_{A\ge a}\sup_{\{n:|n|=A\}} \frac{1}{M_A}\sum_{s=t^1}^{A}\sum_{j=0}^{n_s-1}\left|\sum_{k=0}^{M_s-1} D_{\alpha_1(A,n^{s+1}+jM_s+k)}(x^1)\right.$$
$$\left.\times D_{\alpha_2(A,n^{s+1}+jM_s+k)}(x^2)\right| d\mu(x) \le C.$$

Proof. First, for fixed $t = (t^1, t^2), j, s, A$ we discuss the integral

$$\int_{J_{t^1}\times J_{t^2}} \sup_{|n|=A} \left| \sum_{k=0}^{M_s-1} D_{\alpha_1(A,n^{s+1}+jM_s+k)}(x^1) D_{\alpha_2(A,n^{s+1}+jM_s+k)}(x^2) \right| d\mu(x).$$

Check the function $\sum_{k=0}^{M_s-1} D_{\alpha_1(A,n^{s+1}+jM_s k)}(x^1) D_{\alpha_2(A,n^{s+1}+jM_s+k)}(x^2)$ on the set $J_{t^1}\times J_{t^2}$. Since we have $x^2 \in J_{t^2}$, then by (2) we have $|D_l(x^2)| \le CM_{t^2}$ for each $l \in \mathbb{N}$, and consequently, $|D_{\alpha_2(A,n^{s+1}+jM_s+k)}(x^2)| \le CM_{t^2}$. On the other hand, again by (2) for $x^1 \in J_{t^1}$ we have

$$\begin{aligned} &D_{\alpha_1(A,n^{s+1}+jM_s+k)}(x^1) \\ &\quad = \psi_{[\alpha_1(A,n^{s+1}+jM_s+k)]^{t^1}}(x^1) \\ &\quad\times \left(\sum_{j=0}^{t^1-1} [\alpha_1(A,n^{s+1}+jM_s+k)]_j M_j + \sum_{i=m_{t^1}-[\alpha_1(A,n^{s+1}+jM_s+k)]_{t^1}}^{m_{t^1}-1} r^i_{t^1}(x^1) M_{t^1} \right) \\ &\quad =: \psi_{[\alpha_1(A,n^{s+1}+jM_s+k)]^{t^1}}(x^1) \beta_1(A, n^{s+1}+jM_s+k, t^1, x^1_{t^1}). \end{aligned}$$

The function $\beta_1(A, n^{s+1}+jM_s+k, t^1, x^1_{t^1})$ is $\mathcal{A}_{t^1+1}$ measurable, it depends only on $x^1_{t^1}$ (and not on other coordinates of x^1), and its absolute value is bounded by CM_{t^1}. If it does not cause misunderstanding, we simply abbreviate it by $\beta_1(j)$. That is,

$$\beta_1(j) = \beta_1(A, n^{s+1}+jM_s+k, t^1, x^1_{t^1}).$$

Apply the Cauchy–Bunyakovsky–Schwarz inequality:

$$\begin{aligned} &\int_{J_{t^2}} \left[\int_{J_{t^1}} \sup_{|n|=A} \left| \sum_{k=0}^{M_s-1} D_{\alpha_1(A,n^{s+1}+jM_s+k)}(x^1) D_{\alpha_2(A,n^{s+1}+jM_s+k)}(x^2) \right| d\mu(x^1) \right] d\mu(x^2) \\ &\quad \le \int_{J_{t^2}} \frac{1}{M_{t^1}^{1/2}} \left[\int_{J_{t^1}} \sup_{|n|=A} \left| \sum_{k=0}^{M_s-1} D_{\alpha_1(A,n^{s+1}+jM_s+k)}(x^1) \right.\right. \\ &\quad \left.\left. \times D_{\alpha_2(A,n^{s+1}+jM_s+k)}(x^2) \right|^2 d\mu(x^1) \right]^{1/2} d\mu(x^2) \\ &\quad = \int_{J_{t^2}} \frac{1}{M_{t^1}^{1/2}} \left[\int_{J_{t^1}} \sup_{|n|=A} \sum_{k,l=0}^{M_s-1} \psi_{[\alpha_1(A,n^{s+1}+jM_s+k)]^{t^1}} \right. \\ &\quad \times (x^1) \psi_{[\alpha_1(A,n^{s+1}+jM_s+l)]^{t^1}}(x^1) \\ &\quad \times \beta_1(A, n^{s+1}+jM_s+k, t^1, x^1_{t^1}) \beta_1(A, n^{s+1}+jM_s+l, t^1, x^1_{t^1}) \end{aligned}$$

$$\times D_{\alpha_2(A,n^{s+1}+jM_s+k)}(x^2)D_{\alpha_2(A,n^{s+1}+jM_s+l)}(x^2)d\mu(x^1)\Bigg]^{1/2} d\mu(x^2)$$

$$=: B^1.$$

Since n^{s+1} depends only on $n_{s+1},\dots,n_{A-1},n_A$ (recall that $n_A \neq 0$), then the supremum operator $\sup_{\{n:|n|=A\}}$ above also depends only on $n_{s+1},\dots,n_A$. Thus, by

$$J_{t^1} = \cup_{i=1}^{m_{t^1}-1} I_{t^1+1}(ie_{t^1})$$

we have for B^1:

$$B^1 \le \int_{J_{t^2}} \frac{1}{M_{t^1}^{1/2}} \Bigg[\sum_{n_A=1}^{m_{A-1}-1} \sum_{n_{A-1}=0}^{m_{A-1}-1} \cdots \sum_{n_{s+1}=0}^{m_{s+1}-1} \int_{J_{t^1}} \sum_{k,l=0}^{M_s-1}$$

$$\psi_{[\alpha_1(A,n^{s+1}+jM_s+k)]^{t^1}}(x^1)\psi_{[\alpha_1(A,n^{s+1}+jM_s+l)]^{t^1}}(x^1)$$

$$\times \beta_1(A,n^{s+1}+jM_s+k,t^1,x^1_{t^1})\beta_1(A,n^{s+1}+jM_s+l,t^1,x^1_{t^1})$$

$$\times D_{\alpha_2(A,n^{s+1}+k)}(x^2)D_{\alpha_2(A,n^{s+1}+l)}(x^2)d\mu(x^1)\Bigg]^{1/2} d\mu(x^2)$$

$$= \int_{J_{t^2}} \frac{1}{M_{t^1}^{1/2}} \Bigg[\sum_{i=1}^{m_{t^1}-1} \sum_{n_A=1}^{m_{A-1}-1} \sum_{n_{A-1}=0}^{m_{A-1}} \cdots \sum_{n_{s+1}=0}^{m_{s+1}-1} \sum_{k,l=0}^{M_s-1}$$

$$\beta_1(A,n^{s+1}+jM_s+k,t^1,i)\beta_1(A,n^{s+1}+jM_s+l,t^1,i)$$

$$\times D_{\alpha_2(A,n^{s+1}+k)}(x^2)D_{\alpha_2(A,n^{s+1}+l)}(x^2)$$

$$\times \int_{I_{t^1+1}(ie_{t^1})} \psi_{[\alpha_1(A,n^{s+1}+jM_s+k)]^{t^1}}(x^1)$$

$$\times \psi_{[\alpha_1(A,n^{s+1}+jM_s+l)]^{t^1}}(x^1)d\mu(x^1)\Bigg]^{1/2} d\mu(x^2) =: B^2.$$

That is, we estimate B^1 by B^2 defined above at the end of the previous line. Discuss the integral

$$\int_{I_{t^1+1}(ie_{t^1})} \psi_{[\alpha_1(A,n^{s+1}+jM_s+k)]^{t^1}}(x^1)\psi_{[\alpha_1(A,n^{s+1}+jM_s+l)]^{t^1}}(x^1)d\mu(x^1).$$

If it differs from zero, then the t^1+1-th, t^1+2-th,... coordinates of $\alpha_1(A,n^{s+1}+jM_s+k)$ and $\alpha_1(A,n^{s+1}+jM_s+l)$ should be equal. Since (3) we have that for every k there exists only a bounded number of l's for which $\alpha_1(A,n^{s+1}+jM_s+$

$k) = \alpha_1(A, n^{s+1} + jM_s + l)$. These facts give that for every k there exists – at most – CM_{t^1} number of l's for which this integral is not zero.

Consequently (emphasize that C can depend on m^*),

$$\begin{aligned} B^2 &\leq C \int_{J_{t^2}} M_{t^1}^{\frac{-1}{2}} \left[\sum_{n_A=1}^{m_{A-1}-1} \sum_{n_{A-1}=0}^{m_{A-1}-1} \cdots \sum_{n_{s+1}=0}^{m_{s+1}-1} M_{t^1}^2 M_{t^2}^2 M_s M_{t^1} M_{t^1}^{-1} \right]^{\frac{1}{2}} d\mu \\ &\leq C\sqrt{M_A M_{t^1}}. \end{aligned}$$

This means

$$\begin{aligned} &\int_{J_{t^1} \times J_{t^2}} \sup_{|n|=A} \left| \sum_{k=0}^{M_s-1} D_{\alpha_1(A, n^{s+1}+jM_s+k)}(x^1) D_{\alpha_2(A, n^{s+1}+jM_s+k)}(x^2) \right| d\mu(x) \\ &\quad \leq C\sqrt{M_A M_{t^1}}. \end{aligned}$$

This inequality immediately gives ($a \vee b = \max(a,b)$)

$$\begin{aligned} &\sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1} \times J_{t^2}} \sup_{A \geq a \vee (t^2 - C)} \sup_{|n|=A} \frac{1}{M_A} \\ &\quad \times \sum_{s=t^1}^{A} \sum_{j=0}^{n_s-1} \left| \sum_{k=0}^{M_s-1} D_{\alpha_1(A, n^{s+1}+jM_s+k)}(x^1) D_{\alpha_2(A, n^{s+1}+jM_s k)}(x^2) \right| d\mu(x) \\ &\quad \leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \sum_{A=a \vee (t^2-C)}^{\infty} \sum_{s=t^1}^{A} \sqrt{M_{t^1}/M_A} \\ &\quad \leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \sum_{A=a \vee (t^2-C)}^{\infty} (A - t^1 + 1)\sqrt{M_{t^1}/M_A} \\ &\quad \leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} ((a \vee t^2) - t^1)\sqrt{M_{t^1}/M_{(a \vee t^2)}} \\ &\quad \leq C \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{a} (a - t^1)\sqrt{M_{t^1}/M_a} + C \sum_{t^1=0}^{a-1} \sum_{t^2=a+1}^{\infty} (t^2 - t^1)\sqrt{M_{t^1}/M_{t^2}} \leq C. \end{aligned}$$

This inequality shows that if we want to complete the proof of this lemma, then we have to discuss also the case $\sup_{t^2 - C > A \geq a}$. This follows that t^2 should be at least $a + C$. That is, we have to prove that the following integral is bounded.

$$\sum_{t^1=0}^{a-1}\sum_{t^2=a+C}^{\infty}\int_{J_{t^1}\times J_{t^2}}\sup_{t^2-C>A\geq a}\sup_{|n|=A}\frac{1}{M_A}\sum_{s=t^1}^{A}\sum_{j=0}^{n_s-1}\left|\sum_{k=0}^{M_s-1}\right.$$

$$\left.D_{\alpha_1(A,n^{s+1}+jM_s+k)}(x^1)D_{\alpha_2(A,n^{s+1}+jM_s+k)}(x^2)\right|d\mu(x)=:B^3.$$

The method we are going to use in order to discuss B^3 is the same as we used for the investigation of B^1. The only difference is that in the situation of B^1 we used the estimation $|D_{\alpha_2(A,n^{s+1}+jM_s+k)}(x^2)|\leq CM_{t^2}$, and in the case of B^3 we use – by the help of (4) and the formula of the Dirichlet kernel D_n (2) – the estimation $|D_{\alpha_2(A,n^{s+1}+jM_s+k)}(x^2)|\leq CM_A$. The other steps of this process are the same. Remark that since $j\leq n_s<m_s\leq m^*$, then "we do not have to take too much attention to" j, as constant C can depend on m^*. That is,

$$\begin{aligned}
B^3 &\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=a+C}^{\infty}\int_{J_{t^2}}\sum_{A=a}^{t^2-C}\frac{1}{M_A}\sum_{s=t^1}^{A}M_{t^1}^{-1/2}\\
&\quad\times\left[\sum_{n_A=1}^{m_{A-1}-1}\sum_{n_{A-1}=0}^{m_{A-1}-1}\cdots\sum_{n_{s+1}=0}^{m_{s+1}-1}M_{t^1}^2M_A^2M_sM_{t^1}M_{t^1}^{-1}\right]^{1/2}d\mu(x^2)\\
&=C\sum_{t^1=0}^{a-1}\sum_{t^2=a+C}^{\infty}\sum_{A=a}^{t^2-C}\sum_{s=t^1}^{A}M_{t^2}^{-1}M_A^{-1}M_{t^1}^{-1/2}\sqrt{\frac{M_A}{M_s}M_{t^1}^2M_A^2M_s}\\
&=C\sum_{t^1=0}^{a-1}\sum_{t^2=a+C}^{\infty}\sum_{A=a}^{t^2-C}\sum_{s=t^1}^{A}\frac{\sqrt{M_AM_{t^1}}}{M_{t^2}}\\
&\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=a+C}^{\infty}\sum_{A=a}^{t^2-C}(A-t^1+1)M_A^{1/2}M_{t^1}^{1/2}M_{t^2}^{-1}\\
&\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=a+C}^{\infty}(t^2-t^1+1)M_{t^1}^{1/2}M_{t^2}^{-1/2}\leq C.
\end{aligned}$$

This completes the proof of Lemma 2.4. Remark the fact that the generating sequence m is bounded, that is, $m^*<\infty$ is "heavily used". □

In the sequel we step further, and with the application of Lemma 2.4 we prove the main tool with respect to this investigation issue of the maximal Marcinkiewicz-like kernel, in order to prove that the maximal operator t_*^α is quasi-local (for the definition of quasi-locality, see e.g. [10, page 262]), and consequently, it is of weak type (L^1,L^1).

Lemma 2.5.

$$\int_{G_m^2\setminus(I_a\times I_a)} \sup_{n\geq a-C} |K_n^\alpha(x)| d\mu(x) \leq C.$$

Proof. For $t^1 \leq a-1, t^2 \geq t^1$ and $x \in J_{t^1} \times J_{t^2}$ by (2) and (4) it is clear that $(A = |n|)$

$$|D_{\alpha_1(A,n^{s+1}+jM_s+k)}(x^1) D_{\alpha_2(A,n^{s+1}+jM_s+k)}(x^2)| \leq M_{t^1} M_{(t^2\wedge A)}.$$

This gives

$$\begin{aligned}
&\sum_{t^1=0}^{a-1}\sum_{t^2=t^1}^{\infty}\int_{J_{t^1}\times J_{t^2}} \sup_{A\geq a-C}\sup_{|n|=A}\frac{1}{M_A}\sum_{s=0}^{t^1}\sum_{j=0}^{n_s-1}\left|\sum_{k=0}^{M_s-1} D_{\alpha_1(A,n^{s+1}+jM_s+k)}(x^1)\right. \\
&\quad\left.\times D_{\alpha_2(A,n^{s+1}+jM_s+k)}(x^2)\right| d\mu(x) \\
&\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=t^1}^{\infty}\int_{J_{t^1}\times J_{t^2}} \sup_{A\geq a-C}\frac{1}{M_A}\sum_{s=0}^{t^1} M_s M_{t^1} M_{(t^2\wedge A)} d\mu(x) \\
&\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=t^1}^{a-C}\frac{1}{M_{t^1}M_{t^2}}\sup_{A\geq a-C} M_{t^1}^2 M_{t^2} M_A^{-1} + C\sum_{t^1=0}^{a-1}\sum_{t^2=a-C}^{\infty}\frac{1}{M_{t^1}M_{t^2}} M_{t^1}^2 \\
&\leq C\sum_{t^1=0}^{a-1}\sum_{t^2=t^1}^{a-C}\frac{M_{t^1}}{M_a} + C\sum_{t^1=0}^{a-1}\sum_{t^2=a-C}^{\infty}\frac{M_{t^1}}{M_{t^2}} \leq C.
\end{aligned}$$

This by equality

$$K_n^\alpha(x) = \frac{1}{n}\sum_{s=0}^{A}\sum_{j=0}^{n_s-1}\sum_{k=0}^{M_s-1} D_{\alpha_1(A,n^{s+1}+jM_s+k)}(x^1) D_{\alpha_2(A,n^{s+1}+jM_s+k)}(x^2)$$

and by Lemma 2.4 immediately gives

$$\sum_{t^1=0}^{a-1}\sum_{t^2=t^1}^{\infty}\int_{J_{t^1}\times J_{t^2}} \sup_{n\geq a-C} |K_n^\alpha(x)| d\mu(x) \leq C.$$

Similarly, we can also have

$$\sum_{t^2=0}^{a-1}\sum_{t^1=t^2}^{\infty}\int_{J_{t^1}\times J_{t^2}} \sup_{n\geq a-C} |K_n^\alpha(x)| d\mu(x) \leq C.$$

If we prove the almost everywhere relation

$$G_m^2 \setminus (I_a \times I_a) \subset \left(\bigcup_{t^1=0}^{a-1} \bigcup_{t^2=t^1}^{\infty} J_{t^1} \times J_{t^2} \right) \bigcup \left(\bigcup_{t^2=0}^{a-1} \bigcup_{t^1=t^2}^{\infty} J_{t^1} \times J_{t^2} \right) =: J_{a,1} \bigcup J_{a,2},$$

then the proof of Lemma 2.5 would be complete. This is quite easy, and therefore it is left to the reader. □

Corollary 2.6. *Let $n \in \mathbb{P}$. Then*

$$\|K_n^\alpha\|_1 \le C.$$

Proof. By Lemma 2.5 we have

$$\int_{G_m^2 \setminus (I_{|n|} \times I_{|n|})} |K_n^\alpha| d\mu \le C.$$

Besides, (4) and (2) gives

$$|K_n^\alpha(x)| \le \frac{1}{n} \sum_{k=0}^{n-1} |D_{\alpha_1(|n|,k)}(x^1)||D_{\alpha_2(|n|,k)}(x^2)| \le C \frac{1}{n} \sum_{k=0}^{n-1} M_{|n|} \cdot M_{|n|} \le C M_{|n|}^2.$$

Hence,

$$\int_{I_{|n|} \times I_{|n|}} |K_n^\alpha| d\mu \le C$$

and this completes the proof of Corollary 2.6. □

Now, we can prove that the maximal operator t_*^α is quasi-local (for the definition of quasi-locality, see e.g. [10, page 262]) and then a bit later the fact that it is of weak type (L^1, L^1). In other words:

Lemma 2.7. *Let $f \in L^1(G_m^2)$, $\operatorname{supp} f \subset I_a(u^1) \times I_a(u^2)$, $\int f d\mu = 0$ for some $u \in G_m^2$ and $a \in \mathbb{N}$. Then*

$$\int_{G_m^2 \setminus (I_a(u^1) \times I_a(u^2))} t_*^\alpha f(x) d\mu(x) \le C \|f\|_1.$$

Proof. From the shift invariancy of the Haar measure we can suppose that $u^1 = u^2 = 0$. If $|n| \le a - C$ for some fixed constant $C > 0$ depending on α_1, α_2 (and m^*), then we have by (4) that $\alpha_1(|n|, k), \alpha_2(|n|, k) < M_a$ for every $k < n$. Consequently, the kernel $K_n^\alpha(x^1, x^2)$ (which is a linear combination of two-dimensional Vilenkin functions $\psi_{j,k}$ with $j,k < M_a$) is $\mathcal{A}_{a,a} = \mathcal{A}_a \times \mathcal{A}_a$ measurable. This implies

$$t_n^\alpha f(y) = \int_{I_a \times I_a} f(x) K_n^\alpha(y-x) d\mu(x) = K_n^\alpha(y) \int_{I_a \times I_a} f(x) d\mu(x) = 0.$$

That is, $|n| \geq a - C$ can be supposed. By the theorem of Fubini and Lemma 2.5 we get

$$\begin{aligned}
&\int_{G_m^2 \setminus I_a^2} t_*^\alpha f d\mu \\
&= \int_{G_m^2 \setminus I_a^2} \sup_{|n| \geq a-C} |t_n^\alpha f| d\mu = \int_{G_m^2 \setminus I_a^2} \sup_{|n| \geq a-C} |\int_{I_a^2} f(x) K_n^\alpha(y-x) d\mu(x)| d\mu(y) \\
&\leq \int_{I_a^2} |f(x)| \int_{G_m^2 \setminus I_a^2} \sup_{|n| \geq a-C} |K_n^\alpha(z) d\mu(z)| d\mu(x) \leq C \int_{I_a^2} |f(x)| d\mu(x) = C\|f\|_1.
\end{aligned}$$

This completes the proof of Lemma 2.7. □

Theorem 2.8. *The operator t_*^α is of weak type (L^1, L^1) and it is also of type (L^p, L^p) for all $1 < p \leq \infty$.*

Proof. Now, we know that operator t_*^α is of type (L^∞, L^∞), which is given by Corollary 2.6, and it is quasi-local (Lemma 2.7). Consequently, to prove that operator t_*^α is of weak type (L^1, L^1) is nothing else but to follow the standard argument (see e.g. [10]). Finally, the interpolation lemma of Marcinkiewicz (see e.g. [10]) gives that it is also of type (L^p, L^p) for all $1 < p \leq \infty$. □

Proof of Theorem 2.1. Next, we turn our attention to the proof of the theorem of convergence, that is, to Theorem 2.1. This is also a trivial consequence of the fact that the maximal operator t_*^α is of weak type (L^1, L^1) and the fact that Theorem 2.1 holds for each two-dimensional Vilenkin polynomial (which is also easy to see). □

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