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On Newton–Sobolev spaces

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Abstract. Newton–Sobolev spaces, as presented by N. Shanmugalingam, describe a way to extend Sobolev spaces to the metric setting via upper gradients, for metric spaces with 'sufficient' paths of finite length. Sometimes, as is the case of parabolic metrics, most curves are non-rectifiable. We generalize some of these results to spaces where paths are not necessarily measured by arc length. Under the assumption of a Poincaré-type inequality and an arc-chord property here defined, we obtain the density of some Lipschitz classes, relate Newton–Sobolev spaces to those defined by Hajłasz, and we also get some Sobolev embedding theorems. Finally, we illustrate some non-standard settings where these conditions hold, specifically by adding a weight to arc-length.

1. Introduction

If Ω is an open set in $\mathbb{R}^n$ and f is a smooth function defined on Ω, the Fundamental Theorem of Calculus for line integrals implies that, for every piecewise smooth path γ in Ω with endpoints x, y, we get

$$|f(x) - f(y)| \leq \int_\gamma |\nabla f| d|s|.$$

Nonnegative functions defined in Ω that satisfy this inequality for every x, y and every γ, joining them in place of $|\nabla f|$, are referred to as *upper gradients* (see, for example, [14]).

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In the case $\Omega = \mathbb{R}^n$, one can consider only segments parallel to the coordinate axes instead of more general paths, and those are sufficient to describe partial derivatives and through them gradients. The same can be done if we consider a rotation of these segments, as the Euclidean metric is invariant under rotations, and the same holds for path length. This is not true in a more general setting such as $\mathbb{R}^2$ with the parabolic metric defined further along this section, where only horizontal segments are rectifiable.

In [16], N. Shanmugalingam describes, via upper gradients, a way to characterize Sobolev spaces $W^{1,p}$ in open sets of $\mathbb{R}^n$ that extends to metric measure spaces, defining Newton–Sobolev spaces $N^{1,p}$. If the space has 'sufficient' rectifiable paths (in the sense that the set of rectifiable paths has nonzero p-modulus), an interesting theory of Sobolev functions can be developed, but if the set of rectifiable paths is negligible, this 'Sobolev space' is just L^p.

Easy enough examples of metric measure spaces with no paths of dimension 1 can be constructed. For instance, take $X = \mathbb{R}$ with $d(x, y) = |x - y|^{1/2}$, and we get that paths are either 0-dimensional (trivial paths) or 2-dimensional. While 'classical' Newton–Sobolev theory in such a space would be nonsensical, a good theory could be developed if we measured path 'length' by Hausdorff 2-dimensional measure $\mathcal{H}^2_d$ with respect to the new distance d. Of course, $\mathcal{H}^2_d$ coincides with $\mathcal{H}^1$ with respect to the Euclidean distance, and the above example seems to be just a change of parameters.

In a more interesting scenario, we consider parabolic metrics associated to a matrix, see, for instance, [9]. Take an $n \times n$ diagonal matrix D with eigenvalues satisfying $\alpha_1, \dots, \alpha_n \geq 1$. For $x \in \mathbb{R}^n$ and $\lambda > 0$, we define

$$T_\lambda x = e^{D \log \lambda} x = \begin{pmatrix} \lambda^{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \lambda^{\alpha_n} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

For a norm $\|\cdot\|$ in $\mathbb{R}^n$ it can be shown that, for $x \neq 0$, $\|T_\lambda x\|$ is continuous, strictly increasing in λ, tends to 0 as $\lambda \to 0$ and tends to ∞ as $\lambda \to \infty$. Then there exists a unique $0 < \rho(x) < \infty$ such that $\|T_{1/\rho(x)} x\| = 1$. If we define

$$d(x, y) = \rho(x - y)$$

for $x \neq y$ and $d(x, x) = 0$, then d is a translation invariant metric that also satisfies $d(T_\lambda x, T_\lambda y) = \lambda d(x, y)$ and $d(x, y) = 1$ iff $|x - y| = 1$, $d(x, y) < 1$ iff $|x - y| < 1$, $d(x, y) > 1$ iff $|x - y| > 1$. These metrics thus defined can have different Hausdorff dimensions, see [1].

The word parabolic refers to the case $\alpha_1 = \ldots = \alpha_{n-1} = 1$ and $\alpha_n = 2$, which provides the right dilations for the heat equation and other partial differential equations of parabolic type (see [5]). For example, if we consider $\mathbb{R}^2$ with $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and the maximum norm, we obtain

$$d\left((x,y),(x',y')\right) = \max\left\{|x-x'|, |y-y'|^{1/2}\right\},$$

and it can be shown that balls have Hausdorff dimension 3 (in fact they are Ahlfors 3-regular). Here, the only non-trivial rectifiable paths are horizontal segments, so even though *there are* rectifiable paths, the space is not connected by them. Smooth non-horizontal paths have Hausdorff dimension 2, so we see that this measure is not rotation invariant.

As another example of heterogeneity, we can consider adding a weight ω to arc-length by using the measure $d\mu = \omega d\mathcal{H}^1$. In this case, this path measure will not necessarily be invariant under any kind of isometry.

In this work, following the ideas in [16], we develop a more general theory of Newton–Sobolev spaces by replacing Hausdorff 1-dimensional measure by an arbitrary measure μ as a way of measuring path 'lengths'.

In Sections 2 and 3, we generalize all the machinery needed to construct Newton–Sobolev spaces. In Section 4, we define these spaces and prove that they are complete. In Section 5, we call for some additional properties, such as Poincaré inequality, needed to prove some more interesting results, as Lipschitz density or Sobolev embeddings. We also compare Newton–Sobolev spaces with another kind of Sobolev space in metric spaces: Hajłasz–Sobolev spaces.

2. μ-arc length and upper gradients

Classical definitions of arc length, length function, arc length parametrization and line integrals in the metric setting can be found in [13]. In this section, we modify these concepts so they apply in more general ways to measure path 'lengths'.

Given a metric space (X,d) and a path $\gamma : [a,b] \to X$, i.e. a continuous function from $[a,b]$ into X, its length is defined as $l(\gamma) = \sup \sum_i d(\gamma(t_i), \gamma(t_{i+1}))$, where the supremum is taken over all partitions of $[a,b]$. We say that $\tilde{\gamma}$ is a sub-path of γ if it is the restriction of of γ to a subinterval of $[a,b]$. We say that a path (or subpath) is trivial if it is a constant path (for injective paths this means $a = b$).

The concept of arc length of a path is similar to, but not equal to, Hausdorff one-dimensional measure $\mathcal{H}^1$ of its image, however, they do coincide for injective paths (see [6]). From this result, for injective paths and for Borel nonnegative measurable functions, we get that

$$\int_\gamma g d\sigma = \int_{\mathrm{Im}(\gamma)} g d\mathcal{H}^1,$$

where $d\sigma$ is arc-length, and from this, we can think of exchanging the measure $\mathcal{H}^1$ for another Borel measure, as $\mathcal{H}^s$.

Let μ be a non-atomic Borel measure in X (in the sense that $\mu(\{x\}) = 0$ for each $x \in X$). Define Γ^μ as the set of all non trivial injective paths γ in X such that $0 < \mu(\mathrm{Im}(\tilde{\gamma})) < \infty$ for all non trivial subpaths of γ. For nonnegative Borel functions $g : X \to [0, \infty]$, we define

$$\int_\gamma g = \int_{\mathrm{Im}(\gamma)} g d\mu.$$

Now, for a path $\gamma : [a, b] \to X$ in Γ^μ, we define $h(\gamma) = \mu(\mathrm{Im}(\gamma))$ and its *μ-arc length* $\nu_\gamma : [a, b] \to \mathbb{R}$ as $\nu_\gamma(x) = h(\gamma|_{[a,x]})$.

The following result is immediate.

Lemma 2.1. *For paths $\gamma : [a, b] \to X$ in Γ^μ, we have that ν_γ is strictly increasing, continuous, onto $[0, h(\gamma)]$, and besides*

$$h(\gamma) = h(\gamma|_{[a,x]}) + h(\gamma|_{[x,b]}).$$

We can now recover an analogue to the arc length parametrization.

Theorem 2.2. *For $\gamma : [a, b] \to X$ in Γ^μ, there is a unique $\gamma_h : [0, h(\gamma)] \to X$ such that $\gamma = \gamma_h \circ \nu_\gamma$, which satisfies $\mathrm{Im}(\gamma) = \mathrm{Im}(\gamma_h)$ and $\nu_{(\gamma_h)}(t) = t$ in $[0, h(\gamma)]$ (therefore, $\gamma_h = \gamma_h \circ \nu_{\gamma_h}$). We call this the μ-arc length parametrization of γ.*

Proof. As $\nu_\gamma : [a, b] \to [0, h(\gamma)]$ is strictly increasing and onto, it is a bijection between $[a, b]$ and $[0, h(\gamma)]$, and we can define $\gamma_h = \gamma \circ \nu_\gamma^{-1}$. It is immediate that $\mathrm{Im}(\gamma) = \mathrm{Im}(\gamma_h)$, and

$$\nu_{(\gamma_h)}(t) = \mu(\gamma_h([0, t])) = \mu(\gamma(\nu_\gamma^{-1}([0, t]))) = \mu(\gamma([a, \nu_\gamma^{-1}(t)])) = \nu_\gamma(\nu_\gamma^{-1}(t)) = t. \quad \square$$

As a corollary, we obtain

Theorem 2.3. *If $\gamma : [0,h] \to X$ is a path in Γ^μ parametrized by μ-arc length, then, for every Borel set B of $[0,h]$, we have*

$$\mu(\gamma(B)) = |B|,$$

where $|B|$ is the 1-dimensional Lebesgue measure of B. Furthermore, if $g : X \to \mathbb{R}$ is nonnegative and Borel measurable, then, for each subpath $\tilde{\gamma} = \gamma|_{[a,b]}$, we have

$$\int_{\tilde{\gamma}} g = \int_a^b g \circ \tilde{\gamma}.$$

Finally, we get the same result as with rectifiable curves.

Theorem 2.4. *Given a function $f : X \to \mathbb{R}$ and a path $\gamma : [0,h] \to X$ in Γ^μ parametrized by μ-arc length, if there exists a Borel measurable nonnegative $\rho : X \to \mathbb{R}$ satisfying*

$$|f(\gamma(s)) - f(\gamma(t))| \le \int_{\gamma|_{[s,t]}} \rho < \infty$$

for every $0 \le s < t \le h$, then $f \circ \gamma : [0,h] \to \mathbb{R}$ is absolutely continuous.

Proof. Let $\epsilon > 0$. As $\rho \in L^1(\mathrm{Im}(\gamma), \mu)$, by absolute continuity of the integral, there exists $\delta > 0$ such that for every $E \subset \mathrm{Im}(\gamma)$ with $\mu(E) < \delta$ we have $\int_E \rho d\mu < \epsilon$. Then, if $0 \le a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \le h$ satisfy $\sum_i |b_i - a_i| < \delta$,

$$\mu(\cup_i \gamma([a_i, b_i])) = \sum_i \nu_\gamma(b_i) - \nu_\gamma(a_i) = \sum_i b_i - a_i < \delta,$$

and therefore

$$\sum_i |f \circ \gamma(b_i) - f \circ \gamma(a_i)| \le \sum_i \int_{\gamma|_{[a_i,b_i]}} \rho = \int_{\cup_i \gamma([a_i,b_i])} \rho d\mu < \epsilon. \qquad \square$$

Let now Γ^* be a subset of Γ^μ, closed under taking subpaths (i.e. if $\gamma \in \Gamma^*$ and $\tilde{\gamma}$ is a non-trivial subpath of γ, then $\tilde{\gamma} \in \Gamma^*$). A nonnegative Borel measurable function ρ satisfying

$$|f(x) - f(y)| \le \int_\gamma \rho$$

for every $\gamma \in \Gamma^*$ with endpoints x, y, for every pair of points x, y with $f(x), f(y)$ finite, is called a *μ-upper gradient* for f with respect to Γ^*. As Theorem 2.4 shows, if a function f has an upper gradient with respect to Γ^* that is integrable over each path in Γ^*, then it is absolutely continuous over every path in Γ^*.

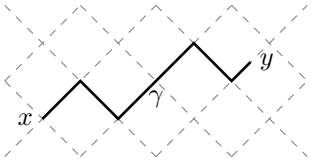
Example 2.5. Let $\mathbb{R}^2$ be equipped with the parabolic distance d discussed in the introduction, and let $\mu = \mathcal{H}^2$. If γ is a segment joining $x = (a, ka + b)$ with $y = (a+h, k(a+h)+b)$ for some $h > 0$, then its measure μ is just its height $|k|h$, while its length is $\sqrt{1+k^2}h$, so in fact we have $d\mu = \frac{k}{\sqrt{1+k^2}}dl$ over these paths (clearly, when $k \to 0$, we get $\mu = 0$, and when $k \to \infty$, $\mu = l$).

Now, for f smooth,

$$|f(y) - f(x)| \le \int_0^{\sqrt{1+k^2}h} \left|\nabla f\left(a + \frac{t}{\sqrt{1+k^2}}, b + \frac{t}{\sqrt{1+k^2}}\right)\right| dt$$
$$= \int_\gamma |\nabla f| ds = \frac{\sqrt{1+k^2}}{k} \int_{\mathrm{Im}(\gamma)} |\nabla f| d\mu,$$

and the same bound can be shown in a similar way for $h < 0$.

Therefore, if we consider Γ_k^* to be the set of all polygonal paths made up of segments of slope $\pm k$ for a fixed $0 < k < \infty$, we obtain that $\frac{\sqrt{1+k^2}}{k}|\nabla f|$ is an upper gradient for f with respect to Γ_k^*. The following picture illustrates a path of Γ_k^* for $k = 1$:



Example 2.6. Now, if we consider $X = \mathbb{R}^n$ with Euclidean distance, but $d\mu = \omega d\mathcal{H}^1$, where ω and $\frac{1}{\omega}$ are locally integrable with respect to $\mathcal{H}^1$, we obtain $\Gamma^\mu = \Gamma^{rect}$, where Γ^{rect} is the set of all non-trivial injective rectifiable paths. For f smooth and $\gamma \in \Gamma^\mu$,

$$|f(y) - f(x)| \le \int_\gamma |\nabla f| = \int_{\mathrm{Im}(\gamma)} |\nabla f| d\mathcal{H}^1 = \int_{\mathrm{Im}(\gamma)} \frac{|\nabla f|}{\omega} d\mu,$$

and in fact the same can be applied to any 'classical' upper gradient of a function f. There is clearly a one-to-one correspondence between upper gradients ρ with $\mathcal{H}^1$, and upper gradients of the form ρ/ω with measure μ.

3. Modulus of a path family and *p*-weak upper gradients

Let now m be a Borel measure on X. As in [16], we adjust the definition of modulus of a set of measures in [8] to path families.

For every family $\Gamma \subset \Gamma^\mu$ and $0 < p < \infty$, we define its *p-modulus* as

$$\mathrm{Mod}_p(\Gamma) = \inf \int_X g^p dm,$$

where the infimum is taken over all nonnegative Borel measurable functions $g : X \to \mathbb{R}$ satisfying $\int_\gamma g \geq 1$ for every $\gamma \in \Gamma$.

The following results can be found in [8], we state them here in the language of paths instead of measures.

Theorem 3.1. Mod_p *is an outer measure on* Γ^μ.

As expected, we say that a property holds for *p-almost every path* $\gamma \in \Gamma^\mu$ if the set Γ where it does not hold has $\mathrm{Mod}_p(\Gamma) = 0$. A useful property of sets of p-modulus zero is the following.

Lemma 3.2. $\mathrm{Mod}_p(\Gamma) = 0$ *if and only if there exists a nonnegative Borel measurable function* g *satisfying* $\int_X g^p dm < \infty$ *and* $\int_\gamma g = \infty$ *for every* $\gamma \in \Gamma$.

Given a set $E \subset X$, we define

$$\Gamma_E = \{\gamma \in \Gamma^\mu : \mathrm{Im}(\gamma) \cap E \neq \emptyset\}, \quad \Gamma_E^+ = \{\gamma \in \Gamma^\mu : \mu(\mathrm{Im}(\gamma) \cap E) > 0\},$$

and we have the following lemma:

Lemma 3.3. *If* $m(E) = 0$, *then* $\mathrm{Mod}_p(\Gamma_E^+) = 0$.

A nonnegative Borel measurable function ρ satisfying

$$|f(x) - f(y)| \leq \int_\gamma \rho$$

for p-almost every $\gamma \in \Gamma^\mu$ with endpoints x, y is called a *p-weak upper gradient* for f.

As in Shanmugalingam's case, we do not lose much by restricting ourselves to weak upper gradients.

Proposition 3.4. *If* ρ *is a p-weak upper gradient for* f *and* $\epsilon > 0$, *there exists an upper gradient* ρ_ϵ *for* f *such that* $\rho_\epsilon \geq \rho$ *and* $\|\rho - \rho_\epsilon\|_p < \epsilon$.

As seen in 2.4, functions with 'small' upper gradients are absolutely continuous on curves. We say that a function f is ACC_p or *absolutely continuous over p-almost every path* if $f \circ \gamma_h : [0, h(\gamma)] \to \mathbb{R}$ is absolutely continuous for p-almost every γ.

Lemma 3.5. *If a function f has a p-weak upper gradient $\rho \in L^p$, it is ACC_p.*

PROOF. Let Γ_0 be the set of all paths γ such that $|f(x) - f(y)| > \int_\gamma \rho$, and let Γ_1 be the set of all paths with a subpath in Γ_0. As ρ is a weak upper gradient, $\operatorname{Mod}_p(\Gamma_0) = 0$, but if g satisfies $\int_\gamma g = \infty$ for every $\gamma \in \Gamma_0$, it clearly satisfies $\int_\gamma g = \infty$ for every $\gamma \in \Gamma_1$, and therefore

$$\operatorname{Mod}_p(\Gamma_1) = 0.$$

Let Γ_2 be the set of all paths γ with $\int_\gamma \rho = \infty$. Then, as $\rho \in L^p$, $\operatorname{Mod}_p(\Gamma_2) = 0$. For paths not in $\Gamma_1 \cup \Gamma_2$, we can apply Theorem 2.4 and we conclude the lemma. □

We will also need the following lemma later on.

Lemma 3.6. *If f is ACC_p, and $f = 0$ m-almost everywhere, then the family*

$$\Gamma = \{\gamma \in \Gamma^\mu : f \circ \gamma \not\equiv 0\}$$

has p-modulus zero.

PROOF. Let $E = \{x : f(x) \neq 0\}$, then $m(E) = 0$ and $\Gamma = \Gamma_E$. As Γ_E^+ has modulus zero (because $m(E) = 0$), we only need to see that $\Gamma_E \backslash \Gamma_E^+$ also has modulus zero. But if $\gamma \in \Gamma_E \backslash \Gamma_E^+$, $\operatorname{Im}(\gamma) \cap E \neq \emptyset$ but $\mu(\operatorname{Im}(\gamma) \cap E) = 0$, therefore, $\gamma_h^{-1}(E)$ has length 0 in $\mathbb{R}$, and $f \circ \gamma_h$ is nonzero in a set of length 0, and if $E \neq \emptyset$, this set is not empty and $f \circ \gamma_h$ cannot be absolutely continuous. Therefore, $\operatorname{Mod}_p(\Gamma_E \backslash \Gamma_E^+) = 0$. □

4. Extended Newton–Sobolev spaces $N^{1,p}$

From now on, we will work on a fixed subset $\Gamma^* \subset \Gamma^\mu$, closed under taking subpaths. Properties defined on the previous section, such as p-weak upper gradients or ACC_p, can be easily adjusted to Γ^* instead of Γ^μ. We will also require that the space X be connected by paths belonging to Γ^*. This is the case of Γ_k^* in the example of $\mathbb{R}^2$ with the parabolic metric. For the Euclidean case, it is sufficient to consider piecewise linear paths made of segments parallel to the coordinate axis instead of all rectifiable paths in order to obtain a theory of Sobolev spaces, but in general this need not be the case.

We define the space $\tilde{N}^{1,p}$ as the space of all functions f having a p-weak upper gradient, both with finite p-norms. We define the $N^{1,p}$ norm as

$$\|f\|_{N^{1,p}} = \|f\|_p + \inf_\rho \|\rho\|_p,$$

where the infimum is taken over all p-weak upper gradients of f.

It immediately follows from definition that $(\tilde{N}^{1,p}, \|\cdot\|_{N^{1,p}})$ is a semi-normed vector space. Moreover, if $f, g \in \tilde{N}^{1,p}$, then

$$|f|, \min\{f, g\}, \max\{f, g\} \in \tilde{N}^{1,p}.$$

As seen before, every function in $\tilde{N}^{1,p}$ is ACC_p.

$\tilde{N}^{1,p}$ is not a normed space, as two distinct functions can be equal almost everywhere, but also because a function may be in $\tilde{N}^{1,p}$, while a function equal almost everywhere to it may not. We do have the following as a corollary of 3.6.

Corollary 4.1. *If $f, g \in \tilde{N}^{1,p}$ and $f = g$ m-a.e., then $\|f - g\|_{N^{1,p}} = 0$.*

Finally, we define the equivalence relation: $f \sim g$ iff $\|f-g\|_{N^{1,p}} = 0$, and the quotient space $N^{1,p} = \tilde{N}^{1,p}/\sim$, called as the *generalized Newton–Sobolev space.*

Example 4.2. In the case of $\mathbb{R}^2$ with the parabolic distance, $\mu = \mathcal{H}^2$ and $m = \mathcal{H}^3$ defined in the Introduction, we saw in Section 2 that if we consider Γ_k^* for a fixed $k > 0$ as our path family, we obtain that $\frac{\sqrt{1+k^2}}{k}|\nabla f|$ is an upper gradient for f. In fact, one can show that $N^{1,p} = W^{1,p}$ with equivalent norms. If we consider the whole of Γ^μ, this will not happen, as we can see by considering that if ρ is a bounded upper gradient for f, for paths γ joining (x, y) and (x', y'), we obtain

$$|f(x, y) - f(x', y')| \leq \int_\gamma \rho \leq \|\rho\|_\infty \mu(\mathrm{Im}(\gamma)).$$

So, for the case of segments in Γ_k^*, we would obtain

$$|f(x, y) - f(x', y')| \leq \int_\gamma \rho \leq \|\rho\|_\infty |y - y'|,$$

and if we allow segments of arbitrarily small height, we obtain that f must be cylindrical, $f(x, y) = g(y)$, so, unless $f \equiv 0$ or $p = \infty$, we cannot obtain $f \in L^p(dm)$. For $p = \infty$, we obtain that $N^{1,\infty}$ consists of cylindrical bounded functions $f(x, y) = g(y)$, with g a Lipschitz-1 function (in the Euclidean sense).

Example 4.3. Back to the example of $X = \mathbb{R}^n$ with Euclidean distance and $d\mu = \omega d\mathcal{H}^1$, where both ω and $\frac{1}{\omega}$ are locally integrable with respect to $\mathcal{H}^1$, so for the measure $dm = \omega^p dx$ (where dx is Lebesgue measure) we get that the space $N^{1,p}(dm)$ consists of those $f \in L^p(dm)$ such that $\frac{|\nabla f|}{\omega} \in L^p(dm)$, or in terms of Lebesgue measure,

$$N^{1,p} = \{f : \omega f, |\nabla f| \in L^p(dx)\}.$$

For the particular case $\omega(x) = 1 + |x|$, this space coincides with the Sobolev–Hermite space $\mathfrak{L}^p_1$, as defined in [2]. More generally, for the Schrödinger operator $L = -\Delta + V(x)$, if V satisfies a certain reverse Hölder condition, for the weight $\omega = V^{1/2}$, our Sobolev space $N^{1,2} = \{f : \omega f, |\nabla f| \in L^2(dx)\}$ is the appropriate space to work with the operator L (see, for instance, [17]).

As in the case of Shanmugalingam, the spaces $N^{1,p}$ turn out to be Banach. We state here the result, its proof is analogous to Theorem 3.7 in [16].

Theorem 4.4. *$N^{1,p}$ is Banach.*

5. Poincaré inequality

If there is no relationship between the 'space measure' m and the 'path measure' μ, most standard results about $N^{1,p}$ cannot be proven. The standard way of relating them is by Poincaré inequality. In our case, we will also need a relationship between the 'path measure' and the distance function.

We say that X supports a $(1,p)$-*Poincaré inequality* of exponent $\beta > 0$ if there exists $C > 0, \lambda \geq 1$ such that for every ball B and every pair f, ρ defined in B such that $f \in L^1(B)$, and an upper gradient of f in B, we have

$$\fint_B |f - f_B| dm \leq C \mathrm{diam}(B)^\beta \left(\fint_{\lambda B} \rho^p dm \right)^{1/p}.$$

The functions f satisfying the above inequality for some $\rho \geq 0$ in L^p belong to the space $P^{\beta,p}_\lambda(X)$ described in [12], where relationships with some generalized Poincaré inequality spaces and classical Newtonian spaces are proven.

In Shanmugalingam's case, this property suffices to prove that Lipschitz functions are dense in $N^{1,p}$. One crucial fact for proving this is that the length of a path is always greater than, or equal to the distance between any pair of points over the curve, but in our context this may not be the case. We say that the family Γ^* has the *μ-arc-chord property* with exponent $\beta \geq 1$ if there exists $C_\mu > 0$, such that for every $\gamma \in \Gamma^*$ (and thus for every subpath of that γ, as Γ^* is closed under taking subpaths), we get that

$$\mathrm{diam}(\mathrm{Im}(\gamma))^\beta \leq C_\mu \mu(\mathrm{Im}(\gamma)).$$

Observe that the usual chord-arc property (see, for instance, [4]) means the opposite inequality: $l(\gamma) \leq C d(x,y)$ if γ is a path joining x and y (which in turn

implies $l(\gamma) \sim d(x,y)$, as the reverse inequality $d(x,y) \le l(\gamma)$ always holds). We do not require this control over the measure of the curves in Γ^*, but the opposite one (thus we reverse the word order in the definition). Observe also that, as mentioned by one of the Referees, the μ-arc-chord property implies the Hausdorff dimension of each $\mathrm{Im}(\gamma)$, $\gamma \in \Gamma^*$ is at most β, so the case $0 < \beta < 1$ makes no sense, as in metric spaces there are no paths of dimension between 0 and 1.

In this section, we will prove some results that arise from these properties, and then we will go back to the example $d\mu = \omega d\mathcal{H}^1$.

First, we will prove a series of lemmas that will give us sufficient conditions for Lipschitz functions to be dense in $N^{1,p}$.

Lemma 5.1. *Let f be ACC_p such that $f|_F = 0$ m-a.e., for F a subset of X. If ρ is an upper gradient of f, then $\rho\chi_{X\setminus F}$ is a p-weak upper gradient of f.*

Proof. Let Γ_0 be the set of paths for which $f \circ \gamma_h$ is not absolutely continuous, and let $E = \{x \in F : f(x) \neq 0\}$, so $\mathrm{Mod}_p(\Gamma_0 \cup \Gamma_E^+) = 0$. Now, if $\gamma \notin \Gamma_0 \cup \Gamma_E^+$ has endpoints x, y,

- if $\mathrm{Im}(\gamma) \subset (X\setminus F)\cup E$, then $|f(x)–f(y)| \le \int_\gamma \rho = \int_\gamma \rho\chi_{X\setminus F}$ as $\mu(\mathrm{Im}(\gamma)\cap E)=0$.
- if $x, y \in F\setminus E$, then $f(x) = f(y) = 0$ and $|f(x) - f(y)| \le \int_\gamma \rho\chi_{X\setminus F}$ holds trivially.
- if $x \in (X\setminus F)\cup E$ (or the same for y) but $\mathrm{Im}(\gamma)$ is not completely in $(X\setminus F)\cup E$, as $(f \circ \gamma_h)^{-1}(\{0\})$ is a closed set of $[0, h(\gamma)]$ ($f \circ \gamma_h$ is continuous), it has a minimum a and maximum b (with $f \circ \gamma_h(a) = f \circ \gamma_h(b) = 0$). Then,

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f(\gamma_h(a))| + |f(\gamma_h(b)) - f(y)| \\ &\le \int_{\gamma_h|_{[0,a]}} \rho + \int_{\gamma_h|_{[b,h(\gamma)]}} \rho \le \int_\gamma \rho\chi_{X\setminus F}, \end{aligned}$$

  as $\gamma_h([0,a])$ and $\gamma_h([b,h(\gamma)])$ intersect F in a set of μ-measure zero. □

As immediate consequences of the μ-arc-chord property, we have the following results (Lemma 5.3 follows a pattern similar to the proof of Lemma 5.1).

Lemma 5.2. *If Γ^* has the μ-arc-chord property with exponent β, then every Lipschitz-β (also called Hölder-β) function is absolutely continuous over every curve of Γ^*.*

Lemma 5.3. *If Γ^* has the μ-arc-chord property with exponent β and $f : X \to \mathbb{R}$ is a Lipschitz-β function with constant L, then $C_\mu L\chi_{\mathrm{supp}(f)}$ is an upper gradient of f. In particular, if* $\mathrm{supp}(f)$ *is compact, we have $f \in \tilde{N}^{1,p}$.*

With the previous results, and also requiring the measure m to be doubling, we get the following.

Theorem 5.4. *If m is doubling, X supports a $(1,p)$-Poincaré inequality of exponent $\beta = 1$, and Γ^* satisfies the μ-arc-chord property with exponent also $\beta = 1$, then Lipschitz-1 functions are dense in $N^{1,p}$.*

Proof. Let $f \in \tilde{N}^{1,p}$, and let $g \in L^p$ be an upper gradient of f. Assume f is bounded (bounded functions are clearly dense in $N^{1,p}$, as a truncated version of a function in $N^{1,p}$ is still in $N^{1,p}$, and convergence of the truncated functions to the original one follows from Lemma 5.1). We define

$$E_k = \{x \in X : Mg^p(x) > k^p\},$$

where M is the noncentered Hardy–Littlewood maximal function. As m is doubling, M is weak type $1,1$, and $m(E_k) \le \frac{C}{k^p}\int_X g^p \to 0$ as $k \to \infty$. Let $F_k = X \backslash E_k$ (which is closed as E_k is open). If $x \in F_k$, $r > 0$ and $B = B(x,r)$,

$$\fint_B |f - f_B| \le Cr\left(\fint_B g^p\right)^{1/p} \le Cr(Mg^p(x))^{1/p} \le Crk.$$

Then if we take $f_n(x) = f_{B(x,2^{-n}r)}$, we have

$$\begin{aligned}|f_{n+j}(x) - f_n(x)| &\le \sum_{i=1}^{j} |f_{n+i+1}(x) - f_{n+i}(x)| \le C\sum_{i=1}^{j} \fint_{B(x,2^{-(n+i)}r)} |f - f_{B(x,2^{-(n+i)}r)}| \\ &\le Ckr2^{-n}\sum_{i=1}^{j} 2^{-i} \le Ckr2^{-n},\end{aligned}$$

and therefore $f_n(x)$ is Cauchy for each $x \in F_k$. Now, we define for $x \in F_k$, $f^k(x) = \lim f_n(x)$. Observe that for Lebesgue points of f in F_k, we have $f^k(x) = f(x)$. Let's verify that f^k is Lipschitz-β. Given $x, y \in F_k$, take $r = d(x,y)$, $B_n = B(x, 2^{-n}r)$, $B'_n = B(y, 2^{-n}r)$, then, as before,

$$\begin{aligned}|f^k(x) - f^k(y)| &\le \sum_{n=0}^{\infty} C\fint_{B_n} |f - f_{B_n}| + C\fint_{2B_0} |f - f_{2B_0}| + \sum_{n=0}^{\infty} C\fint_{B'_n} |f - f_{B'_n}| \\ &\le Ckr\sum_{n=0}^{\infty} 2^{-n} + Crk \le Ckr = Ckd(x,y).\end{aligned}$$

Now, f^k can be extended to all of X as a Lipschitz function with the same Lipschitz constant, and we can assume it is bounded by Ck (see [1]). Then

$$\int_X |f - f^k|^p = \int_{E_k} |f - f^k|^p \le C\int_{E_k} |f|^p + Ck^p m(E_k) \to 0$$

as $k \to \infty$, for $m(E_k) \to 0$ and the weak type of the Hardy–Littlewood maximal implies $k^p m(E_k) \to 0$.

So f^k tends to f in L^p. As f and f^k are ACC_p, $(g+\tilde{C}k)\chi_{E_k}$ is a p-weak upper gradient of $f - f^k$, and as it is in L^p and tends to 0 when $k \to \infty$, $f - f^k \in N^{1,p}$ for every k and $\|f - f^k\|_{N^{1,p}} \to 0$. □

If X is doubling and supports a $(1,q)$ Poincaré inequality of exponent $\beta > 0$ for some $1 \le q < p$, then we have that every function in $N^{1,p}$ has a Hajłasz gradient in L^p, i.e. $N^{1,p} \hookrightarrow M^{\beta,p}$ with $\|\cdot\|_{M^{\beta,p}} \le C\|\cdot\|_{N^{1,p}}$ (see [10], [15], [16], we define $M^{\beta,p}$ to be the space $M^{1,p}$ for the metric d^β). The converse embedding holds true in general for Shanmugalingam's case. In our case, we need the μ-arc-chord property.

Lemma 5.5. *Assume Γ^* satisfies the μ-arc-chord property with exponent β, and let f be a continuous function satisfying*

$$|f(x) - f(y)| \le d(x,y)^\beta (g(x) + g(y))$$

for every x, y, for some nonnegative measurable function g. Then there exists $C > 0$ such that Cg is an upper gradient for f.

Proof. Let $\gamma : [0,h] \to X$ be a path in Γ^*, parametrized by μ-arc length with endpoints x, y. If $\int_\gamma g = \infty$, we are done. Otherwise, for each n we take $\gamma_i = \gamma|_{[\frac{i}{n}, \frac{i+1}{n}]}$, $0 \le i \le n-1$, as γ is a μ-arc length parametrization, we have that $\mu(|\gamma_i|) = \mu(\mathrm{Im}(\gamma))/n = h/n$. For each i, there exists $x_i \in |\gamma_i|$ with $g(x_i) \le \fint_{\gamma_i} g$, and the μ-arc-chord property implies that $d(x_i, x_{i+1})^\beta \le C\mu(|\gamma_i|)$, then

$$\begin{aligned} |f(x_0) - f(x_{n-1})| &\le \sum_i |f(x_i) - f(x_{i+1})| \le \sum_i d(x_i, x_{i+1})^\beta (g(x_i) + g(x_{i+1})) \\ &\le C \sum_i \left(\int_{\gamma_i} g + \int_{\gamma_{i+1}} g \right) \le C \int_\gamma g. \end{aligned}$$

Taking $n \to \infty$, $x_0 \to x, x_{n-1} \to y$ and

$$|f(x) - f(y)| \le C \int_\gamma g,$$

and we have what we needed. □

Corollary 5.6. *If Γ^* satisfies the μ-arc-chord property with exponent β and continuous functions are dense in $M^{\beta,p}$ (which happens, for instance, if $\beta = 1$, see* [10]*), then $M^{\beta,p} \hookrightarrow N^{1,p}$, with $\|\cdot\|_{N_{1,p}} \le C\|\cdot\|_{M^{\beta,p}}$.*

As said before, functions that satisfy a $(1,q)$ Poincaré inequality of exponent $\beta > 0$ for some $1 \leq q < p$ are automatically in $M^{\beta,p}$ (see [15] for the case $\beta = 1$, the general case follows the same proof). Together with the previous corollary, we obtain the following theorem.

Theorem 5.7. *If X is doubling and supports a $(1,q)$ Poincaré inequality with exponent $\beta = 1$ for some $1 \leq q < p$, and Γ^* satisfies the μ-arc-chord property with the same exponent $\beta = 1$, then $M^{1,p} = N^{1,p}$, with equivalent norms.*

As in [16], we have the following versions of the classical Sobolev embedding theorems. In Shanmugalingam's case they are proven for $\beta = 1$, but the same proof can be applied for other β in our case.

Theorem 5.8. *If m is doubling and satisfies*

$$m(B(x,r)) \geq Cr^N$$

for C, N independent of $x \in X, 0 < r < 2\operatorname{diam}(X)$, and if X supports a $(1,p)$ Poincaré inequality of exponent $\beta \leq 1$ for $p > N/\beta$, then functions in $N^{1,p}$ are Lipschitz-α with $\alpha = \beta - N/p$.

Proof. Let g be an upper gradient of $f \in \tilde{N}^{1,p}$, that is in L^p. The telescoping argument as in the proof of Theorem 5.4, together with Poincaré inequality, give, for x, y Lebesgue points of f and $r = d(x,y)$, B_n, B'_n, f_n, as before,

$$\begin{aligned} |f(x) - f(y)| &\leq C \sum_{n=0}^{\infty} (2^{-n}r)^{\beta} \left(\fint_{2B_n} g^p \right)^{1/p} + C \sum_{n=0}^{\infty} (2^{-n}r)^{\beta} \left(\fint_{B'_n} g^p \right)^{1/p} \\ &\leq C \sum_{n=0}^{\infty} (2^{-n}r)^{\beta - N/p} \|g\|_p \leq C \|g\|_p d(x,y)^{\alpha}. \end{aligned}$$

Now, if L is the set of Lebesgue points of f, $m(X \backslash L) = 0$, then $f|_L$ is Lipschitz-α with $\alpha \leq 1$, so it can be extended to a Lipschitz-α function $\tilde{f}$ in all X. $\Gamma^+_{X \backslash L}$ has p-modulus zero, and the set Γ_0 of paths where f is not absolutely continuous has p-modulus zero. If $\gamma \notin (\Gamma^+_{X \backslash L} \cup \Gamma_0)$, as $\mu(\operatorname{Im}(\gamma) \cup X \backslash L) = 0$, taking γ_h its μ-arc-chord parametrization, we obtain that $\gamma_h^{-1}(X \backslash L)$ has measure zero as a subset of $\mathbb{R}$, and as $f \circ \gamma_h$ and $\tilde{f} \circ \gamma_h$ coincide outside $\gamma_h^{-1}(X \backslash L)$, and they are both continuous, they coincide in all $[0, h(\gamma)]$, so f and $\tilde{f}$ coincide in all of $\operatorname{Im}(\gamma)$. Finally, if $E = \{x \in X : f(x) \neq \tilde{f}(x)\}$, then Γ_E has p-modulus equal to zero, and therefore f and $\tilde{f}$ are the same function in $N^{1,p}$. □

Theorem 5.9. *If X is bounded (and thus $m(X) < \infty$) and satisfies*

$$cr^N \le m(B(x,r)) \le Cr^N$$

with c, C, N independent of $x \in X, 0 < r < 2\operatorname{diam}(X)$ (i.e. X is Ahlfors N-regular), and if X supports a $(1,q)$ Poincaré inequality of exponent β for $q > 1/\beta$, then for p satisfying $q < p < Nq$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{Nq}$ we have that every $f \in N^{1,p}$ with upper gradient g,

$$\|u - u_X\|_{p^*} \le C \operatorname{diam}(X)^{\beta - 1/q} \|g\|_p.$$

Proof. Let $f \in \tilde{N}^{1,p}$, and x be a Lebesgue point of f. Take $r = 2\operatorname{diam}(X)$, and the sequence of balls $B_n = B(x, 2^{-n}r)$. Then, by the telescoping argument as in the proof of Theorem 5.8,

$$\begin{aligned}
|f(x) - f_X| &\le C \sum_{n=0}^{\infty} (2^{-n}r)^{\beta} \left(\frac{C}{(2^{-n}r)^N} \int_{B_n} g^q \right)^{1/q} \\
&\le C \sum_{n=0}^{\infty} (2^{-n}r)^{\beta - 1/q} \left(\int_{B_n} \frac{g^q(z)}{d(x,z)^{N-1}} dm(z) \right)^{1/q} \\
&\le C \operatorname{diam}(X)^{\beta - 1/q} \left(\int_X \frac{g^q(z)}{d(x,z)^{N-1}} dm(z) \right)^{1/q}.
\end{aligned}$$

As for Ahlfors spaces, the Riesz potential $I_1(h)(x) = \int_X \frac{h(z)}{d(x,z)^{N-1}} dm(z)$ is bounded from L^s into L^{s^*} for $s^* = Ns/(N-s)$ and $1 < s < N$, and $g^q \in L^{p/q}$, we have that $\|I_1(g^q)\|_{s^*} \le C\|g^q\|_{p/q} = C\|g\|_p^q$, for $s = p/q$. Finally,

$$\| |f - f_X|^q \|_{s^*} \le C \left(\operatorname{diam}(X)\right)^{\beta q - 1} \|I_1(g^q)\|_{s^*} \le C \left(\operatorname{diam}(X)\right)^{\beta q - 1} \|g\|_p^q,$$

and as $qs^* = Nqp/(Nq - p) = p^*$, the theorem follows. □

Example 5.10. We finish this work with the example $X = \mathbb{R}^n$ with Euclidean distance, $d\mu = \omega d\mathcal{H}^1$, $dm = \omega^p dx$, where ω and $\frac{1}{\omega}$ are locally integrable. First, we consider when a Poincaré inequality holds.

If ω is bounded, as Poincaré inequality is true for dx, we get

$$\begin{aligned}
\fint_B |f - f_B| dm &\le 2 \fint_B |f - f_{B,dx}| dm \le 2 \left(\frac{|B|}{m(B)} \right)^{1/p} \left(\frac{1}{|B|} \int_B |f - f_{B,dx}|^p \omega^p dx \right)^{1/p} \\
&\le C \|\omega\|_{\infty} \operatorname{diam}(B) \left(\fint_B \left(\frac{|\nabla f|}{\omega} \right)^p dm \right)^{1/p},
\end{aligned}$$

where $f_{B,dx} = \fint_B f dx$.

Instead of asking for ω to be bounded, we may use a two-weight Poincaré inequality as found in [11]. Let $1 < p < n$, $\omega^p \in A_\infty$, and

$$\int_Q \omega^p dx \leq C|Q|^{q(\frac{1}{p}-\frac{1}{n})}$$

for each cube Q, with C independent of Q, and some q such that $p < q < \frac{np}{n-p}$. If $p = q = 2$, this would be Fefferman–Phong's condition (see [7]).

In our case, the pair $1, \omega^p$ satisfies condition $A_{p,q}^{1/n}$, where we say two weights w_1, w_2 satisfy condition $A_{p,q}^{\alpha}$ if there exists $C > 0$ such that

$$\left(\int_Q w_1^{-p'/p}\right)^{1/p'} \left(\int_Q w_2\right)^{1/q} \leq C|Q|^{1-\alpha}$$

for each cube Q, for $0 \leq \alpha < 1$, $1 < p, q < \infty$, $1/p - \alpha \leq 1/q$.

E. Harboure proves in [11] that these conditions imply that there exist constants $C > 0$ and $\delta > 0$ (depending on the A_∞ and $A_{p,q}^{1/n}$ constants) such that the following Poincaré inequality holds:

$$\int_Q |f - f_{Q,dx}|^p \omega^p dx \leq C \left(\int_Q \omega^p dx\right)^{\delta} \int_Q |\nabla f|^p dx.$$

From this condition, our $(1, p)$ Poincaré inequality follows,

$$\begin{aligned}
⨍_Q |f - f_Q| dm &\leq C \frac{1}{m(Q)^{1/p}} \left(\int_Q |f - f_{Q,dx}|^p \omega^p dx\right)^{1/p} \\
&\leq C \frac{1}{m(Q)^{1/p}} \left(m(Q)^{\delta} \int_Q \left(\frac{|\nabla f|}{\omega}\right)^p dm\right)^{1/p} \\
&\leq C \mathrm{diam}(Q)^{\beta} \left(⨍_Q \left(\frac{|\nabla f|}{\omega}\right)^p dm\right)^{1/p},
\end{aligned}$$

for $\beta = \frac{\delta q}{p}\left(\frac{n}{p} - 1\right)$, where the last inequality follows from the fact that, by our assumption, $m(Q) \leq C\mathrm{diam}(Q)^{q(n/p-1)}$.

As an example of such ω, we may consider $\omega(x) = |x|^{-\lambda}$, for some $0 \leq \lambda < 1$. Then, $\omega^p \in A_\infty$ if $p\lambda < n$ and the pair $1, \omega^p$ satisfies condition $A_{p,q}^{1/n}$ for $q = \frac{n-\lambda p}{n-p} p$.

As a special case, we can consider $\lambda = 0$, so the weight $\omega = 1$, which gives classical Sobolev spaces $W^{1,p}$, is included in our result.

This particular example of $\omega(x) = |x|^{-\lambda}$ can also be obtained using a result by S. Chanillo and R. Wheeden, see [3], and we get the β-Poincaré inequality of the kind we require, for $0 \leq \lambda < 1$ and $p > 1$, with $\beta = 1 - \lambda$.

With Poincaré inequality, Theorems 5.8 and 5.9 hold, provided the other conditions are met. We also obtain one half of Theorem 5.7, as a Poincaré inequality is sufficient to obtain $N^{1,p} \hookrightarrow M^{\beta,p}$.

If there exists $c > 0$ such that $\omega(x) \geq c$ for all x, we also get the arc-chord property,

$$\operatorname{diam}(\operatorname{Im}(\gamma)) \leq \mathcal{H}^1(\operatorname{Im}(\gamma)) \leq \frac{1}{c} \int_{\operatorname{Im}(\gamma)} \omega d\mathcal{H}^1 = \frac{1}{c} \mu(\operatorname{Im}(\gamma)).$$

As an example, the weight $\omega(x) = |x|^{-\lambda}$ satisfies this restriction in the usual case $\lambda = 0$, or if we take our space to be $X = Q_0$ for some fixed cube Q_0 (here we consider only cubes $Q \subset Q_0$ that may contain the origin, so ω is not necessarily bounded), and satisfies the $A_{p,q}^{1/n}$ condition restricted to cubes in Q_0. This case allows for both a Poincaré inequality and an arc-chord property, even though the exponents in each case may not coincide.

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