Publ. Math. Debrecen
90/1-2 (2017), 251–268
DOI: 10.5486/PMD.2017.7712

An observation on Asanov's Unicorn metrics

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Abstract. Finsleroid–Finsler metrics form an important class of singular (y-local) Finslerian metrics. They were introduced by G. S. Asanov in 2006. As a special case, Asanov produced examples of Landsberg spaces of dimension at least three that are not of Berwald type. These are called Unicorns [5]. The existence of regular (y-global) Landsberg metrics that are not of Berwald type is an open problem up to this day. In this paper, we prove that Asanov's Unicorns belong to the class of generalized Berwald manifolds. More precisely, we prove the following theorems: a Finsleroid–Finsler space is a generalized Berwald space if and only if the Finsleroid charge is constant. Especially, a Finsleroid–Finsler space is a Landsberg space if and only if it is a generalized Berwald manifold with a semi-symmetric compatible linear connection.

Introduction

Finsleroid–Finsler metrics were introduced by G. S. Asanov in 2006. As a special case, Asanov produced singular (y-local) examples of Landsberg spaces of dimension at least three that are not of Berwald type. The existence of regular (y-global) Landsberg metrics that are not of Berwald type is an open problem up to this day, see D. Bao [5]. The cronology of the basic steps:

- 1998 – the central symmetric version of the Finsleroid–Finsler metric in Asanov [3].
- 2003 – the non-symmetric version of the Finsleroid–Finsler metric in Vincze [15] as *Asanov-type Finslerian metric functions.*

Mathematics Subject Classification: 53C60.
Key words and phrases: Finsler spaces, conformality, Finsleroid–Finsler metrics, generalized Berwald manifolds.
Supported by the University of Debrecen's internal research project RH/885/2013.

- 2006 – Asanov's examples for (non-symmetric) Finsleroid–Finsler metrics that are of Landsberg but not of Berwald type in Asanov [4];
- 2016 – non-symmetric Finsleroid–Finsler metrics with closed Finsleroid axis 1-forms as the solutions of a conformal rigidity problem, Vincze [26].

In this paper, we prove that Asanov's Unicorns belong to the class of generalized Berwald manifolds. More precisely, we prove the following theorems: a Finsleroid–Finsler space is a generalized Berwald space if and only if the Finsleroid charge is constant. Especially, a Finsleroid–Finsler space is a Landsberg space if and only if it is a generalized Berwald manifold with a semi-symmetric compatible linear connection.

1. Notations and terminology

Let M be a manifold with local coordinates $u^1, \dots, u^n$. The induced coordinate system of the tangent manifold TM consists of the functions $x^1, \dots, x^n$ and $y^1, \dots, y^n$, where x's refer to the coordinates of the base point, and y's denote the coordinates of the direction. A *Finslerian metric* is a continuous function $F\colon TM \to \mathbb{R}$ satisfying the following conditions: F is smooth on the complement of the zero section (regularity), $F(tv) = tF(v)$ for all $t > 0$ (positive homogeneity), and the Hessian

$$g_{ij} = \frac{\partial^2 E}{\partial y^i \partial y^j} \quad \text{of the Finslerian energy} \quad E = \frac{1}{2}F^2$$

is positive definite at all nonzero elements $v \in TM$ (strong convexity). It is called the Riemann–Finsler metric of the Finsler manifold. The Riemann–Finsler metric makes each tangent space (except at the origin) a Riemannian manifold with standard canonical objects such as the *volume form*

$$d\mu = \sqrt{\det g_{ij}}\; dy^1 \wedge \dots \wedge dy^n,$$

the *Liouville vector field*

$$C := y^1 \partial/\partial y^1 + \dots + y^n \partial/\partial y^n,$$

and the *induced volume form on the indicatrix hypersurface*

$$\partial K_p := F^{-1}(1) \cap T_pM \quad (p \in M).$$

The coordinate expression is

$$\mu = \sqrt{\det g_{ij}} \sum_{i=1}^{n} (-1)^{i-1} \frac{y^i}{F} dy^1 \wedge \cdots \wedge dy^{i-1} \wedge dy^{i+1} \cdots \wedge dy^n.$$

As a general reference of Finsler geometry, see [6]; we will use the following notations and terminology:

$$l_i = \frac{\partial F}{\partial y^i}, \quad g^{ij} = (g_{ij})^{-1}, \quad C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}, \quad \text{where} \quad \mathcal{C}^l_{ij} = g^{lk} \mathcal{C}_{ijk}$$

is the so-called *first Cartan tensor*. The first Cartan tensor is totally symmetric and $y^k \mathcal{C}_{ijk} = 0$. The *geodesic spray coefficients* and the *horizontal sections* are given by

$$G^l = \frac{1}{2} g^{lm} \left(y^k \frac{\partial^2 E}{\partial y^m \partial x^k} - \frac{\partial E}{\partial x^m} \right) \quad \text{and} \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G^l_i \frac{\partial}{\partial y^l}, \tag{1}$$

where

$$G^l_i = \frac{\partial G^l}{\partial y^i}.$$

The *second Cartan tensor* or *Landsberg tensor* is

$$P^l_{ij} = \frac{1}{2} g^{lm} \left(\frac{\partial g_{jm}}{\partial x^i} - 2G^k_i \mathcal{C}_{jkm} - G^k_{ij} g_{km} - G^k_{im} g_{jk} \right), \quad \text{where} \quad G^l_{ij} = \frac{\partial G^l_i}{\partial y^j}.$$

The *mixed curvature of the Berwald connection* is defined as

$$P^l_{ijk} = -G^l_{ijk}, \quad \text{where} \quad G^l_{ijk} = \frac{\partial G^l_{ij}}{\partial y^k}.$$

By some direct computations, we get the identity

$$P^l_{ij} = -\frac{F}{2} l_m g^{kl} P^m_{ijk}. \tag{2}$$

Definition 1. Let M be a Finsler manifold; a linear connection ∇ on the base manifold M is called *compatible* to the Finslerian metric if the parallel transports with respect to ∇ preserve the Finslerian length of tangent vectors. Finsler manifolds admitting compatible linear connections are called generalized Berwald manifolds. Berwald manifolds are generalized Berwald manifolds with torsion-free compatible linear connections.

Definition 2. A Finsler manifold is a Landsberg manifold if the Landsberg tensor vanishes.

The notion of generalized Berwald manifolds goes back to V. Wagner [27]. The basic questions of the theory are the unicity of the compatible linear connection and its expression in terms of the canonical data of the Finsler manifold (intrinsic characterization). In the case of a classical Berwald manifold (compatible linear connection with zero torsion), the intrinsic characterization is the vanishing of the mixed curvature tensor of the Berwald connection. This means that the quantities G^l_{ij}'s depend only on the position. They constitute the coefficients of the compatible linear connection on the base manifold. In general, the intrinsic characterization of the compatible linear connection is based on the so-called *averaged Riemannian metric*

$$\gamma_p(v,w) := \int_{\partial K_p} g(v,w)\,\mu. \tag{3}$$

Using average processes is a new and important trend with a rapidly increasing number of papers in Finsler geometry: R. G. Torromé [14], T. Aikou [2], M. Crampin [7] and [8], V. S. Matveev *et al.* [9], [10] and [11], Cs. Vincze [16], [17], [22] and [24]. For further references, see also [20], [21] and [25].

Theorem 1 ([17])**.** *If a linear connection on the base manifold is compatible with the Finslerian metric function, then it must be metrical with respect to the averaged Riemannian metric γ.*

It is well known that a metric connection is uniquely determined by the torsion tensor. Following Agricola–Friedrich [1], consider the decomposition

$$T(X,Y) := T_1(X,Y) + T_2(X,Y),$$

where

$$T_1(X,Y) := T(X,Y) - \frac{1}{n-1}\big(\tilde{T}(X)Y - \tilde{T}(Y)X\big),$$

$\tilde{T}$ is the trace tensor of the torsion, and

$$T_2(X,Y) := \frac{1}{n-1}\left(\tilde{T}(X)Y - \tilde{T}(Y)X\right).$$

Note that the torsion tensor of a metric linear connection on a manifold of dimension 2 is automatically of the form $T = \tilde{T}(X)Y - \tilde{T}(Y)X$ (cf. Definition 3). Otherwise, the trace-less part can be divided into two further components

$$T_1(X,Y) = A_1(X,Y)+S_1(X,Y) \;\Rightarrow\; T(X,Y) = A_1(X,Y)+S_1(X,Y)+T_2(X,Y),$$

by separating the totally anti-symmetric/axial part A_1. Therefore, we have eight possible classes of generalized Berwald manifolds depending on the surviving terms, such as classical Berwald manifolds admitting torsion-free compatible linear connections Szabó [12] (we have no surviving terms) or Finsler manifolds admitting semi-symmetric compatible linear connections Vincze [22].

Definition 3. A linear connection is said to be semi-symmetric if the torsion tensor is of the form

$$T(X,Y) = \lambda(Y)X - \lambda(X)Y, \tag{4}$$

where λ is a one-form on the manifold.

The problem of the intrinsic characterization of compatible semi-symmetric linear connections is completely solved [22]: it can be expressed in terms of metrics and differential forms given by averaging. For the special case of exact (or at least closed) differential forms in the torsion (4), see [16].

Theorem 2 ([22])**.** *A non-Riemannian Finsler manifold is a generalized Berwald manifold admitting a semi-symmetric compatible linear connection if and only if $\sigma(p) > 0$ for any $p \in M$, and the linear connection*

$$\bar{\nabla}_X Y = \nabla^*_X Y + \frac{1}{2\sigma}\left(\eta^*(Y)X - \gamma(X,Y)\eta^{*\sharp}\right)$$

is compatible with the Finslerian metric function, where E^* and ∇^* are the Riemannian energy and the Lévi–Civita connection of the averaged Riemannian metric, h^* and S^* are the horizontal endomorphism and the canonical spray associated with ∇^*; see formula (1) *with substitution of the Riemannian energy function E^*,*

$$\rho^* := \frac{d_{h^*}E}{E} - \frac{1}{2}\frac{S^*E}{E}\frac{d_J E^*}{E^*} \quad \text{and} \quad f := \log\frac{E^*}{E},$$

where

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i} \quad \text{and} \quad J\left(\frac{\partial}{\partial y^i}\right) = 0$$

is the canonical vertical endomorphism/almost tangent structure on the tangent manifold. Furthermore,

$$\eta^*(X_p) := \int_{\partial K^*_p} d_J\rho^*(\Theta, X^c) + \frac{1}{2}\frac{S^*E}{E}X^v f\,\mu^* \quad \text{and} \quad \sigma(p) := \int_{\partial K^*_p}\frac{1}{2E^*}\|J\Theta\|^2\,\mu^*,$$

where X^c and X^v denote the complete and the vertical lifts of the vector field X, respectively. The vector field $J\Theta$ is defined by the formula

$$g^*(J\Theta, X^v) = E^* d_J f(X^c),$$

where g^* is the Riemann–Finsler metric of E^* as a special Finslerian energy, the norm of $J\Theta$ is also taken with respect to g^*, and μ^* is the induced volume form on the indicatrix hypersurface ∂K_p^* as a Riemannian submanifold in T_pM.

Remark 1. Note that $J\Theta$ is enough to be defined to compute the integrand, because of

$$d_J\rho^*(\Omega_1, \Omega_2) := d\rho^*(J\Omega_1, J\Omega_2),$$

where Ω_1 and Ω_2 are vector fields on the tangent manifold.

1.1. Randers metrics, (α, β)-metrics, the sign property. The complete solution of the intrinsic characterization is also given in the special case of Randers manifolds, without any special requirement for the torsion tensor. Let (M, α) be a connected Riemannian manifold, and suppose that the one-form β in $\wedge^1(M)$ satisfies condition

$$\sup_{\alpha(v,v)=1} \beta(v) < 1. \tag{5}$$

The Randers metric on the manifold M is defined as $F(v) = \sqrt{\alpha(v, v)} + \beta(v)$.

Theorem 3 ([23])**.** *A Randers manifold is a generalized Berwald manifold if and only if there exists a linear connection ∇ such that $\nabla\alpha = 0$ and $\nabla\beta = 0$.*

The following theorem formulates a necessary and sufficient condition for a Randers manifold to be a generalized Berwald manifold in terms of the dual vector field $\alpha(\beta^\sharp, X) = \beta(X)$.

Theorem 4 ([23])**.** *A Randers manifold is a generalized Berwald manifold if and only if $\beta^\sharp$ is of constant Riemannian length.*

The compatible linear connection is given as

$$\nabla_X Y = \nabla^*_X Y + \frac{\alpha(\nabla^*_X \beta^\sharp, Y)\beta^\sharp - \alpha(Y, \beta^\sharp)\nabla^*_X \beta^\sharp}{\alpha(\beta^\sharp, \beta^\sharp)}. \tag{6}$$

If the compatible linear connection is semi-symmetric, then we also have a structure theorem for the Riemannian manifold admitting a perturbation β such that the Randers manifold is a generalized Berwald manifold with a semi-symmetric compatible linear connection [23]. By the main result in [23], the manifold carries

a warped product metric structure; for the special case of exact (or at least closed) differential forms in the torsion (4), see VINCZE [18], and also VINCZE [19]. These results have been generalized by TAYEBI and BARZEGARI in [13] for (α, β)-metrics satisfying the *sign property*

$$\Phi'(-s)\Phi(s) + \Phi(-s)\Phi'(s) > 0 \quad \text{or} \quad \Phi'(-s)\Phi(s) + \Phi(-s)\Phi'(s) < 0, \tag{7}$$

where $F = \alpha\Phi\left(\frac{\beta}{\alpha}\right)$ is a Finslerian metric function, and $\Phi\colon (-b_0, b_0) \to \mathbb{R}^+$. According to the positivity of Φ the sign property, (7) is equivalent to

$$\varphi'(-s)\varphi(s) + \varphi(-s)\varphi'(s) > 0 \quad \text{or} \quad \varphi'(-s)\varphi(s) + \varphi(-s)\varphi'(s) < 0, \tag{8}$$

where $\varphi = \Phi^2$.

Theorem 5 ([13]). *A Finsler manifold with an (α, β)-metric satisfying the sign property is a generalized Berwald manifold if and only if there exists a linear connection ∇ such that $\nabla\alpha = 0$ and $\nabla\beta = 0$.*

Theorem 6 ([13]). *A Finsler manifold with an (α, β)-metric satisfying the sign property is a generalized Berwald manifold if and only if $\beta^\sharp$ is of constant Riemannian length.*

2. Asanov's Finsleroid–Finsler metrics

Using ASANOV's original notations in [4], the general form of Finsleroid–Finsler metrics is given by

$$F = e^{G\Phi/2}\sqrt{b^2 + gqb + q^2}, \tag{9}$$

where $b = b_i y^i$ is the Finsleroid axis 1-form, $q = \sqrt{r_{ij}y^i y^j}$, $r_{ij} = a_{ij} - b_i b_j$, and a_{ij} is a Riemannian metric such that $a^{ij}b_i b_j = 1$,

$$g = g(p) \quad \text{and} \quad -2 < g(p) < 2 \quad \text{(the Finsleroid charge)},$$

$$h = \sqrt{1 - \frac{g^2}{4}}, \quad G = g/h, \quad \Phi = \begin{cases} +\frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\frac{q + \frac{g}{2}b}{hb} & \text{if } b > 0 \\ -\frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\frac{q + \frac{g}{2}b}{hb} & \text{if } b < 0. \end{cases}$$

The common limit of the right hand sides as $b \to 0$ is $\arctan\dfrac{G}{2}$.

2.1. An alternative formulation. In what follows, we present the metric in a more compact form. If

$$f(b) := \arctan \frac{q + \frac{g}{2}b}{hb},$$

then

$$\lim_{b\to\pm\infty} f(b) = \arctan \frac{g}{2h} = \arctan \frac{G}{2} \quad \text{and} \quad f'(b) = -\frac{qh}{b^2 + q^2 + bgq}, \tag{10}$$

because $h^2 + g^2/4 = 1$. In a similar way, if

$$\tilde{f}(b) := \arctan \frac{2b + gq}{2hq},$$

then

$$\lim_{b\to\pm\infty} \tilde{f}(b) = \pm\frac{\pi}{2} \quad \text{and} \quad \tilde{f}'(b) = \frac{qh}{b^2 + q^2 + bgq}. \tag{11}$$

Therefore, $f + \tilde{f}$ is constant on the connected parts of the domain. Taking the limits $b \to \infty$ and $b \to -\infty$, respectively, we have

$$\arctan \frac{q + \frac{g}{2}b}{hb} + \arctan \frac{2b + gq}{2hq} = \begin{cases} +\frac{\pi}{2} + \arctan \frac{G}{2} & \text{if } \ b > 0 \\ -\frac{\pi}{2} + \arctan \frac{G}{2} & \text{if } \ b < 0. \end{cases} \quad \Rightarrow$$

$$\Phi = \arctan \frac{2b + gq}{2hq}. \tag{12}$$

Using the notations $\alpha = \sqrt{a_{ij}y^i y^j}$, $\beta = b_i y^i$ (Finsleroid axis 1-form) and $g = \dfrac{K}{2}$ (Finsleroid charge), it follows that

$$q = \sqrt{\alpha^2 - \beta^2}, \quad h := \sqrt{1 - \frac{g^2}{4}} = \sqrt{1 - \frac{K^2}{16}} = \frac{\sqrt{16 - K^2}}{4},$$

and

$$G = g/h = \frac{2K}{\sqrt{16 - K^2}}.$$

Definition 4. The Finslerian energy $E = (1/2)F^2$ of a Finsleroid–Finsler metric is

$$E = \frac{1}{2}\left(\alpha^2 + \frac{K}{2}\beta\sqrt{\alpha^2 - \beta^2}\right) e^{\frac{2K}{\sqrt{16-K^2}} \arctan \frac{1}{\sqrt{16-K^2}}\left(\frac{4\beta}{\sqrt{\alpha^2-\beta^2}} + K\right)}. \tag{13}$$

The metric (13) is formally an (α, β)-metric

$$E = \frac{1}{2}\alpha^2 \varphi\left(\frac{\beta}{\alpha}\right), \tag{14}$$

where

$$\varphi(s) = \left(1 + \frac{K}{2} s\sqrt{1-s^2}\right) e^{\frac{2K}{\sqrt{16-K^2}} \arctan \frac{1}{\sqrt{16-K^2}}\left(\frac{4s}{\sqrt{1-s^2}}+K\right)}, \tag{15}$$

$s \in [-1, 1]$, and the value at ± 1 is defined by the continuous extension

$$\varphi(\pm 1) = \lim_{s\to\pm 1} \varphi(s) = e^{\pm\frac{2K}{\sqrt{16-K^2}}\frac{\pi}{2}} \quad \Rightarrow \quad E(\pm\beta^\sharp) = \frac{1}{2} e^{\pm\frac{2K}{\sqrt{16-K^2}}\frac{\pi}{2}}. \tag{16}$$

Actually, (14) represents a more general form of metrics, because φ depends on the position by the Finsleroid charge $K/2$, too. In the case of a standard (α, β)-metric, φ is a function of the single variable s.

Lemma 1. *The function φ is of class $\mathcal{C}^1$ with respect to the variable s.*

Proof. Fix a point $p \in M$; in what follows, we prove that φ is of class $\mathcal{C}^1$ at ± 1. We discuss the case of $s = 1$ in details. The case of $s = -1$ is similar. By definition,

$$\varphi'(1) = \lim_{s\to 1} \frac{\varphi(s) - \varphi(1)}{s - 1}.$$

For any fixed $s < 1$, we can use the Lagrange mean value theorem

$$\varphi(s) - \varphi(1) = \varphi'(t)(s - 1),$$

because φ is continuous on the closed interval $[s, 1]$ (see formula (16) of the continuous extension), and differentiable on $]s, 1[$. Therefore,

$$\varphi'(1) = \lim_{t\to 1} \varphi'(t) = 0 \tag{17}$$

as a simple calculation shows:

```
> f:=s->(1+K*s*sqrt(1-s^2)/2)*
>    exp((2*K/sqrt(16-K^2))*tan^(-1)((1/sqrt(16-K^2))*
>    ((4*s/sqrt(1-s^2))+K)));
> g:=s->diff(f(s),s);
> simplify(g(s));
> sqrt(1-s^2)*K*
>    exp((2*K/sqrt(16-K^2))*tan^(-1)((1/sqrt(16-K^2))*
>    ((4*s/sqrt(1-s^2))+K)));
```

i.e.

$$\varphi'(s) = K\sqrt{1-s^2}e^{\frac{2K}{\sqrt{16-K^2}}\arctan\frac{1}{\sqrt{16-K^2}}\left(\frac{4s}{\sqrt{1-s^2}}+K\right)}. \tag{18}$$

□

Theorem 7. *The function φ satisfies the sign property*

$$a(s) := \varphi'(-s)\varphi(s) + \varphi(-s)\varphi(-s) = 2K\sqrt{1-s^2}e^{2A(-s)+2A(s)} \begin{cases} > 0 \text{ if } K > 0, \\ < 0 \text{ if } K < 0, \end{cases}$$

where

$$A(s) := \frac{K}{\sqrt{16-K^2}}\arctan\frac{1}{\sqrt{16-K^2}}\left(\frac{4s}{\sqrt{1-s^2}}+K\right).$$

Proof. The proof is a straightforward calculation:

```
> h:=s->sqrt(1-s^2)*K*
>    exp((2*K/sqrt(16-K^2))*tan^(-1)((1/sqrt(16-K^2))*
>    ((4*s/sqrt(1-s^2))+K)));
> a:=s->f(-s)*h(s)+f(s)*h(-s);
> simplify(a(s));
> 2*sqrt(1-s^2)*K*
>    exp((2*K/sqrt(16-K^2))*(tan^(-1)((1/sqrt(16-K^2))*
>    ((4*s/sqrt(1-s^2))+K)))*
>    exp((2*K/sqrt(16-K^2))*tan^(-1)((1/sqrt(16-K^2))*
>    ((-4*s/sqrt(1-s^2))+K)));
```

The worksheet is the continuation of the previous one. □

Theorem 8 ([4])**.** *A Finsleroid–Finsler space is a Landsberg space if and only if the function K is constant and*

$$(\nabla^*\beta)(X,Y) = \frac{\operatorname{div}\beta^\sharp}{n-1}\left(\alpha(X,Y) - \beta(X)\beta(Y)\right). \tag{19}$$

Remark 2. The original formulation of Theorem 8 in [4, Theorem 3, p. 278] is that a Finsleroid–Finsler space is a Landsberg space if and only if the Finsleroid axis 1-form β is closed, the Finsleroid charge is constant, and

$$(\nabla^*\beta)(X,Y) = k\left(\alpha(X,Y) - \beta(X)\beta(Y)\right) \tag{20}$$

for some scalar field $k\colon M \to \mathbb{R}$. Note that if (20) holds, then the closedness of β is redundant because

$$d\beta(X,Y) = \nabla^*\beta(X,Y) - \nabla^*\beta(Y,X) \overset{(20)}{=} 0.$$

On the other hand, condition (20) is obviously equivalent to

$$\alpha(\nabla^*_X\beta^\sharp, Y) = k\,(\alpha(X,Y) - \beta(X)\beta(Y)) \quad \Rightarrow \quad k = \frac{\operatorname{div}\beta^\sharp}{n-1},$$

because of the unit length of $\beta^\sharp$ with respect to α.

3. The main results

Theorem 9. *A connected Finsleroid–Finsler space is a generalized Berwald space if and only if the Finsleroid charge is constant.*

Proof. Suppose that there exists a linear connection such that it is compatible to the Finsleroid–Finsler metric. Since the parallel transports preserve the Finslerian norm of tangent vectors and they are linear between the tangent spaces, it follows that they preserve the Riemann–Finsler metric g_{ij}, and the indicatrices with the induced Riemannian metric are isometric. Asanov's Finsleroid–Finsler metric has indicatrices of constant curvature

$$1 - \frac{K^2(p)}{4};$$

see [4, formula (2.32)]. Therefore, the Finsleroid charge K must be constant on a connected manifold. Conversely, suppose that the function K is constant. Then the Finsleroid–Finsler metric is an (α,β)-metric of the form

$$F = \alpha\Phi\left(\frac{\beta}{\alpha}\right), \quad \text{where} \quad \Phi^2 = \varphi,$$

because φ does not depend on the position; see formula (14) and Definition 4. We have two possible cases:

- If $K = 0$, then the space is Riemannian as a special generalized Berwald manifold.
- If $K \neq 0$, then Theorem 7 implies that the function φ satisfies the sign property (8), and, by Theorem 6, it is a generalized Berwald manifold, because the Finsleroid axis 1-form β is of constant (unit) length with respect to α. □

Theorem 10. *A connected Finsleroid–Finsler space is a Landsberg space if and only if it is a generalized Berwald space with a semi-symmetric compatible linear connection such that the torsion tensor is of the form*

$$T(X,Y) = \lambda(Y)X - \lambda(X)Y, \quad \textit{where} \quad \lambda = -\frac{\operatorname{div}\beta^\sharp}{n-1}\beta.$$

Proof. Suppose that a Finsleroid–Finsler space is a Landsberg space. Then, by Theorem 8, we have that the Finsleroid charge K is constant, i.e. the space is a generalized Berwald space in the sense of Theorem 9. By formula (6), the compatible linear connection is

$$\begin{aligned}\nabla_X Y &= \nabla^*_X Y + \frac{\alpha(\nabla^*_X\beta^\sharp, Y)\beta^\sharp - \alpha(Y,\beta^\sharp)\nabla^*_X\beta^\sharp}{\alpha(\beta^\sharp,\beta^\sharp)}\\ &= \nabla^*_X Y + \alpha(\nabla^*_X\beta^\sharp, Y)\beta^\sharp - \alpha(Y,\beta^\sharp)\nabla^*_X\beta^\sharp,\end{aligned}$$

because $\beta^\sharp$ is of unit length with respect to α. On the other hand,

$$\begin{aligned}\alpha(\nabla^*_X\beta^\sharp, Y) = (\nabla^*\beta)(X,Y) &\overset{(19)}{=} \frac{\operatorname{div}\beta^\sharp}{n-1}\left(\alpha(X,Y) - \beta(X)\beta(Y)\right) \quad \Rightarrow\\ \nabla^*_X\beta^\sharp &= \frac{\operatorname{div}\beta^\sharp}{n-1}\left(X - \beta(X)\beta^\sharp\right)\end{aligned}$$

and, consequently,

$$\nabla_X Y = \nabla^*_X Y - \frac{\operatorname{div}\beta^\sharp}{n-1}\beta(Y)X + \frac{\operatorname{div}\beta^\sharp}{n-1}\alpha(X,Y)\beta^\sharp, \tag{21}$$

i.e.

$$\nabla_X Y = \nabla^*_X Y + \lambda(Y)X - \alpha(X,Y)\lambda^\sharp, \quad \text{where} \quad \lambda = -\frac{\operatorname{div}\beta^\sharp}{n-1}\beta. \tag{22}$$

Formula (22) determines the only metric linear connection with torsion

$$T(X,Y) = \lambda(Y)X - \lambda(X)Y.$$

Conversely, suppose that we have a Finsleroid–Finsler space such that it is a generalized Berwald manifold with ∇ in formula (21) as a compatible linear connection. Then, by Theorem 9, the Finsleroid charge is constant, and we have an (α,β)-metric. We have two possible cases:

- If $K = 0$, then the space is Riemannian as a special Landsberg manifold.
- If $K \neq 0$, then Theorem 7 implies that the function φ satisfies the sign property (8), and, by Theorem 5, $\nabla\beta = 0$. This implies the special form (19) of $\nabla^*\beta$, and the statement follows by Theorem 8. □

4. Appendix: regularity properties of Finsleroid–Finsler metrics

Finsleroid–Finsler metrics belong to the class of y-local Finslerian metrics, because the third order partial derivatives with respect to the variables y's are singular at $\pm\beta^\sharp$. In what follows, we prove that the partial derivatives with respect to the variables y's exist, and continuous up to order 2, i.e. Finsleroid–Finsler metrics are of class C^2 on the complement of the zero section.

4.1. The first y-derivatives of a Finsleroid–Finsler energy function. By Lemma 1, the function φ is of class C^1. Therefore,

$$\frac{\partial E}{\partial y^i} \overset{(14)}{=} \frac{1}{2}\left(2y^i\varphi\left(\frac{\beta}{\alpha}\right)+\alpha^2\varphi'\left(\frac{\beta}{\alpha}\right)\frac{\partial\beta/\alpha}{\partial y^i}\right)$$
$$= \frac{1}{2}\left(2y^i\varphi\left(\frac{\beta}{\alpha}\right)+\alpha^2\varphi'\left(\frac{\beta}{\alpha}\right)\frac{b_i\alpha-(\beta/\alpha)y^i}{\alpha^2}\right) \quad (23)$$

is of class $\mathcal{C}^1$ on the complement of the zero section. For the sake of simplicity, we can use an orthonormal coordinate system

$$y^1,\ldots,y^n \quad \text{such that} \quad y^n=\beta \text{ i.e. } b_i=\delta^n_i \quad \text{and} \quad \alpha^2=(y^1)^2+\cdots+(y^n)^2. \quad (24)$$

In terms of an orthonormal coordinate system (24),

$$\frac{\partial E}{\partial y^i} \overset{(15),(18),(23)}{=} e^{\frac{2K}{\sqrt{16-K^2}}\arctan\frac{1}{\sqrt{16-K^2}}\left(\frac{4\left(\frac{\beta}{\alpha}\right)}{\sqrt{1-\left(\frac{\beta}{\alpha}\right)^2}}+K\right)}\left(y^i+\frac{K}{2}\delta^n_i\alpha\sqrt{1-\left(\frac{\beta}{\alpha}\right)^2}\right),$$

$$\frac{\partial E}{\partial y^i} = e^{\frac{2K}{\sqrt{16-K^2}}\arctan\frac{1}{\sqrt{16-K^2}}\left(\frac{4\beta}{\sqrt{\alpha^2-\beta^2}}+K\right)}\left(y^i+\frac{K}{2}\delta^n_i\sqrt{\alpha^2-\beta^2}\right). \quad (25)$$

By (16), (17) and (23),

$$\frac{\partial E}{\partial y^i}_{(0,\ldots,0,\pm1)} = \delta^n_i e^{\pm\frac{2K}{\sqrt{16-K^2}}\frac{\pi}{2}}. \quad (26)$$

4.2. The second y-derivatives of a Finsleroid–Finsler energy function. To compute the second order partial derivatives, it is useful to introduce the function

$$A := \frac{K}{\sqrt{16-K^2}}\arctan\frac{1}{\sqrt{16-K^2}}\left(\frac{4\beta}{\sqrt{\alpha^2-\beta^2}}+K\right).$$

It can be expressed as the function of the variable $s = \beta/\alpha$

$$A(s) := \frac{K}{\sqrt{16-K^2}} \arctan \frac{1}{\sqrt{16-K^2}} \left(\frac{4s}{\sqrt{1-s^2}} + K \right).$$

Since formula (25) can be written as

$$\frac{\partial E}{\partial y^i} = e^{2A\left(\frac{\beta}{\alpha}\right)} \left(y^i + \frac{K}{2} \delta_i^n \sqrt{\alpha^2 - \beta^2} \right),$$

it follows that

$$\frac{\partial^2 E}{\partial y^j \partial y^i} = \frac{\partial e^{2A}}{\partial y^j} \left(y^i + \frac{K}{2} \delta_i^n \sqrt{\alpha^2 - \beta^2} \right) + e^{2A\left(\frac{\beta}{\alpha}\right)} \left(\delta_j^i + \frac{K}{2} \delta_i^n \frac{y^j - \beta \delta_j^n}{\sqrt{\alpha^2 - \beta^2}} \right), \quad (27)$$

where

$$\begin{aligned} \frac{\partial e^{2A}}{\partial y^j} &= e^{2A\left(\frac{\beta}{\alpha}\right)} (2A)' \left(\frac{\beta}{\alpha} \right) \frac{\partial \beta/\alpha}{\partial y^j} \\ &= K \frac{e^{2A\left(\frac{\beta}{\alpha}\right)}}{2\sqrt{1 - \left(\frac{\beta}{\alpha}\right)^2} \left(1 + \frac{K}{2} \frac{\beta}{\alpha} \sqrt{1 - \left(\frac{\beta}{\alpha}\right)^2} \right)} \frac{b_j \alpha - (\beta/\alpha) y^j}{\alpha^2} \end{aligned}$$

as a straightforward calculation shows:

```
> f0:=s->
>     exp((2*K/sqrt(16-K^2))*tan^(-1)((1/sqrt(16-K^2))*
>     ((4*s/sqrt(1-s^2))+K)));
> f1:=s->diff(f0(s),s);
> simplify(f1(s));
> (K/2)*(1/(sqrt(1-s^2)*(1+K*s*sqrt(1-s^2)/2)))*
>     exp((2*K/sqrt(16-K^2))*tan^(-1)((1/sqrt(16-K^2))*
>     ((4*s/sqrt(1-s^2))+K)));
```

Therefore,

$$\frac{\partial e^{2A}}{\partial y^j} = K \frac{e^{2A\left(\frac{\beta}{\alpha}\right)}}{2\sqrt{\alpha^2 - \beta^2} \left(\alpha^2 + \frac{K}{2} \beta \sqrt{\alpha^2 - \beta^2} \right)} (\delta_j^n \alpha^2 - \beta y^j), \quad (28)$$

and we have

$$\frac{\partial^2 E}{\partial y^j \partial y^i} \overset{(27),(28)}{=} \frac{e^{2A\left(\frac{\beta}{\alpha}\right)}}{\alpha^2 + \frac{K}{2} \beta \sqrt{\alpha^2 - \beta^2}} K \frac{\alpha^2 \delta_j^n - \beta y^j}{2\sqrt{\alpha^2 - \beta^2}} \left(y^i + \frac{K}{2} \delta_i^n \sqrt{\alpha^2 - \beta^2} \right)$$

$$+\frac{e^{2A\left(\frac{\beta}{\alpha}\right)}}{\alpha^2+\frac{K}{2}\beta\sqrt{\alpha^2-\beta^2}}\left(\delta^i_j+\frac{K}{2}\delta^n_i\frac{y^j-\beta\delta^n_j}{\sqrt{\alpha^2-\beta^2}}\right)\left(\alpha^2+\frac{K}{2}\beta\sqrt{\alpha^2-\beta^2}\right),$$

where

$$\begin{aligned}
&K\frac{\alpha^2\delta^n_j-\beta y^j}{2\sqrt{\alpha^2-\beta^2}}\left(y^i+\frac{K}{2}\delta^n_i\sqrt{\alpha^2-\beta^2}\right)\\
&\quad+\left(\delta^i_j+\frac{K}{2}\delta^n_i\frac{y^j-\beta\delta^n_j}{\sqrt{\alpha^2-\beta^2}}\right)\left(\alpha^2+\frac{K}{2}\beta\sqrt{\alpha^2-\beta^2}\right)\\
&=K\frac{(\alpha^2\delta^n_j-\beta y^j)y^i+\delta^n_i(y^j-\beta\delta^n_j)\alpha^2}{2\sqrt{\alpha^2-\beta^2}}+\delta^j_i\left(\alpha^2+\frac{K}{2}\beta\sqrt{\alpha^2-\beta^2}\right)\\
&\quad+\frac{K^2}{4}(\delta^n_j\alpha^2-\beta y^j)\delta^n_i+\frac{K^2}{4}\beta\delta^n_i(y^j-\beta\delta^n_j)\\
&=K\frac{\alpha^2(\delta^n_j y^i+\delta^n_i y^j)-\beta(y^iy^j+\alpha^2\delta^n_i\delta^n_j)}{2\sqrt{\alpha^2-\beta^2}}+\delta^j_i\left(\alpha^2+\frac{K}{2}\beta\sqrt{\alpha^2-\beta^2}\right)\\
&\quad+\frac{K^2}{4}\delta^n_i\delta^n_j(\alpha^2-\beta^2).
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{\partial^2E}{\partial y^j\partial y^i}&=\frac{e^{2A\left(\frac{\beta}{\alpha}\right)}}{\alpha^2+\frac{K}{2}\beta\sqrt{\alpha^2-\beta^2}}K\frac{\alpha^2(\delta^n_j y^i+\delta^n_i y^j)-\beta(y^iy^j+\alpha^2\delta^n_i\delta^n_j)}{2\sqrt{\alpha^2-\beta^2}}\\
&+\frac{e^{2A\left(\frac{\beta}{\alpha}\right)}}{\alpha^2+\frac{K}{2}\beta\sqrt{\alpha^2-\beta^2}}\left(\delta^j_i\left(\alpha^2+\frac{K}{2}\beta\sqrt{\alpha^2-\beta^2}\right)+\frac{K^2}{4}\delta^n_i\delta^n_j(\alpha^2-\beta^2)\right). \quad (29)
\end{aligned}$$

4.3. The continuity of the second order partial derivatives.

4.3.1. *The case of* $i=1,\dots,n-1$ *and* $j=1,\dots,n-1$. By (29),

$$\frac{\partial^2E}{\partial y^j\partial y^i}=\frac{e^{2A\left(\frac{\beta}{\alpha}\right)}}{\alpha^2+\frac{K}{2}\beta\sqrt{\alpha^2-\beta^2}}\left(\frac{-K\beta y^iy^j}{2\sqrt{\alpha^2-\beta^2}}+\delta^j_i\left(\alpha^2+\frac{K}{2}\beta\sqrt{\alpha^2-\beta^2}\right)\right), \quad (30)$$

and

$$\left|\frac{y^i}{\sqrt{\alpha^2-\beta^2}}\right|=\left|\frac{y^i}{\sqrt{(y^1)^2+\dots+(y^i)^2+\dots(y^{n-1})^2}}\right|\le 1 \quad \text{(boundedness)}.$$

Therefore,

$$\frac{-K\beta y^iy^j}{2\sqrt{\alpha^2-\beta^2}} \;\mapsto\; 0 \quad \text{as} \quad y^j\to 0$$

and, consequently,

$$\lim_{v\to(0,\dots,0,\pm1)} \frac{\partial^2 E}{\partial y^j \partial y^i}_v \overset{(30)}{=} e^{\pm\frac{2K}{\sqrt{16-K^2}}\frac{\pi}{2}}\delta_i^j. \tag{31}$$

By definition,

$$\frac{\partial^2 E}{\partial y^j \partial y^i}_{(0,\dots,0,\pm1)} := \lim_{s\to 0} \frac{\frac{\partial E}{\partial y^i}_{(0,\dots,0,s,0,\dots,0,\pm1)} - \frac{\partial E}{\partial y^i}_{(0,\dots,0,\pm1)}}{s}.$$

Using that the first order partial derivatives are continuous (Subsection 4.1), the Lagrange mean value theorem shows that the second order partial derivatives at $(0,\dots,0,\pm1)$ are just given by the limit (31):

$$\frac{\partial E}{\partial y^i}_{(0,\dots,0,s,0,\dots,0,\pm1)} - \frac{\partial E}{\partial y^i}_{(0,\dots,0,\pm1)} = \frac{\partial^2 E}{\partial y^j \partial y^i}_{(0,\dots,0,t,0,\dots,0,\pm1)} (s-0),$$

where t is between 0 and s, i.e.

$$\frac{\partial^2 E}{\partial y^j \partial y^i}_{(0,\dots,0,\pm1)} = \lim_{t\to 0} \frac{\partial^2 E}{\partial y^j \partial y^i}_{(0,\dots,0,t,0,\dots,0,\pm1)} \overset{(31)}{=} e^{\pm\frac{2K}{\sqrt{16-K^2}}\frac{\pi}{2}}\delta_i^j,$$

and we have the continuity of the second order partial derivatives.

4.3.2. *The case of* $i = 1,\dots,n-1$ *and* $j = n$. Using that $y^n = \beta$, formula (29) implies that

$$\begin{aligned}\frac{\partial^2 E}{\partial y^n \partial y^i} &= \frac{e^{2A\left(\frac{\beta}{\alpha}\right)}}{\alpha^2 + \frac{K}{2}\beta\sqrt{\alpha^2-\beta^2}}\left(K\frac{\alpha^2 y^i - \beta^2 y^i}{2\sqrt{\alpha^2-\beta^2}} + \delta_i^n\left(\alpha^2 + \frac{K}{2}\beta\sqrt{\alpha^2-\beta^2}\right)\right)\\ &= \frac{e^{2A}}{\alpha^2 + \frac{K}{2}\beta\sqrt{\alpha^2-\beta^2}}\frac{K}{2}y^i\sqrt{\alpha^2-\beta^2}.\end{aligned}$$

Therefore,

$$\lim_{v\to(0,\dots,0,\pm1)} \frac{\partial^2 E}{\partial y^n \partial y^i}_v = 0.$$

The computation of the second order partial derivative at $(0,\dots,0,\pm1)$ by definition needs the same step based on the Lagrange mean value theorem as in Subsection 4.3.1.

4.3.3. *The case of $i = j = n$.* Since $y^n = \beta$, it follows by (29) that

$$\begin{aligned}\frac{\partial^2 E}{\partial y^n \partial y^n} &= \frac{e^{2A\left(\frac{\beta}{\alpha}\right)}}{\alpha^2 + \frac{K}{2}\beta\sqrt{\alpha^2-\beta^2}} \\ &\quad \times \left(K\frac{2\alpha^2\beta - \beta(\beta^2+\alpha^2)}{2\sqrt{\alpha^2-\beta^2}} + \alpha^2 + \frac{K}{2}\beta\sqrt{\alpha^2-\beta^2} + \frac{K^2}{4}(\alpha^2-\beta^2)\right) \\ &= \frac{e^{2A}}{\alpha^2 + \frac{K}{2}\beta\sqrt{\alpha^2-\beta^2}} \left(K\beta\sqrt{\alpha^2-\beta^2} + \alpha^2 + \frac{K^2}{4}(\alpha^2-\beta^2)\right).\end{aligned}$$

Therefore,

$$\lim_{v\to(0,\ldots,0,\pm 1)} \frac{\partial^2 E}{\partial y^n \partial y^n}_{v} = e^{\pm\frac{2K}{\sqrt{16-K^2}}\frac{\pi}{2}}.$$

The computation of the second order partial derivative at $(0,\ldots,0,\pm 1)$ by definition needs the same step based on the Lagrange mean value theorem as in Subsection 4.3.1.

Acknowledgements. The paper was motivated by the oral communication with Professor David Bao at the 50th Symposium on Finsler Geometry (21–25 October 2015, Hiroshima, Japan). I am very grateful for his human and professional encouragement.

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(Received May 14, 2016)